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THE EFFECT OF STATISTICAL FLUCTUATIONS ON CONFINEMENT AND ON THE VACUUM STRUCTURE OF THE $\mathbb{C}P^{n-1}$ MODELS

G. LAZARIDES*
II. Institut für Theoretische Physik, Universität Hamburg, BRD.

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We study the two-dimensional $\mathbb{C}P^{n-1}$ non-linear σ-models at a finite temperature, T, within the $1/n$ expansion. We show that *permanent* confinement of their fundamental particles is a strictly zero-temperature phenomenon. The quantum statistical fluctuations suppress the non-trivial topological structure of the classical $\mathbb{C}P^{n-1}$ models and consequently the background topological density vanishes at $T \neq 0$ for an infinite space "volume". These models do not depend on the vacuum angle θ at any $T \neq 0$.

1. Introduction

Recently, a great amount of interest has been spent on the properties of the $\mathbb{C}P^{n-1}$ non-linear σ-models in two space-time dimensions. These models have been introduced by Eichenherr [1] and their most attractive feature is their similarity with the four-dimensional SU(n) gauge theories. At the classical level, they are conformally invariant and exhibit a non-trivial topological structure. In their quantum version, effects due to instantons are accessible [2] to the powerful $1/n$ expansion, a method far more rigorous than the infrared-divergent dilute-gas approximation. Using this expansion, it has been proved [2] that the $\mathbb{C}P^{n-1}$ models are asymptotically free and exhibit dimensional transmutation as well as a non-trivial vacuum structure characterized by a parameter θ. Moreover, it has been shown [2], that their fundamental particles are permanently confined by a topological Coulomb force.

In this article, we will study the properties of the $\mathbb{C}P^{n-1}$ models at finite temperatures within the $1/n$ expansion. We find that the quantum statistical fluctuations profoundly alter the main characteristics of these theories. In particular, at every non-zero temperature, their topological structure is suppressed and consequently they become θ-independent. In addition, their fundamental particles

* Alexander von Humboldt fellow. On leave of absence from the Physics Department of the University of Ioannina, Greece.

cease to be permanently confined. These results are strictly quantum statistical, since at every non-zero temperature there exist topologically non-trivial classical field configurations with finite action.

We point out that the critical temperature above which the unconfined phase is realized is exactly zero in agreement with the general arguments [3] on phase transitions for infinite systems in one space dimension. In spite of that, the energy needed to separate a particle and an antiparticle is extremely high for low temperatures, so, in practice, one only sees their bound states. On the contrary, at temperatures of the order of the hadronic masses, i.e., $T \simeq 10^{13}$ K, which were realized in the early stages of the evolution of the universe, an appreciable thermal liberation of particles should occur.

θ-independence of the $\mathbb{C}P^{n-1}$ models at every non-zero temperature suggests that the θ-angle is physically irrelevant. The expectation value of the topological density for an infinite system is exactly zero even at infinitesimally small temperatures. This fact does not exclude the possibility of creating a non-zero topological density locally.

This article is organized as follows. In sect. 2, the basic properties of the $\mathbb{C}P^{n-1}$ models are summarized for later use. In sect. 3, we study the $\mathbb{C}P^{n-1}$ models at finite temperatures and discuss the physical consequences. In sect. 4, we give the physical interpretation of our results. Finally, in sect. 5, we summarize our conclusions.

2. The $\mathbb{C}P^{n-1}$ model in two dimensions

This model [2] is an SU(n) invariant theory of fields, $[z](x)$, in two-dimensional space-time, which take values in the $n-1$ dimensional complex projective space, $\mathbb{C}P^{n-1}$, i.e., the space of all equivalence classes $[z]$ of complex vectors $z = (z_1, \cdots, z_n) \neq 0$, two of which being equivalent, if $z' = \lambda z$, $\lambda \in \mathbb{C}$. The action of SU(n) on $\mathbb{C}P^{n-1}$ is defined by

$$g[z]=[gz]\,; \qquad (gz)_\alpha = g_{\alpha\beta} z_\beta\,, \qquad g \in \mathrm{SU}(n)\,. \tag{2.1}$$

The sphere S^{2n-1} of vectors z, $\bar{z}_\alpha z_\alpha = 1 (\alpha = 1, \cdots, n)$, can be looked upon as a U(1) bundle with base space $\mathbb{C}P^{n-1}$. The corresponding projection is the Hopf map

$$\pi: S^{2n-1} \to \mathbb{C}P^{n-1}\,, \qquad \pi(z) = [z]\,. \tag{2.2}$$

A smooth field $[z]$: $R^2 \to \mathbb{C}P^{n-1}$ can then be used to pull back this bundle to a U(1) bundle over R^2 which is always trivial and consequently admits a smooth section $z: R^2 \to S^{2n-1}$. Thus, any field $[z](x)$ can be represented by a field of complex unit vectors $z(x)$ keeping in mind that two such z's should be considered equivalent if they are related by a gauge transformation

$$z'_\alpha(x) = e^{i\Lambda(x)} z_\alpha(x)\,. \tag{2.3}$$

The Euclidean action of the model can then be written as

$$S=\frac{n}{2f}\int d^2x\,\overline{D_\mu z}D_\mu z\,, \tag{2.4}$$

where

$$\begin{aligned} D_\mu &= \partial_\mu + iA_\mu\,,\\ A_\mu &= \tfrac{1}{2}i\bar{z}\overleftrightarrow{\partial}_\mu z = \tfrac{1}{2}i\{\bar{z}_\alpha\partial_\mu z_\alpha-(\partial_\mu\bar{z}_\alpha)z_\alpha\}\,, \end{aligned} \tag{2.5}$$

and is invariant under the gauge transformation (2.3).

The smooth fields $[z](x)$ with finite action can be classified by the topological charge

$$Q\equiv\int d^2x\,q(x)=\frac{i}{2\pi}\int d^2x\,\varepsilon_{\mu\nu}\overline{D_\mu z}D_\nu z\,,\qquad \varepsilon_{12}=1\,. \tag{2.6}$$

The quantum version of this model in the $1/n$ expansion describes an SU(n) vector z of scalar, charged particles with mass m. These particles, called "partons", interact by exchanging scalar (α) and vector (λ_μ) quanta. The non-zero parton mass is entirely due to quantum fluctuations.

The α-interaction is short ranged and does not correspond to an exchange of a physical particle. On the other hand, the λ_μ propagator,

$$D^\lambda_{\mu\nu}(p)=\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)D^\lambda(p)\,, \tag{2.7}$$

has a pole at $p^2=0$, i.e.,

$$D^\lambda(p)\underset{p^2\to 0}{\sim}\frac{12\pi m^2}{p^2}\,. \tag{2.8}$$

Therefore, in the static limit, the λ_μ exchange gives rise to a linear Coulomb potential that confines partons permanently. This phenomenon is also strictly quantum mechanical. We should point out that the pole in eq. (2.8) does not correspond to a physical zero-mass particle, since it does not appear in the two-point function of the gauge-invariant operator $\varepsilon_{\mu\nu}\partial_\mu\lambda_\nu$.

The theory constructed on a $\theta\neq 0$ vacuum is defined by the modified action

$$S^\theta=S-i\theta Q\,. \tag{2.9}$$

It is important to notice that the corresponding quantum theory in the $1/n$ expansion depends on θ only because of the existence of the pole (2.8) in the λ_μ propagator. In particular, the topological density $q(x)$ has a non-zero vacuum expectation value for $\theta\neq 0$.

3. The statistical fluctuations and their consequences

The transition from zero to a finite temperature T is effected [4–6] by the substitution $x_0 \to -ix_2$, where x_0 is the real time and x_2 an angular variable, which ranges from 0 to $\beta = 1/kT$ (k is the Boltzmann constant) and covers simply an S^1. The fields $[z](x)$ are restricted to be periodic functions of x_2, i.e.,

$$[z](x_1, x_2+\beta) = [z](x_1, x_2)\,. \tag{3.1}$$

Thus, they are defined on the Euclidean manifold $R \times S^1$. A smooth field $[z]$: $R \times S^1 \to \mathbb{C}P^{n-1}$ can always be represented by a smooth unit complex vector field z: $R \times S^1 \to S^{2n-1}$. This can easily be seen by repeating the corresponding argument of sect. 2 and using the fact that the U(1) bundles over $R \times S^1$ are always trivial. Of course, two fields z and z' should be considered equivalent if they are related by a gauge transformation,

$$z'(x) = g(x)z(x) \equiv e^{i\Lambda(x)}z(x)\,, \tag{3.2}$$

where

$$g: R \times S^1 \to U(1)\,. \tag{3.3}$$

The periodicity of $g(x)(g(x_1, x_2+\beta) = g(x_1, x_2))$ implies that

$$\Lambda(x_1, x_2+\beta) = \Lambda(x_1, x_2) + 2\pi k, \qquad k \in Z\,. \tag{3.4}$$

Therefore, the space of all gauge transformations (3.3), G, consists of a denumerable number of homotopy classes, $G_k (k \in Z)$.

The generating functional for the temperature-dependent Green function is given by [2, 4]

$$\mathbb{Z}(J, \bar{J}) = \int \mathcal{D}z\mathcal{D}\bar{z} \prod_x \delta\left(|z(x)|^2 - \frac{n}{2f}\right) \cdot \times \exp\left\{-S + \int_{R\times S^1} d^2x\, [\bar{J}(x)z(x) + \bar{z}(x)J(x)]\right\}, \tag{3.5}$$

where

$$S = \int_{R\times S^1} d^2x \left\{\partial_\mu \bar{z}\partial_\mu z + \frac{f}{2n}(\bar{z}\overleftrightarrow{\partial}_\mu z)(\bar{z}\overleftrightarrow{\partial}_\mu z)\right\}. \tag{3.6}$$

Here, the fields $z_\alpha(x)$ have been rescaled by a factor $(n/2f)^{1/2}$ and the integrals are understood over the Euclidean manifold $R \times S^1$.

Field configurations $[z](x)$ on $R \times S^1$ with finite action must obey the following boundary conditions

$$[z](x) \xrightarrow[x_1\to\infty]{} [z_1] = \text{const.}\,,$$

$$[z](x) \xrightarrow[x_1\to-\infty]{} [z_2] = \text{const.}\,. \tag{3.7}$$

Thus, $R\times S^1$ can be compactified to a sphere S^2 by identifying its points at $x_1=\infty$ and $x_1=-\infty$ separately. The fields $[z](x)$ can be considered as mappings

$$[z]: S^2 \to \mathbb{C}P^{n-1} \tag{3.8}$$

and by virtue of the relation [2] $\pi_2(\mathbb{C}P^{n-1})=Z$ (π_2 is the second order homotopy group) are classified in a denumerable number of homotopy classes labeled by an integer topological charge Q.

Eq. (3.7) implies that

$$\begin{aligned} z(x) &\xrightarrow[x_1\to\infty]{} e^{i\Lambda_1(x_2)}z_1\,, \qquad z_1=\text{const}\,, \\ z(x) &\xrightarrow[x_1\to-\infty]{} e^{i\Lambda_2(x_2)}z_2\,, \qquad z_2=\text{const}\,, \end{aligned} \tag{3.9}$$

where $\Lambda_i(x_2)(i=1,2)$ satisfy the relations

$$\Lambda_i(x_2+\beta)=\Lambda_i(x_2)+2\pi k_i\,, \qquad k_i\in Z\,. \tag{3.10}$$

The topological number Q can be written as

$$\begin{aligned} Q &= \frac{1}{2\pi}\int_{R\times S^1} d^2x\,\varepsilon_{\mu\nu}\partial_\mu A_\nu \equiv \int_{R\times S^1} d^2x\,q(x)=k_2-k_1\,, \\ A_\mu &= \frac{f}{n} i\bar{z}\overleftrightarrow{\partial}_\mu z\,. \end{aligned} \tag{3.11}$$

It is easy to show that there exist classical field configurations $z(x)$ on $R\times S^1$ with finite action and any topological number. Thus, the non-trivial topological structure of the classical $\mathbb{C}P^{n-1}$ models persists at any finite temperature at the classical level.

The partition function for the quantum $\mathbb{C}P^{n-1}$ models in the $1/n$ expansion can be constructed from eq. (3.5) by introducing Lagrange multiplier fields $\alpha(x)$ and $\lambda_\mu(x)$ defined on $R\times S^1$ to make the action quadratic in z. Performing the Gaussian z-integral, we obtain [2]

$$\mathbb{Z}\equiv\mathbb{Z}(J=\bar{J}=0)=\int \mathcal{D}\alpha\mathcal{D}\lambda_\mu \exp\{-S_{\text{eff}}(\alpha,\lambda_\mu)\}\,, \tag{3.12}$$

where

$$\begin{aligned} S_{\text{eff}}(\alpha,\lambda_\mu) &= n\,\text{Tr}\ln_{R\times S^1}(\Delta)+\frac{i\sqrt{n}}{2f}\int_{R\times S^1} d^2x\,\alpha(x)\,, \\ \Delta &= -D_\mu D_\mu+m_\beta^2-\frac{i}{\sqrt{n}}\alpha\,, \qquad D_\mu=\partial_\mu+\frac{i}{\sqrt{n}}\lambda_\mu\,, \end{aligned} \tag{3.13}$$

and the operator Δ is restricted to act on functions $z(x)$ defined on $R\times S^1$. The invariance of the theory under the transformation in eq. (3.2) is now reflected by the invariance of $S_{\rm eff}$ under the transformation

$$\alpha(x)\to\alpha(x)\,,$$
$$\lambda_\mu(x)\to\lambda_\mu(x)+\sqrt{n}ig^{-1}\partial_\mu g\,,\qquad g=e^{i\Lambda(x)}:\ {\rm R}\times{\rm S}^1\to{\rm U}(1)\,. \tag{3.14}$$

$S_{\rm eff}$ can be expanded in a power series of $1/\sqrt{n}$

$$S_{\rm eff}=\sum_{\nu=1}^{\infty}(n)^{1-\nu/2}S^{(\nu)}\,. \tag{3.15}$$

The coefficient of the first term is given by

$$\begin{aligned}S^{(1)}&=\frac{i}{2f}\int_{{\rm R}\times{\rm S}^1}{\rm d}^2x\,\alpha(x)-i\,{\rm Tr}_{{\rm R}\times{\rm S}^1}\{(-\Box+m_\beta^2)^{-1}\alpha\}\\&=i\tilde\alpha(0)\left\{\frac{1}{2f}-\beta^{-1}\sum_{l=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{{\rm d}q}{2\pi}(m_\beta^2+q^2+\omega_l^2)^{-1}\right\},\qquad \omega_l=2\pi\beta^{-1}l\,,\end{aligned} \tag{3.16}$$

where

$$\tilde\alpha(p,\omega_l)=\int_{{\rm R}\times{\rm S}^1}{\rm d}^2x\,e^{-ipx_1}\,e^{-i\omega_l x_2}\alpha(x)\,. \tag{3.17}$$

The sum appearing in eq. (3.16) can be evaluated by the usual trick [7]

$$\begin{aligned}\beta^{-1}\sum_{l=-\infty}^{\infty}f(\omega_l)&=\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}f(\omega)\\&+\frac{1}{2\pi}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{{\rm d}z\,f(z)}{e^{i\beta z}-1}+\frac{1}{2\pi}\int_{i\varepsilon-\infty}^{i\varepsilon+\infty}\frac{{\rm d}z\,f(z)}{e^{-i\beta z}-1}\,,\qquad 0<\varepsilon<m_\beta\,,\end{aligned} \tag{3.18}$$

where the first term in the right-hand side of the equation coincides with the corresponding zero-temperature expression with m replaced by m_β and the rest is the finite-temperature contribution which is always finite. Then, regularizing the zero-temperature part of eq. (3.16), calculating the second and the third integral in the right-hand side of eq. (3.18) by distorting the contour of the z-integration so as to pick the contribution of the poles at $z=-i(m_\beta^2+q^2)^{1/2}$ and $z=i(m_\beta^2+q^2)^{1/2}$ respectively, and imposing the saddlepoint condition [2] $S^{(1)}=0$, we obtain the effective parton mass at a temperature T, m_β:

$$\ln\left(\frac{m_\beta^2}{m^2}\right)=4\int_0^{\infty}\frac{{\rm d}q}{\gamma[\exp(\beta\gamma)-1]}>0\,,\qquad \gamma=(m_\beta^2+q^2)^{1/2}>0\,. \tag{3.19}$$

Here, $m=m_\infty$ is the parton mass at $T=0$. Eq. (3.19) implies that

$$m_\beta\geqslant m\,,\qquad m_\beta\xrightarrow[\beta\to0]{}\infty\,,$$
$$\beta m_\beta\xrightarrow[\beta\to0]{}0\,. \tag{3.20}$$

The coefficient of the second term in the expansion (3.15) can be written as

$$S^{(2)}=\frac{1}{2}\int_{R\times S^1} d^2x \int_{R\times S^1} d^2y\, \{\alpha(x)\Gamma^{\alpha}(x-y)\alpha(y)+\lambda_\mu(x)\Gamma^{\lambda}_{\mu\nu}(x-y)\lambda_\nu(y)\}\,, \tag{3.21}$$

where the Fourier transform of $\Gamma^{\lambda}_{\mu\nu}$ is

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=2\delta_{\mu\nu}\beta^{-1}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}(m_\beta^2+q^2+\omega_k^2)^{-1}$$

$$-\beta^{-1}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{(m_\beta^2+q^2+\omega_k^2)(m_\beta^2+(q+p)^2+(\omega_k+\omega_l)^2)}\,. \tag{3.22}$$

Here, $\omega_l=2\pi\beta^{-1}l$, $\omega_k=2\pi\beta^{-1}k$, $k, l\in Z$, $p_\mu=(p,\omega_l)$ and $q_\mu=(q,\omega_k)$. Using eq. (3.18), we now write $\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)$ as follows:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=\tilde{\Gamma}^{\lambda,0}_{\mu\nu}(p,\omega_l)+\Delta\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)\,, \tag{3.23}$$

where $\tilde{\Gamma}^{\lambda,0}_{\mu\nu}$ is the zero-temperature polarization operator with m replaced by m_β and

$$\Delta\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=\frac{1}{2\pi}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}\frac{dz}{e^{i\beta z}-1}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left\{\frac{2\delta_{\mu\nu}}{m_\beta^2+q^2+z^2}\right.$$

$$\left.-\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{(m_\beta^2+q^2+z^2)(m_\beta^2+(q+p)^2+(z+\omega_l)^2)}\right\}$$

$$+\frac{1}{2\pi}\int_{i\varepsilon-\infty}^{i\varepsilon+\infty}\frac{dz}{e^{-i\beta z}-1}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left\{\frac{2\delta_{\mu\nu}}{m_\beta^2+q^2+z^2}\right.$$

$$\left.-\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{(m_\beta^2+q^2+z^2)(m_\beta^2+(q+p)^2+(z+\omega_l)^2)}\right\},$$

$$q_\mu=(q,z)\,, \tag{3.24}$$

is the finite temperature contribution to $\tilde{\Gamma}^{\lambda}_{\mu\nu}$.

$\Delta\tilde{\Gamma}^{\lambda}_{\mu\nu}$ is finite and can be calculated by distorting the contours of the z-integrals in eq. (3.24) so as to pick the contributions of the poles at $z=\pm i(m_\beta^2+q^2)^{1/2}$ and

$z=-\omega_l \pm i(m_\beta^2+(q+p)^2)^{1/2}$. The result is

$$\Delta\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{\gamma(e^{\beta\gamma}-1)}\left\{2\delta_{\mu\nu}-\frac{\Pi_{\mu\nu}}{\omega_l^2-2i\omega_l\gamma+p(p+2q)}\right.$$
$$\left.-\frac{\bar{\Pi}_{\mu\nu}}{\omega_l^2+2i\omega_l\gamma+p(p+2q)}\right\},$$
$$\gamma=(m_\beta^2+q^2)^{1/2}>0,\qquad \Pi_{11}=(p+2q)^2,\qquad \Pi_{12}=\Pi_{21}=(p+2q)(\omega_l-2i\gamma),$$
$$\Pi_{22}=(\omega_l-2i\gamma)^2\,. \tag{3.25}$$

It is easy to see from eq. (3.25) that

$$p_\mu\Delta\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=0\,. \tag{3.26}$$

$S_{\rm eff}$ must be invariant under infinitesimal gauge transformations (see eq. (3.14)). Thus

$$p_\mu\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=0\,, \tag{3.27}$$

which implies that

$$p_\mu\tilde{\Gamma}^{\lambda,0}_{\mu\nu}(p,\omega_l)=0\,. \tag{3.28}$$

$\tilde{\Gamma}^{\lambda,0}_{\mu\nu}$ contains two divergent loop integrals and, therefore, must be renormalized so as to obey eq. (3.28). This leads to the unambiguous expression of ref. [2] with m replaced by m_β. $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ is not a Lorentz tensor for $T\neq 0$ [8]. However, the lack of transverse directions in one-dimensional space implies that any object satisfying eq. (3.27) must be of the form

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(p,\omega_l)=\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{\omega_l^2+p^2}\right)\tilde{\Gamma}^{\lambda}(p,\omega_l)\,,\qquad (p_\mu\neq 0)\,,$$
$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(0,0)=\delta_{\mu 2}\delta_{\nu 2}\tilde{\Gamma}^{\lambda}(0,0)\,. \tag{3.29}$$

Eq. (3.25), then, gives

$$\Delta\tilde{\Gamma}^{\lambda}(p,\omega_l)=\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{\gamma(e^{\beta\gamma}-1)}\left\{4-\frac{(p+2q)^2+(\omega_l-2i\gamma)^2}{\omega_l^2-2i\omega_l\gamma+p(p+2q)}\right.$$
$$\left.-\frac{(p+2q)^2+(\omega_l+2i\gamma)^2}{\omega_l^2+2i\omega_l\gamma+p(p+2q)}\right\}. \tag{3.30}$$

In the static limit, $\omega_l=0$, we obtain

$$\Delta\tilde{\Gamma}^{\lambda}(p,\omega_l=0)=\frac{p^2+4m_\beta^2}{p}\mathbb{P}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{1}{\gamma(e^{\beta\gamma}-1)(q+\frac{1}{2}p)}, \tag{3.31}$$

where $\mathbb{P}$ denotes the proper part of the integral. For $p=0$, this equation becomes

$$\Delta\tilde{\Gamma}^{\lambda}(p=\omega_l=0)=2m_\beta^2\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{1}{\gamma^3(e^{\beta\gamma}-1)}$$

$$+2m_\beta^2\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{\beta\, e^{\beta\gamma}}{\gamma^2(e^{\beta\gamma}-1)^2}\,. \tag{3.32}$$

Therefore $\tilde{\Gamma}^{\lambda}(p=\omega_l=0)\neq 0$ for $T\neq 0$. This can be understood by noticing that condition (3.27) allows for a non-vanishing $\tilde{\Gamma}_{22}(p=\omega_l=0)$, since the energy variable, ω_l, is discrete.

Now, we would like to comment on gauge invariance of the renormalized S_{eff}. A typical term of S_{eff} can be written as

$$\Sigma=\int_{R\times S^1} d^2x\,\lambda_\mu(x)\ldots\Gamma_{\mu\ldots}(x,\ldots)\,. \tag{3.33}$$

The condition $\partial_\mu\Gamma_{\mu\ldots}(x,\ldots)=0$ is not sufficient to automatically insure invariance of S_{eff} under gauge transformations (3.14) which are not homotopic to the identity, since

$$e^{i\Lambda(x)}:\Sigma\to\Sigma-\sqrt{n}(2\pi k\beta^{-1})\ldots\tilde{\Gamma}_{2\ldots}(p=\omega_l=0,\ldots)+\ldots\,,$$
$$e^{i\Lambda(x)}\in G_k,\qquad \tilde{\Gamma}_{2\ldots}(p=\omega_l=0,\ldots)\neq 0\,. \tag{3.34}$$

In the diagrammatic language, this means that the gauge transform of any diagram representing a term of S_{eff} will consist of the original diagram plus all diagrams obtained from it by replacing an external λ_μ line, $\int_{R\times S^1} d^2x\,\lambda_\mu(x)$, by $-\sqrt{n}(2\pi k\beta^{-1})\delta_{\mu 2}$, Fourier transforming with respect to x and putting $p=\omega_l=0$. Therefore, to each order in $1/\sqrt{n}$, we will have an infinite set of diagrams. The zero-temperature part of $\tilde{\Gamma}_{2\ldots}(p=\omega_l=0,\ldots)$ is always zero. Thus, we can restrict ourselves to the finite-temperature parts of these diagrams which are given by convergent expressions of the type (3.24). One can, then, convince himself that all these parts sum up to give us the finite-temperature part of the original diagrams to the given order in $1/\sqrt{n}$ with the internal energy variable z replaced by $z-2\pi\beta^{-1}k$. Changing variables, $z'=z-2\pi\beta^{-1}k$, we see that the renormalized S_{eff} is invariant under all gauge transformations.

To calculate the partition function (3.12) in the $1/n$ expansion we must integrate over the Gaussian fluctuations of $\alpha(x)$ and $\lambda_\mu(x)$. One can show that λ_μ configurations with a non-vanishing topological charge,

$$Q=\frac{1}{2\pi\sqrt{n}}\int_{R\times S^1} d^2x\,\varepsilon_{\mu\nu}\partial_\mu\lambda_\nu(x)\,, \tag{3.35}$$

do not contribute to this integral for $T \neq 0$, since they give an infinite $S^{(2)}$. To see this, let us notice that eq. (3.35) implies that

$$\tilde{\lambda}_2(p, \omega_l = 0) \underset{p\to 0}{\sim} \frac{2\pi\sqrt{n}Q}{ip} + \dots . \tag{3.36}$$

$S^{(2)}$ contains a term

$$\mathscr{F} = \tfrac{1}{2}\beta^{-1} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \tilde{\lambda}_2(-p, \omega_l = 0)\tilde{\Gamma}^{\lambda}_{22}(p, \omega_l = 0)\tilde{\lambda}_2(p, \omega_l = 0) . \tag{3.37}$$

So, eqs. (3.36) and (3.32) imply that $\mathscr{F} = +\infty$ for $Q \neq 0$. Therefore, the non-trivial topological structure of the classical $\mathbb{C}P^{n-1}$ models is suppressed by the quantum statistical fluctuations in the $1/n$ expansion for every $T \neq 0$, and consequently we can restrict the functional integral in eq. (3.12) to λ_μ configurations with $Q = 0$.

Now we are ready to fix the gauge in eq. (3.12) for the partition function. We split the space of λ_μ configurations into a denumerable set of classes, $L_k (k \in Z)$, defined by

$$\lambda_\mu(x) \in L_k , \qquad \text{if and only if } \lambda_\mu \underset{x_1\to\infty}{\sim} -\sqrt{n}\partial_\mu \Lambda(x_2) ,$$
$$\Lambda(x_2 + \beta) = \Lambda(x_2) + 2\pi k . \tag{3.38}$$

Gauge invariance of S_{eff} implies that

$$\mathbb{Z} = \sum_{k=-\infty}^{\infty} \int_{L_k} \mathscr{D}\lambda_\mu \mathscr{D}\alpha \, e^{-S_{\text{eff}}} = \int_{L_0} \mathscr{D}\lambda_\mu \mathscr{D}\alpha \, e^{-S_{\text{eff}}} \tag{3.39}$$

(we do not keep track of infinite constants). We are now left with the freedom to perform gauge transformations which are homotopic to the identity. Therefore, the gauge condition $\partial_\mu \lambda_\mu = 0$, supplemented with the assumption that $\Lambda(x)$ tends to a regular function of x_2 as $|x_1| \to \infty$, fixes the gauge completely. The partition function reads

$$\mathbb{Z} = \int_{L_0} \mathscr{D}\lambda_\mu \mathscr{D}\alpha \, e^{-S_{\text{eff}}(\alpha, \lambda_\mu)} \prod_x \delta(\partial_\mu \lambda_\mu) . \tag{3.40}$$

The temperature-dependent λ_μ propagator

$$D^{\lambda}_{\mu\nu}(p, \omega_l) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2 + \omega_l^2}\right) D^{\lambda}(p, \omega_l) , \qquad D^{\lambda} = (\tilde{\Gamma}^{\lambda})^{-1} , \tag{3.41}$$

does not have a pole at $p = \omega_l = 0$ for $T \neq 0$. Thus, the topological Coulomb force due to the quantum fluctuations at $T = 0$ is screened by the quantum statistical fluctuations at any $T \neq 0$. In particular, in the static limit, $\omega_l = 0$, and for $\beta m_\beta \gg 1$, eqs. (3.32) and (2.8) give

$$\tilde{\Gamma}^\lambda(p, \omega_l = 0) \underset{p \to 0}{\sim} \frac{p^2}{12\pi m_\beta^2} + 2m_\beta^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{\beta\, e^{\beta\gamma}}{\gamma^2(e^{\beta\gamma} - 1)^2} . \tag{3.42}$$

Therefore, the screening radius, r, for $\beta m_\beta \gg 1$ is given by

$$r^{-2} \simeq 12\pi m_\beta^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{2m_\beta^2 \beta\, e^{\beta\gamma}}{\gamma^2(e^{\beta\gamma} - 1)^2}$$

$$\underset{\beta m_\beta \to \infty}{\sim} c m_\beta^2 (\beta m_\beta)^\nu\, e^{-\beta m_\beta} , \qquad c > 0 , \qquad 0 \leq \nu \leq \tfrac{1}{2} . \tag{3.43}$$

The transition from the $\theta = 0$ to a $\theta \neq 0$ vacuum is effected by the replacement

$$S_{\text{eff}} \to S_{\text{eff}}^\theta = S_{\text{eff}} - i \frac{\theta}{2\pi\sqrt{n}} \int_{R \times S^1} d^2x\, \varepsilon_{\mu\nu} \partial_\mu \lambda_\nu(x) . \tag{3.44}$$

The temperature-dependent λ_μ propagator does not have a pole at $p = \omega_l = 0$. Thus, the quantum $\mathbb{C}P^{n-1}$ model in the $1/n$ expansion is θ-independent at any $T \neq 0$. In particular, the expectation value of the topological density is always zero for $T \neq 0$, i.e.,

$$\langle q(x) \rangle_\theta = 0 , \qquad \forall \theta , \qquad T \neq 0 . \tag{3.45}$$

4. Physical interpretation

The long-range behaviour of the quantum $\mathbb{C}P^{n-1}$ model in the $1/n$ expansion at $T = 0$ can be mimicked by the phenomenological Euclidean Lagrangian density [9]

$$\mathcal{L} = \overline{D_\mu z} D_\mu z + m^2 \bar{z} z + \frac{1}{12\pi m^2}\left(\frac{1}{2} F^2\right) ,$$

$$D_\mu = \partial_\mu + \frac{i}{\sqrt{n}} \lambda_\mu , \qquad F = \varepsilon_{\mu\nu} \partial_\mu \lambda_\nu , \qquad n \to \infty . \tag{4.1}$$

So, we have a system of n partons (antipartons) with mass m and electric charge $1/\sqrt{n}(-1/\sqrt{n})$ interacting through an instantaneous Coulomb force.

The following rough argument can give us some insight into our result. One can say that, at a non-zero temperature, the mean parton (antiparton) density, which in the absence of interaction is

$$\frac{N}{L} = n \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{e^{\beta\gamma} - 1} , \qquad \gamma = (m^2 + q^2)^{1/2} , \qquad n \to \infty , \tag{4.2}$$

provides a screening of the Coulomb potential between two external static oppositely charged particles. The screening radius, for $n \to \infty$, is given by the classical Debye-Hückel formula [6]

$$r^{-2} = \sum_i e_i^2 \left[\frac{\partial (n_i/L)}{\partial \mu_i} \right]_{\mu_i = 0}, \tag{4.3}$$

where we sum over all species of particles in the system. e_i is the charge, n_i the mean number and μ_i the chemical potential of the particles of the ith kind. Eqs. (4.2) and (4.3) give

$$r^{-2} = 12\pi m^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{2\beta\, e^{\beta\gamma}}{(e^{\beta\gamma}-1)^2}, \qquad (\text{for } n \to \infty), \tag{4.4}$$

which agrees with eq. (3.43) for $q^2 \ll m_\beta^2$. The electric field between two oppositely charged condensor plates placed at $x_1 = \infty$ and $x_1 = -\infty$ vanishes. Thus, for $T \neq 0$, there exists no background electric field (topological density) in the "vacuum".

It is physically instructive to study the spectrum of excitations of our system at $T \neq 0$. For that, we need the retarded time-dependent λ_μ propagator [5, 6]. This is constructed from the temperature-dependent λ_μ propagator, $D^\lambda(p, \omega_l)$ ($\omega_l > 0$), by analytic continuation with respect to the energy, $\omega_l \to -i\omega$ ($\omega \in \mathbb{C}$), and is analytic for Im $\omega > 0$. Its poles for Im $\omega \leq 0$ provide us with the energy and attenuation of the excitations.

To this end, let us analytically continue ($\omega_l \to -i\omega$, Im $\omega > 0$) $\Delta\tilde{\Gamma}^\lambda(p, \omega_l)$. Eq. (3.30) gives

$$\Delta\tilde{\Gamma}^{\lambda,R}(p,\omega) = 2(p^2 + 4m_\beta^2 - \omega^2) \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{\gamma(e^{\beta\gamma}-1)} \times \frac{p(p+2q)-\omega^2}{(p(p+2q)-\omega^2)^2 - 4\omega^2\gamma^2}. \tag{4.5}$$

It is easy to see that, for $p^2 < \omega^2 < 4m_\beta^2 + p^2$, this function is analytic and Im $\Delta\tilde{\Gamma}^{\lambda,R} = 0$. In the long-wavelength limit, $p = 0$, and for $\omega^2 \ll m_\beta^2$, $\beta m_\beta \gg 1$, one obtains from eqs. (4.5) and (2.8) that

$$\tilde{\Gamma}^{\lambda,R}(\omega, p=0) \underset{\omega \to 0}{\sim} \frac{-\omega^2}{12\pi m_\beta^2} + \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{2m_\beta^2}{\gamma^3(e^{\beta\gamma}-1)}. \tag{4.6}$$

Therefore, in the long-wavelength limit and for $T \neq 0$, there exists an undamped plasma oscillation [8] of our system whose frequency is given by

$$\omega_0^2 \simeq 12\pi m_\beta^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{2m_\beta^2}{\gamma^3(e^{\beta\gamma}-1)} \underset{\beta m_\beta \to \infty}{\sim} c' m_\beta^2 (\beta m_\beta)^{-\rho}\, e^{-\beta m_\beta}, \qquad c' > 0, \qquad \tfrac{1}{2} \leq \rho \leq 1. \tag{4.7}$$

This is a real mode of the system, since the corresponding pole appears in the retarded time-dependent two-point function of the gauge-invariant operator $F = \varepsilon_{\mu\nu}\partial_\mu\lambda_\nu$.

One can easily understand this plasma oscillation by an elementary argument [10]. To this end, we return to the phenomenological description of the $\mathbb{C}P^{n-1}$ model (see eq. (4.1)) and we confine our system in a box of length L. Displacement of the partons by x and the antipartons by $-x$ creates an effective electric field through the system

$$F = -2x\left(\frac{N}{L}\right)\frac{12\pi m^2}{\sqrt{n}}. \tag{4.8}$$

Then, the equation of motion of a parton (antiparton),

$$\frac{1}{\sqrt{n}}F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \tag{4.9}$$

implies that the system oscillates with a frequency ω_0 given by

$$\omega_0^2 = 12\pi m^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi}\frac{2}{m(\mathrm{e}^{\beta\gamma}-1)}. \tag{4.10}$$

This equation coincides with eq. (4.7) for $q^2 \ll m_\beta^2$. The plasma oscillation can be thought as an oscillation of the dipole moment of the system with frequency ω_0 and exists only for $T \neq 0$. One can say that two particles interact in the presence of the system by exchanging a virtual plasmon.

5. Conclusions

We have studied the properties of the two dimensional $\mathbb{C}P^{n-1}$ non-linear σ-models at finite temperatures within the $1/n$ expansion. We showed that *permanent* confinement of their fundamental particles is a strictly zero-temperature phenomenon and that these theories become θ-independent at every $T \neq 0$. These conclusions depend heavily on the fact that the λ_μ propagator does not have a pole at $p_\mu = 0$ for $T \neq 0$, i.e., temperature acts like an infrared cutoff so there are no infrared singularities at $T \neq 0$.

The non-trivial topological structure of the classical $\mathbb{C}P^{n-1}$ models is now suppressed by the quantum statistical fluctuations at $T \neq 0$ and the topological Coulomb force is screened. Consequently, the background effective electric field (topological density) vanishes for an infinite system at $T \neq 0$. All these results are strictly *quantum statistical.*

In addition, we have argued that at $T \neq 0$ there exists a real undamped plasma oscillation. One can say that particles interact in the presence of the system at $T \neq 0$ by exchanging a virtual plasmon.

Our results can be understood qualitatively as follows: an external static charge impurity in the system at $T \neq 0$ is clothed by an oppositely charged statistical parton cloud, which extends up to a distance r. Thus, the effective Coulomb field of this impurity is screened. Two oppositely charged condensor plates placed at $x_1 = \infty$ and $x_1 = -\infty$ produce no effective electric field at any finite space point for $T \neq 0$. Consequently, the expectation value of the topological density is zero and the theory becomes θ-independent for any $T \neq 0$. The parameter θ, which appears at zero temperature, now becomes physically irrelevant.

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