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MORE ABOUT THE *S*-MATRIX OF THE CHIRAL SU(*N*) THIRRING MODEL

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We fix the bound state poles of the *S*-matrix of the chiral SU(*N*) Thirring model by general arguments. Avoiding an infrared problem by using a modified $1/N$ expansion, the result is confirmed in leading order.

The exact *S*-matrix of the chiral SU(*N*) invariant Thirring model described by the Lagrangian

$$\mathscr{L} = i\bar{\psi}\gamma\,\partial\psi - \tfrac{1}{2}(\sigma^2+\pi^2) + g\bar{\psi}(\sigma + i\gamma^5\pi)\psi, \tag{1}$$

was proposed in ref. [1].

Let us define $2\to 2$ particle scattering amplitudes involving the scattering of particles P_α belonging to the fundamental representation of SU(*N*) and the antiparticles A_α by

$$^{\mathrm{out}}\langle \mathrm{P}_\beta(p_1')\mathrm{A}_\delta(p_2')|\mathrm{P}_\alpha(p_1)\mathrm{A}_\delta(p_2)\rangle^{\mathrm{in}} = {}_{\alpha\beta}F_{\beta\delta}(\theta)\,\delta_{\mathrm{if}} - {}_{\alpha\gamma}B_{\delta\beta}(\theta)\bar{\delta}_{\mathrm{if}}\,, \tag{2a}$$

$$^{\mathrm{out}}\langle \mathrm{P}_\beta(p_1')\mathrm{P}_\delta(p_2')|\mathrm{P}_\alpha(p_1)\mathrm{P}_\delta(p_2)\rangle^{\mathrm{in}} = {}_{\alpha\gamma}S_{\beta\delta}(\theta)\,\delta_{\mathrm{if}} - {}_{\alpha\gamma}S_{\delta\beta}(\theta)\bar{\delta}_{\mathrm{if}}\,, \tag{2b}$$

where

$$\delta_{\mathrm{if}} \equiv \delta(\boldsymbol{p}_1' - \boldsymbol{p}_1)\,\delta(\boldsymbol{p}_2' - \boldsymbol{p}_2)\,,$$
$$\bar{\delta}_{\mathrm{if}} \equiv \delta(\boldsymbol{p}_1' - \boldsymbol{p}_2)\,\delta(\boldsymbol{p}_2' - \boldsymbol{p}_1)\,, \tag{2c}$$

and

$${}_{\alpha\gamma}F_{\beta\delta}(\theta) = t_1(\theta)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + t_2(\theta)\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,,$$
$${}_{\alpha\gamma}B_{\delta\beta}(\theta) = r_1(\theta)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + r_2(\theta)\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,,$$
$${}_{\alpha\gamma}S_{\beta\delta}(\theta) = u_1(\theta)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + u_2(\theta)\,\delta_{\alpha\delta}\,\delta_{\gamma\beta}\,, \tag{2d}$$

with $\mathrm{ch}\theta = p_1 p_2/m^2$. Crossing implies $u_1(\theta) = t_1(i\pi - \theta)$, $u_2(\theta) = t_2(i\pi - \theta)$ and $r_1(\theta) = r_2(i\pi - \theta)$.

The main result of ref. [1] was the proposal that the S-matrix of the chiral SU(N) Thirring model factorises and belongs to the class II of the SU(N) S-matrices which were computed previously [2]. This S-matrix has vanishing backward scattering and the amplitudes are given by the following formulas

$$t_1(\theta) = (\text{pole factor})\, t_1^{(0)}(\theta)\,, \tag{3a}$$

$$u_2(\theta) = -\frac{2}{N}\frac{i\pi}{\theta}u_1, \qquad r_1(\theta) = r_2(\theta) = 0\,, \tag{3b}$$

with

$$t_1^{(0)}(\theta) = \frac{\Gamma(\frac{1}{2} + \theta/2i\pi)\Gamma(\frac{1}{2} + 1/N - \theta/2i\pi)}{\Gamma(\frac{1}{2} - \theta/2i\pi)\Gamma(\frac{1}{2} + 1/N + \theta/2i\pi)} \tag{3c}$$

the minimal solution with no poles and zeros in the physical sheet and

$$\text{pole factor} = \prod_{i=1}^{l} \frac{\mathrm{sh}\,\frac{1}{2}(\theta + \theta_i)}{\mathrm{sh}\,\frac{1}{2}(\theta - \theta_i)} \prod_{j=1}^{l'} \frac{\mathrm{ch}\,\frac{1}{2}(\theta - \theta'_j)}{\mathrm{ch}\,\frac{1}{2}(\theta + \theta'_j)}\,. \tag{3d}$$

For the identification of this S-matrix with the chiral Thirring model a variety of classical arguments and a check of the amplitudes r_1, r_2, u_2 and t_2 in leading order of the usual $1/N$ expansion were presented in ref. [1]. Furthermore it was noted that the multiparticle S-matrix factorises in leading order $1/N$ in terms of two-particle amplitudes [3, 1] (assuming that the infrared divergent parts make sense).

There was, however, an infrared problem involved in the evaluation of the amplitudes $u_1(\theta)$, $t_1(\theta)$ within the ordinary $1/N$ expansion. Knowledge of these amplitudes is crucial if one wishes to check the precise form of the pole factor (3d) in leading order $1/N$. Therefore using an *ad hoc* regularisation procedure and relying on consistency arguments with semiclassical calculations [4] only speculations were made on the form of t_1. These speculations are, however, inconsistent.

In the present note we apply a general argument of Karowski [5] to this model, which (under the assumption of a reasonable non-relativistic limit) leaves only one possibility for the pole factor (3d) open. The result is confirmed within an improved $1/N$ expansion which avoids the infrared problem in computing $t_1(\theta)$ to lowest order.

The semiclassical analysis [4] of the model indicates a rich spectrum of multi-fermion and boson bound-states. As in the treatment of the Gross-Neveu model by Zamolodchikov and Zamolodchikov [6] the spectrum has to be "fitted" by an appropriate choice of the pole factor (3d). Recently Karowski [5] has clarified the question when the insertion of a pole factor (with correct residuum in an appropriate channel) has a bound-state interpretation. It turns out that there are only a very limited number of possibilities to insert a bound-state pole. Poles appearing in several channels would correspond to ghosts in channels with the wrong residuum.

The consistency requirement is that no bound-state ghost transition is allowed. This is, for example, fulfilled in the massive Thirring model where at points of the coupling corresponding to vanishing backward scattering redundant poles exist [7]. In the present model redundant poles are not allowed because they are in contradiction with the consistency requirements. For example if we would try to fit the boson part of the semiclassical spectrum by introducing a pole in $t_1(\theta)$ at $\theta - i\pi\alpha/N$ this implies poles in both parity +, − adjoint channels $M_{2\pm} = t_1$ only one of which has the correct residuum [1, 5]. Consistency requires

$$0 = {}^{\text{out}}\langle M^i_- f^\alpha | M^i_+ f^\beta \rangle^{\text{in}} \propto \lambda^i_{\beta'\alpha'} \{ \langle \alpha' \bar{\beta}' \gamma' | S_{23} S_{13} | \alpha \bar{\beta} \gamma \rangle - \langle \bar{\beta}' \alpha' \gamma' | S_{23} S_{13} | \bar{\beta} \alpha \gamma \rangle \} \lambda^i_{\alpha\beta} , \tag{4}$$

leading to the equations

$$t_2' u_1 - u_1' t_2 = t_1' u_2 - u_2' t_1 = t_1' u_1 - u_1' t_1 , \qquad \text{for } N = 2 , \tag{5a}$$

$$\frac{u_1}{u_1'} = \frac{t_1}{t_1'} = \frac{u_2}{u_2'} = \frac{t_2}{t_2'} , \qquad \text{for } N \geq 3 , \tag{5b}$$

where

$$u = u(\theta + i\alpha\pi/2N) , \qquad u' = u(\theta - i\alpha\pi/2N) ,$$

which are in disagreement with the S-matrix (2). In fact the semiclassical boson bound states are explained by noting $t_1(0) = 1$. In the massive Thirring model one has $t(0) = 0$ except at those points of the coupling where a boson bound state is just at threshold where one has $t(0) = 1$. Therefore we claim that in the exact solution of the present model the poles interpreted in the semiclassical treatment as boson bound states remain at threshold.

By the above argument one is left with precisely one bound-state pole possible in the chiral Thirring model in the antisymmetric channel

$$\text{pole factor} = \frac{\text{ch}\, \frac{1}{2}(\theta - 2\pi i/N)}{\text{ch}\, \frac{1}{2}(\theta + 2\pi i/N)} . \tag{6}$$

This pole is so arranged that it is not present in the symmetric channel (*cf.*, ref. [1]).
It implies a spectrum in exact agreement with the semiclassical multifermion spectrum

$$m_n = m \frac{\sin(\pi n/N)}{\sin(\pi/N)} . \tag{7}$$

All other possible poles are CDD poles in the real sense of the word, i.e., they have no possible bound-state interpretation. It has been noted very early that CDD poles are excluded in the S-matrices of field theories which have a reasonable non-relativistic limit in the sense of potential scattering [8].

Another interesting contribution to the better understanding of our S-matrix has been recently made by Kurak and Swieca [9]. These authors claim that since the physical fermions have lost not only the chiral U(1) but also the charge U(1) symmetry, they transform according to pure SU(N). They propose an interpretation of the antiparticles as a bound state of $n-1$ particles and perform consistency checks at the S-matrix level. The condition of vanishing backward scattering is an immediate consistency requirement. At present we are not able to comment on this approach in general. We remark, however, the fact that the fermions have lost their U(1) charge has already been demonstrated in leading order $1/N$ in ref. [1]. There it was shown that the matrix element of the chiral current $\tilde{j}^0_\mu = \epsilon^\nu{}_\mu j^0_\nu$ between the physical particles vanishes in leading order $1/N$. So far we have found no inconsistencies concerning the opinion [10, 1] that the $1/N$ expansion is reliable as long as no infrared problems are involved. There remains only the question of the relevance of the abnormal statistics mentioned in [9].

Now we come to the check of the bound-state factor (6) within a new $1/N$ expansion.

There are several ways of formulating the infrared improved $1/N$ expansion [10]. The method we present, using bosonization has certain advantages. Introducing fields ρ, θ defined through:

$$\sigma + i\pi = \rho\, e^{i\theta}, \tag{8}$$

in the Lagrangian we obtain:

$$\mathscr{L} = \bar{\psi} i \not\partial \psi - \tfrac{1}{2}\rho^2 + g\rho\bar{\psi}\, e^{i\theta\gamma_5}\psi. \tag{9}$$

The fermions in (9) can be bosonized by the usual formulae [10] and we obtain

$$\mathscr{L} = \sum \tfrac{1}{2}(\partial_\mu\varphi_i)^2 - \tfrac{1}{2}\rho^2 + g\rho \sum \cos(\theta + \varphi_i\sqrt{4\pi}). \tag{10}$$

We can use the bosonization formulae once more and introduce the field $\tilde{\psi}$ by

$$\bar{\tilde{\psi}}_i i \tilde{\psi}_i = \tfrac{1}{2}\left[\partial_\mu\left(\varphi_i + \frac{\theta}{\sqrt{4\pi}}\right)\right]^2, \qquad \text{etc.},$$

to obtain

$$\mathscr{L} = \bar{\tilde{\psi}} i \not\partial \tilde{\psi} - \tfrac{1}{2}\rho^2 + g\rho\bar{\tilde{\psi}}\tilde{\psi} + \tfrac{1}{2}\partial_\mu\theta\bar{\tilde{\psi}}\gamma^\mu\nu_5\tilde{\psi} + \frac{N}{8\pi}(\partial_\mu\theta)^2. \tag{11}$$

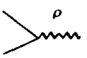
The important point is that the massless θ field only appears in the Lagrangian as a derivative $\epsilon_{\mu\nu}\,\partial^\nu\theta = A_\mu$: thus infrared divergence problems are manifestly improved. In the process the ψ field describing infra-particles in the sense of Schroer [11] has been replaced by the field $\tilde{\psi}$ which is an interpolating field for the physical particles.

The $1/N$ perturbation theory is now set up in the standard way: expanding the action around the stationary point $\rho = m$, $A_\mu = 0$, with the dynamically generated mass. The ρ and A_μ propagators are given by:

$$D^\rho = -\frac{2\pi i}{N}\frac{\operatorname{th}\frac{1}{2}\phi}{\phi}, \tag{12a}$$

$$D^A_{\mu\nu} = -\frac{2\pi i}{Nm^2}\frac{1}{\phi \,\mathrm{th}\,\frac{1}{2}\phi}(K_\mu K_\nu - g_{\mu\nu}K^2) \underset{K^2\sim 0}{\sim} \frac{4\pi i}{N}\frac{K_\mu K_\nu - g_{\mu\nu}K^2}{K^2}, \tag{12b}$$

for $K^2 = -4m^2\,\mathrm{sh}^2\,\frac{1}{2}\phi$, and the vertices are



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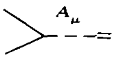

$= -\frac{1}{2}i\gamma_\mu\,.$

It is clear that to lowest order the results of ref. [1] are reproduced for amplitudes u_2, t_2, r_1, r_2 which are free of infrared problems. The advantage now is that the amplitudes t_1, u_1 can be calculated without any difficulty. The ρ-exchange diagram yields for u_1 a term $\pi i/N\,\mathrm{sh}\,\theta$ as before. The A_μ exchange yields:

$$\frac{(-\frac{1}{2}i)^2}{4m\,\mathrm{sh}\,\theta}\bar{u}\gamma^\mu u\bar{u}'\gamma_\mu u'\left(-\frac{4\pi i}{N}\right) = \frac{\pi i}{N}\coth\theta\,. \tag{13}$$

The fact that the $K_\mu K_\nu$ in the A_μ propagator gives zero contribution can be seen by taking the particles on-shell successively. We thus obtain:

$$u_1 = 1 + \frac{i\pi}{N}\coth\tfrac{1}{2}\theta + \dots, \tag{14}$$

in agreement with the assumed form of the S-matrix.

We also have checked that the multiparticle scattering amplitudes factorize again in terms of two-particle amplitudes in leading order, $1/N$.

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