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Z_2 Monopoles in the Standard $SU(2)$ Lattice Gauge Theory Model[1]

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Abstract. The standard $SU(2)$ lattice gauge theory model without fermions may be considered as a Z_2 model with monopoles and fluctuating coupling constants. At low temperatures β^{-1} (= small bare coupling constant) the monopoles are confined.

1. Introduction and Summary of Results

We consider the standard model of an $SU(2)$ gauge field theory without fermions on a hypercubic lattice Λ in 4-dimensional Euclidean space time. It was introduced by Wilson [1] and is defined as follows (notations and definitions as in [2]).

The (random) variables of the theory are the socalled "string bit variables" $U[b]\in G=SU(2)$: A configuration U is a map which assigns an element $U[b]\in G$ to every directed link b between nearest neighbour vertices x, y on the lattice in such a way that $U[b]\to U[b]^{-1}$ under reversal of direction of the link b.

If C is an oriented path (with prescribed initial point if it is closed) which consists of links $b_1 \dots b_n$ then we write

$$U[C]=U[b_n]\dots U[b_1] \tag{1.1a}$$

In particular, a plaquette p(= 2-dimensional unit cell) has a boundary $\dot{p}=\partial p$ consisting of four links $b_1 \dots b_4$. So

$$U[\dot{p}]=U[b_4]\dots U[b_1] \tag{1.1b}$$

$U[C]$ is called the parallel transporter around C.

The Euclidean action of the model is

$$L(U)=\sum_p \mathscr{L}(U[\dot{p}]) \quad \text{with } \mathscr{L}(V)=\beta \operatorname{tr} V$$
$$\text{for } V\in SU(2). \tag{1.2}$$

Sum over p is over all plaquettes in the lattice. Their orientation is immaterial since $\mathscr{L}(V)=\mathscr{L}(V^{-1})$. Boundary conditions will be specified in Sect. 2.

Observables are (real) functions $F(U)$ of the random variables $U[b]$. Their expectation value in the standard model is

$$\langle F\rangle=\int d\mu F(U), \tag{1.3}$$

$$d\mu=\frac{1}{Z}e^{L(U)}\prod_b dU[b]; \quad Z=\int e^{L(U)}\prod_b dU[b]. \tag{1.4}$$

Integrations over $U[b]$ are always over G; $dU[b]$ is normalized Haar measure on G. (Normalized means $\int_G dU[b]=1$). The product over b runs over all links in the lattice.

The center of the gauge group G will be denoted by Γ. It consists of matrices $\pm\mathbb{1}$. We will not distinguish them in notation from numbers ± 1. $\Gamma\approx Z_2$ (the integers 0, 1 with addition modulo 2).

We will show that the model can be reinterpreted as a Z_2 gauge theory with monopoles and fluctuating coupling constants.* This new theory has as its variables

$$\dot{U}[b]\in G/\Gamma \quad \text{and} \quad \sigma[p]=\pm 1\in\Gamma \tag{1.5}$$

They are associated with links b resp. plaquettes p. They are not completely independent. The variables $\dot{U}[b]$ specify the values of a function $\rho[c]=\pm 1\in\Gamma$ for every 3-cell (cube) c of the lattice. Given $\dot{U}$, the variables $\sigma[p]$ must satisfy

$$\prod_{p\in\partial c}\sigma[p]=\rho[c] \quad \text{for every cube } c \tag{1.6}$$

Product is over the 6 plaquettes in the boundary ∂c of a cube c. The meaning of this equation becomes clear if we go over to additive language. Let e_μ be the unit lattice vector in μ-direction, and $e_{-\mu}=-e_\mu$. Let $p=p_{\mu\nu}(x)$ be the plaquette with corner points $x, x+e_\mu, x+e_\mu+e_\nu, x+e_\nu$. Similarly, let $c=c_{\mu\nu\lambda}(x)$ be the cube with corner points $x, x+e_\mu, \dots,$

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* Another model with monopoles has recently been studied by Yoneya [3]

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$x+e_\mu+e_\nu+e_\lambda$. We define field strength F and current j taking values in the field $Z_2=\{0,1\}$ by the formulae

$$\sigma[p]=\exp i\pi F_{\mu\nu}(x) \quad \text{if } p=p_{\mu\nu}(x) \tag{1.7}$$

$$\rho[c]=\exp i\pi j_{\mu\nu\lambda}(x) \quad \text{if } c=c_{\mu\nu\lambda}(x)$$

Then (1.6) takes the form

$$\Delta_\mu F_{\nu\lambda}+\Delta_\lambda F_{\mu\nu}+\Delta_\nu F_{\lambda\mu}=j_{\mu\nu\lambda}(x) \tag{1.6'}$$

Δ_μ is the difference operator on the lattice (viz. $\Delta_\mu f(x)=f(x+e_\mu)-f(x)$). Equation (1.6′) is to be regarded as an equation between elements of Z_2, i.e. integers modulo 2.

Equation (1.6′) is of the form of 2nd Maxwell equations in the presence of a magnetic current j. This current is a gauge invariant function of the variables $\dot{U}[b]$. It is conserved. The conservation law reads

$$\prod_{c\in\partial h}\rho[c]=1 \quad \text{for every hypercube } h \tag{1.8}$$

Or, equivalently, if we set $j_{\mu\nu\lambda}(x)=\tilde{j}_\rho(x)$ for $\mu\nu\lambda\rho$ a cyclic permutation of 1234. then

$$\Delta_\mu \tilde{j}_\mu(x)=0 \tag{1.8'}$$

Explicit expressions for ρ or j and for the path measure in terms of the new variables will be derived in the next section.

Let us draw some tentative conclusions from the observed existence of monopoles in the standard $SU(2)$ model, and compare with the modified $SU(2)$ model studied in [2]*.

The modified model is obtained by ruling the monopoles out of existence. This is done by including in the path measure a factor $\prod_c \theta(\prod_{p\in\partial c} \operatorname{tr} U[\dot{p}])$. Product over c runs over all cubes in the lattice. This amounts to admitting only configurations U such that $\rho[c]=1$ for all cubes c, cp. (2.3) below.

Both theories have formally the same continuum limit, and the monopoles in the standard model become unimportant as $\beta\to\infty$ in the following sense. Let X be any nonempty set of $|X|$ cubes. Then

$$\langle\prod_{c\in X}\theta(-\rho[c])\rangle \leqq D(\beta)^{(1/4)|X|} \tag{1.9}$$

with $D(\beta)\leqq \text{const}\cdot e^{-\beta/13}\to 0$ as $\beta\to\infty$.

This follows from inequalities (1.23) of [2] since $\rho[c]=-1$ implies by (2.3) below that $\operatorname{tr} U[\dot{p}]<0$ for at least one plaquette $p\in\partial c$, and at most four cubes in X can have a common plaquette in their boundaries.

Next, let us look at the 't Hooft disorder parameter [5]. Let Σ be the time $t=0$ hyperplane in the lattice Λ and S a set of links in Σ. Its coboundary $\hat{\partial}S$ consists of those plaquettes p in Σ which have an odd number of links of S in their boundary ∂p. We are mainly interested in S, $\hat{\partial}S$ of the form shown in Fig. 1; $\hat{\partial}S$ is a closed loop bounding the surface S in the dual lattice of Σ. The 't Hooft disorder parameter $\langle B[S]\rangle$ is defined as in [2]. We give another definition which will be shown to be equivalent. The 't Hooft disorder parameter is equal to the expectation value of the Wilson loop integral [1] for a monopole transported around $\hat{\partial}S$. Equivalence is proven by performing a duality transformation on the Z_2 theory. In place of the variables $\sigma[p]$ one is then dealing with new variables $\omega[c]=\pm 1$ attached to cubes c of the lattice Λ. These cubes may be considered as links on the dual lattice of Λ. We may go over to additive language as in (1.7). To this end one introduces vector potentials $\tilde{A}_\mu(x)$ taking values in the field $Z_2=\{0,1\}$ by

$$\omega[c]=\exp i\pi\, \tilde{A}_\mu(x)\equiv \exp i\pi\, \tilde{A}[c] \tag{1.10}$$

if $c=c_{\nu\lambda\rho}(x)$, $\mu\nu\lambda\rho$ = permutation of 1234

(Signs are unimportant since $-A=A$ for elements A of Z_2). The duality transformation interchanges the role of electric and magnetic fields. As a result, the current j now appears as an electric current coupled to the vector potential $\tilde{A}$, cp. (3.10b) of Sect. 3 for the new action, and the discussion following it.

Let C be the set of cubes protruding from plaquettes $p\in\hat{\partial}S$ in positive time direction. C is a closed path on the dual lattice of Λ. It will be shown that the 't Hooft disorder parameter takes the form

$$\langle B[S]\rangle=\langle \exp i\pi \oint_c \tilde{A}\rangle \tag{1.11}$$

where

$$\oint_c \tilde{A}=\sum_{c\in C}\tilde{A}[c]$$

Let the closed path $\hat{\partial}S$ bound a rectangular area S of $L\cdot T$ lattice squares as in Fig. 1, with $T\gg L$. The Wilson loop formula (1.11) implies—according to arguments due to Wilson [1]—that

$$\langle B[S]\rangle \sim \text{const}\, e^{-TV(L)} \quad \text{for } T\gg L, \tag{1.12}$$

and $V(L)$ may be interpreted as potential energy of a pair of external monopoles a distance L apart.

Since dynamical monopoles exist in the standard model, it seems reasonable to expect that $V(L)$ stays bounded as $L\to\infty$. For even if strings form that tend to confine the external monopoles, they can break, creating a monopole pair out of the vacuum. Since the number of plaquettes in $\hat{\partial}S$ is $|\hat{\partial}S|=2L+2T$ this would amount to a bound, valid for all values of β, of the form

$$\langle B[S]\rangle \geqq c(L)e^{-\alpha|\hat{\partial}S|} \quad \text{for } T\to\infty \tag{1.13}$$

i.e. a perimeter law. α may depend on β.

't Hooft has argued that bound (1.13) together with a mass gap should be a sufficient condition for confinement of static quarks [5]. Unfortunately it is not at all clear that this assertion applies to theories with dynamical monopoles. Therefore, even if the bound (1.13) is indeed true for the standard $SU(2)$ lattice gauge

* The proof of inequality (1.12) of [2] for the modified model has been extended to the standard model by Fröhlich [4]

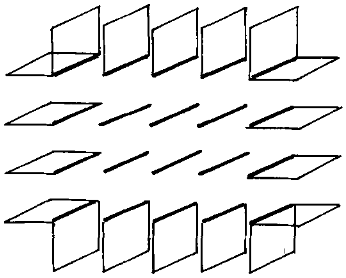

Fig. 1. A set S of links (heavy lines) in the time 0 plane Σ and plaquettes in ∂S (squares)

theory model, nothing can be concluded from that.

For the modified $SU(2)$ model mentioned above it was shown in [2] that the bound (1.13) is *not* satisfied for large enough β. Instead one finds an area law

$$\langle B[S] \rangle \leqq c'(L) e^{-\alpha'|S|} \tag{1.14}$$

for large enough β in the modified model (low temperature phase).

One may ask how validity of (1.13) for the standard model and (1.14) for the modified model can be compatible with the assertion made earlier that the monopoles become unimportant as $\beta \to \infty$. A likely answer is that $V(L)$ increases linearly with L in the standard model before it bends over, and is approximately equal to $V(L)$ in the modified model so long as $L \leqq L_0(\beta)$, and $L_0(\beta) \to \infty$ as $\beta \to \infty$.

One may also ask whether the Z_2 monopoles are confined or not in the low temperature phase of the standard model (β large). It is instructive to inquire first whether the analog of the Wilson criterium for quark confinement is satisfied. This amounts to looking at the corresponding theory without the monopoles and determining the behaviour of the expectation value of the Wilson loop integral (for external monopoles) in this theory. The theory without monopoles is just our modified model. In view of (1.11), the question is then answered by the bound (1.13) for that model: The monopoles are confined. Appeal to a Wilson criterion seems reasonable for large β since the result (1.9) assures then that the monopole pairs are dilute.

It remains yet to be investigated whether the onset of dissociation of monopole pairs corresponds to a phase transition in the classical sense (point of nonanalyticity in β of the Gibbs potential). We remark that one may interpolate between the standard model and the modified model by using action

$$L = \beta \left\{ \sum_p \operatorname{tr} U[\dot{p}] + \sum_c \ln \tfrac{1}{2} \left(1 + \tanh \lambda \prod_{p \in \partial c} \operatorname{tr} U[\dot{p}] \right) \right\}$$

For large values of λ the monopole pairs will still be dilute when they start to dissociate. It should be remembered, however, that an elaborate argument was needed for the two dimensional plane rotator model [6] to show that the onset of dissociation of vortex pairs produces observable consequences of a phase transition.

In his recent paper [7] 't Hooft argues that one cannot have both electric and magnetic confinement. However, confinement of the monopoles discussed here until now does not imply magnetic confinement in the sense of 't Hooft, and—as we have pointed out already in our papers [2, 8]—one cannot conclude from our result that static quarks are not confined when β is sufficiently large. In fact, the monopoles discussed so far are merely the smallest ones of a family of monopoles of increasing size. This will be discussed in Sect. 5. It turns out that monopoles of any given size are confined (in a sense that will be made precise in Sect. 5) for sufficiently large β. But it depends on the monopole's size how large β has to be and the possibility remains open that for any β monopoles of sufficiently large size (depending on β) are not confined.

2. Z_2 Theory with Monopoles

Let U be a configuration of the standard model. We introduce auxiliary variables

$$\dot{U}[b] = U[b]\Gamma \in G/\Gamma \approx SO(3) \tag{2.1a}$$

$$\sigma[p] = \operatorname{sign} \operatorname{tr} U[\dot{p}] = \pm 1 \in \Gamma \tag{2.1b}$$

We admit free boundary conditions, or a mixture of free and cyclic boundary conditions in which $\dot{U}[b]$ satisfies cyclic boundary conditions, (and also $\sigma[p]$ if one so chooses) but not the variables $U[b]$ themselves.

The variables (2.1) are invariant under gauge transformations by elements of Γ, viz.

$$U[b] \to \gamma[x_2] U[b] \gamma[x_1]^{-1} \quad \text{for } b = (x_2, x_1) \tag{2.2}$$
$$\gamma[x] = \pm 1 \in \Gamma$$

Let us introduce

$$\rho[c] = \prod_{p \in \partial c} \operatorname{sign} \operatorname{tr} U[\dot{p}] \tag{2.3}$$

It follows from the definition (2.1b) of $\sigma[p]$ that

$$\prod_{p \in \partial c} \sigma[p] = \rho[c] \tag{2.4}$$

We will show that $\rho[c]$ depends on the configuration U only through cosets $\dot{U}[b]$. This implies that (2.4) is a relation between variables $\sigma[p]$ and $\dot{U}[b]$. As is explained in the introduction, it is of the form of a field equation for field strengths $\sigma[p]$.

Let U and U' be two configurations such that $\dot{U}[b] = \dot{U}'[b]$ for all b. Then $U'[b] = U[b]\gamma[b]$ with $\gamma[b] \in \Gamma$. Therefore $U'[\dot{p}] = U[\dot{p}]\gamma[\dot{p}]$ with

$$\gamma[\dot{p}] = \prod_{b \in \partial p} \gamma[b] \tag{2.5}$$

It follows from this definition that

$$\prod_{p \in \partial c} \gamma[\dot{p}] = 1 \quad \text{for every cube } c. \tag{2.6}$$

Upon substituting U' for U, $\rho[c]$ changes to $\rho[c]\prod_{p\in\partial c}\gamma[\dot{p}]=\rho[c]$. Thus $\rho[c]$ remains unchanged and depends therefore on the configuration U only through the cosets $\dot{U}[b]$, as was to be shown.

It follows from the definition (2.3) of $\rho[c]$ that

$$\prod_{c\in\partial h}\rho[c]=1 \quad \text{for every hypercube } h. \tag{2.7}$$

since every factor sign tr $U[\dot{p}]$ appears twice. As was explained in the introduction, this expresses conservation of the magnetic current.

Let us now suppose that we are given any collection of elements $\dot{U}[b]$ of G/Γ and $\sigma[p]$ of Γ which is such that relation (2.4) is satisfied for all cubes c. We show that there exists then a configuration U—i.e. a collection of elements $U[b]\in G$—such that (2.1) hold, and U is unique up to gauge transformations (2.2) by elements of Γ.

To compute $\rho[c]$ from given $\dot{U}$, one selects in an arbitrary way representatives $U_1[b]\in G$ of the cosets $\dot{U}[b]$ so that $\dot{U}[b]=U_1[b]\Gamma$. Then $\rho[c]$ is computed from (2.3), with U_1 substituted for U. The result does not depend on the choice of representative by the argument given earlier.

Let $\sigma_1[p]=\text{sign tr } U_1[\dot{p}]$, whence $\rho[c]=\prod_{p\in\partial c}\sigma_1[p]$. By hypothesis, (2.4) is fulfilled. Therefore

$$\prod_{p\in\partial c}\sigma[p]=\prod_{p\in\partial c}\sigma_1[p]$$

Let us write

$$\sigma[p]=\sigma_1[p]\gamma[p]$$

with $\gamma[p]=\pm 1\in\Gamma$. Then $\gamma[p]$ satisfies the relation

$$\prod_{p\in\partial c}\gamma[p]=1 \quad \text{for every cube } c \tag{2.8}$$

It follows that there exists for every link b a $\gamma_1[b]=\pm 1\in\Gamma$ such that $\gamma[p]=\prod_{b\in\partial p}\gamma_1[b]$ for all plaquettes p. $\gamma_1[b]$ need not satisfy cyclic boundary conditions even if $\gamma[p]$ do. Obviously, $U[b]=U_1[b]\gamma_1[b]$ fulfills relations (2.1). This proves existence of the configuration U.

Now we turn to uniqueness. Suppose configurations U and U' produce the same values of the auxiliary variables defined by (2.1). This requires that $U'[b]=U[b]\gamma[b]$ with $\gamma[b]=\pm 1\in\Gamma$, and

$$\prod_{b\in\partial p}\gamma[b]=1$$

for every plaquette p. The last requirement implies that $\gamma[b]$ is a pure gauge, and therefore U and U' are related by a gauge transformation (2.2) by elements of Γ as was to be shown. The gauge transformation need not obey cyclic boundary conditions.

Let us also note that $\rho[c]$ is invariant under $SO(3)$ gauge transformations

$$\dot{U}[b]\to\dot{V}[x_2]\dot{U}[b]\dot{V}[x_1]^{-1} \quad \text{with} \quad \dot{V}[x]\in G/\Gamma\approx SO(3) \tag{2.9}$$

With a one to one correspondence between old and new variables established, we can rewrite the path measure. We introduce

$$K_p(\dot{U})=\beta|\text{tr } U[\dot{p}]|\geq 0 \tag{2.10}$$

Clearly, because of the absolute signs it depends only on cosets $\dot{U}[b]$, and it is invariant under $SO(3)$ gauge transformations (2.9).

The action (1.2) becomes

$$L=L(\dot{U},\sigma)=\sum_p K_p(\dot{U})\sigma[p] \tag{2.11}$$

Of course the path measure also has to include δ-functions to take care of the constraint (2.4).

Any gauge invariant function $F(U)$ may be regarded as a function of the new variables

$$F(U)=F_1(\dot{U},\sigma)$$

(For local observables, defined as in [2], this is always true. For more general functions, which may depend on variables attached to cells on the boundary of the lattice Λ, it follows from our choice of boundary conditions). Expectation values take the form

$$\langle F\rangle=\int d\mu F_1(\dot{U},\sigma) \tag{2.12}$$

with

$$d\mu=\frac{1}{Z}e^{L(\dot{U},\sigma)}\prod_c\delta\Big(\rho[c]^{-1}\prod_{p\in\partial c}\sigma[p]\Big)\prod_b d\dot{U}[b]\prod_p d\sigma[p] \tag{2.13}$$

A new expression for the partition function Z results from $\langle 1\rangle=\int d\mu=1$. Notations are as follows. The δ-function is a Kronecker-δ defined by

$$\delta(1)=1, \qquad \delta(-1)=0 \tag{2.14}$$

$d\dot{U}$ is normalized Haar measure on G/Γ, and

$$\int d\sigma(\dots)\equiv\tfrac{1}{2}\sum_{\sigma=\pm 1}(\dots). \tag{2.15}$$

Products over b, p, c in (2.13) run over all links, plaquettes, and cubes in the lattice Λ, respectively.

3. Duality Transformation

Electric-magnetic duality has been extensively studied in the literature, see e.g. Mandelstams recent paper [9, 10]. It was also noted there and before that the 't Hooft disorder parameter may be viewed as expectation value of a Wilson loop integral for monopoles. Nevertheless it appears necessary to give proof of assertion (1.11) in the introduction. It is based on performing a duality transformation on the Z_2 theory of Sect. 2. The duality transformation is performed in the same way as for the modified model of [2]. It amounts to a Fourier transformation on the Abelian group Γ.

The variables of the dual model take values in the dual group $\hat{\Gamma}$ = group of characters of unitary irreducible representations of Γ. For $\Gamma \approx Z_2$ there are two such characters, and $\hat{\Gamma} \approx Z_2$ again. We identify them with numbers $\omega = \pm 1$. The corresponding characters are functions on Γ given by

$$\tilde{\omega}(\gamma) = \begin{cases} 1 \text{ if } \omega = 1 \\ \gamma \text{ if } \omega = -1 \end{cases} \quad \text{for } \gamma = \pm 1 \in \Gamma \tag{3.1}$$

A variable $\omega[c]$ is assigned to every 3-cell c of the lattice. It takes values $\omega[c] = \pm 1$. The corresponding characters will be denoted by $\tilde{\omega}_c(\gamma)$.

It will be convenient to use the coboundary operator $\hat{\partial}$ (= boundary operator on the dual lattice). It is defined by saying that a 3-cell

$$c \in \hat{\partial} p \text{ if and only if } p \in \partial c, \text{ etc.} \tag{3.2}$$

p is a plaquette. One writes accordingly

$$\omega[\hat{\partial} p] \equiv \prod_{c \in \hat{\partial} p} \omega[c] = \prod_{\substack{c \\ p \in \partial c}} \omega[c] \tag{3.3}$$

For convenience we shall use a non-normalized Haar measure on $\hat{\Gamma}$,

$$\int d\omega(\ldots) \equiv \sum_{\omega = \pm 1} (\ldots) \tag{3.4}$$

Temporarily we shall neglect indicating $\dot{U}$-dependence of functions (such as K_p, $\hat{\mathscr{L}}_p$, F below) explicitly.

One expands in a Fourier series on Γ.

$$e^{K_p \sigma[p]} = \int d\omega_p e^{\hat{\mathscr{L}}_p(\omega_p)} \tilde{\omega}_p(\sigma[p])^{-1}, \tag{3.5a}$$

$$\begin{aligned} \delta(\rho[c]^{-1} \prod_{p \in \partial c} \sigma[p]) &= \int d\omega[c] \tilde{\omega}_c(\rho[c] \prod_{p \in \partial c} \sigma[p]^{-1}) \\ &= \int d\omega[c] \tilde{\omega}_c(\rho[c]) \prod_{p \in \partial c} \tilde{\omega}_c(\sigma[p])^{-1} \end{aligned} \tag{3.5b}$$

And, for a function F that depends on variables $\sigma[p]$ with $p \in Y$,

$$F(\{\sigma[p]\}_{p \in Y}) = \int \hat{F}(\{\omega'_p\}_{p \in Y}) \prod_{p \in Y} \{\tilde{\omega}'_p(\sigma[p])^{-1} d\omega'_p\} \tag{3.5c}$$

One inserts these expansions into the definition of $\langle F \rangle$. Summations over variables $\sigma[p]$ may then be performed with the help of orthogonality relations of characters. They produce δ-functions. The ω_p-summations can be performed next, making use of the presence of these δ-functions. As a result one obtains ($\omega[\hat{\partial} p] \equiv \prod_{c \in \hat{\partial} p} \omega[c]$)

$$\begin{aligned} \langle F \rangle = &\frac{1}{Z} \int \prod_b d\dot{U}[b] \prod_c d\omega[c] \int \prod_{p \in Y} d\omega'_p \\ &\cdot \hat{F}(\{\omega'_p\}_{p \in Y}) \prod_c \tilde{\omega}_c(\rho[c]) \exp \sum_p \hat{\mathscr{L}}_p(\omega'^{-1}_p \omega[\hat{\partial}_p]) \end{aligned} \tag{3.6}$$

If $p \notin Y$ one is to put $\omega'_p = 1$. $\hat{\mathscr{L}}_p$, $\rho[c]$ and, in general, also $\hat{F}$ depend also on variables $\dot{U}[b]$. An explicit formula for $\hat{\mathscr{L}}_p$ is obtained from its definition (3.5a)

$$\hat{\mathscr{L}}_p(\omega) = \hat{M}_p(\dot{U}) + \hat{K}_p(\dot{U})\omega \quad \text{for} \quad \omega = \pm 1 \in \hat{\Gamma} \tag{3.7}$$

$$\hat{K}_p = \tfrac{1}{2} \ln \coth K_p(\dot{U}) \geqq 0$$

$$\hat{M}_p = \tfrac{1}{2} \ln (\sinh K_p(\dot{U}) \cosh K_p(\dot{U}))$$

Expression (3.6) involves the new path measure

$$d\hat{\mu} = \frac{1}{Z} \prod_c \tilde{\omega}_c(\rho[c]) e^{\sum_p \hat{\mathscr{L}}_p(\omega[\hat{\partial} p])} \prod_b d\dot{U}[b] \prod_c d\omega[c] \tag{3.8}$$

Because of the factors $\tilde{\omega}_c(\rho[c]) = \pm 1$ this measure is not positive.

It is amusing to see formula (3.8) translated into additive language. It involves then the vector potential $\tilde{A}$ and the current $\tilde{j}$. They take values in the field $Z_2 = \{0, 1\}$ and were defined in terms of $\omega[c]$ resp. $\rho[c]$ in (1.10) resp. (1.7) and f. of the introduction.

We introduce

$$\tilde{F}_{\mu\nu}(x) = \Delta_\mu \tilde{A}_\nu - \Delta_\nu \tilde{A}_\mu \tag{3.9}$$

It is defined as an element of the field $Z_2 = \{0, 1\}$; addition is addition in Z_2, i.e. modulo 2.

In this language one finds

$$d\hat{\mu} = \frac{1}{Z} e^{\hat{L}} \prod_b d\dot{U}[b] \prod_{x,\mu} d\tilde{A}_\mu(x) \tag{3.10a}$$

with action $\hat{L}$ that can be written in the form

$$\begin{aligned} \hat{L} = &\sum \{\hat{M}_p(\dot{U}) + \hat{K}_p(\dot{U}) - 2\hat{K}_p(\dot{U}) \tilde{F}_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x)\} \\ &+ i\pi \sum \tilde{A}_\mu(x) \tilde{j}_\mu(x). \end{aligned} \tag{3.10b}$$

The current $\tilde{j}$ is also a function of $\dot{U}$. The first sum goes over all plaquettes $p = p_{\mu\nu}(x)$, and the second one over all links $(x, x + e_\mu)$ (they are in one to one correspondence with cubes $c_{\nu\lambda\rho}(x)$). Finally

$$\int d\tilde{A}(\ldots) \equiv \sum_{\tilde{A} = 0,1} (\ldots)$$

Apart from the fluctuating coupling constant $2K_p(\dot{U})$, expression (3.10b) has a familiar look, *except* for the strange imaginary factor $i\pi$ multiplying the last term. This factor would not have been expected from analogy with electrodynamics. We note however that complete analogy cannot hold. In electrodynamics, the Coulomb force between like charges is repulsive, between opposite charges it is attractive. In a Z_2 gauge field theory there can be no distinction between like and opposite charges, since $-1 = +1$ in Z_2. The factor $i\pi$ in (3.10b) makes $e^{\hat{L}}$ invariant under $\tilde{j}_\mu(x) \to -\tilde{j}_\mu(x)$. At the same time it is responsible for the lack of positivity of the measure $d\hat{\mu}$.

The new path measure $d\hat{\mu}$ is that of a Z_2 gauge field theory. We show that it is invariant under the following local gauge transformations

$$\omega[c] \to v[h_2] \omega[c] v[h_1]^{-1} \quad \text{if} \quad \hat{\partial} c = h_2 - h_1 \tag{3.11}$$

$$\text{for} \quad v[h] = \pm 1 \in \hat{\Gamma}$$

Interpreted as elements of the dual lattice, c is a link and h_1, h_2 are the two endpoints of this link. (In the original lattice they are 4-dimensional hypercubes touching each other along the cube c).

In terms of the vector potential $\tilde{A}$, these gauge transformations are of the familiar form

$$A_\mu(x) \to A_\mu(x) + \Delta_\mu f(x) \tag{3.12}$$

with $f(x) \in Z_2 = \{0, 1\}$. Gauge invariance of the measure (3.10) follows therefore from the fact that the current $\tilde{j}$ is identically conserved. That is, for any configuration $\dot{U} = \{\dot{U}[b]\}$ the values of the functions $\tilde{j}_\mu(x)$ of the random variable $\dot{U}[b]$ satisfy (1.8').

We conclude this section with a comment on the lack of positivity of the measure $d\hat{\mu}$. Physical positivity in the quantum field theoretic sense—also known as reflection positivity or Osterwalder Schrader (OS) positivity [11]—holds in spite of it. Basically this is a consequence of the fact that the measure is obtained by a duality transformation from a model that is known to respect OS-positivity [12].

Let F be a local observable which is a real gauge invariant function of the variables $\omega[c]$, $\dot{U}[b]$ attached to cubes c resp. links b in the halfspace $t \equiv x^4 \geqq 0$ only. Define

$$(\Theta F)(\{\omega[c], \dot{U}[b]\}) = F(\{\omega[\Theta c], \dot{U}[\Theta b]\}) \tag{3.13}$$

Time reflection $\Theta : (x^1 x^2 x^3, x^4) \to (x^1 x^2 x^3, -x^4)$ acts on cells c, b in the obvious way.

Any such function F can be regarded as a function which depends on variables $\omega[c]$ only through gauge invariants $\omega[\hat{\partial} p]$ associated with plaquettes p in the halfspace $x^4 \geqq 0$. By doing the duality transformation backwards and using OS-positivity of the original model one can show that

$$\langle(\Theta F)F\rangle \equiv \int d\hat{\mu}(\Theta F)F \geqq 0. \tag{3.14}$$

4. Wilson Loop for Z_2 Monopoles

Let S be a set of links in the time $t = 0$ hyperplane $\sum$. The 't Hooft operator $B[S]$ is defined by its action on wave functions in the quantum field theoretic Hilbert space of states, and $\langle B[S]\rangle$ is the vacuum expectation value of this operator, cp. [2]. By using the path integral formula for the vacuum state it was shown there (cp. (4.11) of [2]) that $\langle B[S]\rangle$ is equal to the expectation value of a multiplication operator, viz.

$$\langle B[S]\rangle = \langle F\rangle = \int d\mu F(U) \tag{4.1}$$

$$F(U) = \exp \sum_{b \in S} \{\mathscr{L}(-U[\dot{p}_b]) - \mathscr{L}(U[\dot{p}_b])\}$$

This holds generally, both for the standard and the modified model. p_b is the plaquette protruding from the spacelike link b in positive time direction. In the variables of Sect. 2

$$F = \exp -2 \sum_{b \in S} K_{p_b}(\dot{U})\sigma[p_b] \tag{4.2}$$

Now one can use formula (3.6) to rewrite $\langle F\rangle$ in the language of the dually transformed model. A short computation (the same as is carried out in Eqs. (4.12)...(4.13) of [2]) gives the result

$$\langle B[S]\rangle = \langle \prod_{p \in \partial S} \omega[c_p]\rangle \equiv \int d\hat{\mu} \prod_{p \in \hat{\partial} S} \omega[c_p] \tag{4.3}$$

c_p is the cube protruding from the spacelike plaquette p in positive time direction. In additive language (1.10) this is the desired formula (1.11).

5. Z_2 Monopoles of Larger Size

Let us for a moment restrict attention to the $t = 0$ plane Σ in the lattice Λ, and consider a configuration $U = \{U[b]\}_{b \in \Sigma}$. Let c be a cube in Σ. If $\rho[c] = -1$ then there is a monopole located at c. By (1.6) it is end point of a string of magnetic flux, cp. Fig. 2. We attribute size $1a^3$ to the monopole (a = unit of length given by the lattice spacing in Λ.) The string has cross section $1a^2$.

Besides these small monopoles there are monopoles of larger size, they are end points of strings of magnetic flux of larger cross section. To discuss them we introduce sublattices Λ_N of Λ with larger lattice spacing $2^N a$. We consider Λ_N as cell complexes consisting of vertices x', links b', plaquettes p', cubes c', and hypercubes h'. The links b' in Λ_N are paths in Λ consisting of 2^N links of Λ. A plaquette p' in Λ_N is composed of 4^N plaquettes of Λ. Its boundary $\dot{p}' = \partial p'$ is a path in Λ and so the parallel transporter $U[\dot{p}']$ can be defined by (1.1a).

We may now proceed as in Sect. 2. For plaquettes $p' \in \Lambda_N$ we define

$$\sigma_N[p'] = \operatorname{sign} \operatorname{tr} U[\dot{p}'] \tag{5.1}$$

For cubes $c' \in \Lambda_N$ we consider the monopole distribution function

$$\rho_N[c'] = \prod_{p' \in \partial c'} \operatorname{sign} \operatorname{tr} U[\dot{p}'] \tag{5.2}$$

It follows that

$$\prod_{p' \in \partial c'} \sigma_N[p'] = \rho_N[c'] \tag{5.3}$$

Products are of course over plaquettes in Λ_N.

A substitution of variables $U[b] \to U[b]\gamma[b]$ ($\gamma[b] = \pm 1$, b links in the original lattice Λ) takes $U[b'] \to U[b']\gamma[b']$ for $b' \in \Lambda_N$ with $\gamma[b'] = \pm 1$. Therefore, by the same argument as in Sect. 2, the monopole distribution function ρ_N is a function of the cosets $\dot{U}[b] = U[b]\Gamma(b \in \Lambda)$ only. Thus the configuration $\dot{U}$ also determines the distribution of monopoles of size $8^N a^3$.

A monopole of size $8^N a^3$ in the $t = 0$ plane Σ is end point of a string of magnetic flux of cross section $4^N a^2$, cp. Fig. 2.

Next we derive an estimate for the cost in action of quanta of magnetic flux.

Let p' be a rectangle of $2^{N_1} \times 2^{N_2}$ plaquettes $p \subset p'$ in Λ. We will show that

$$\operatorname{tr} U[\dot{p}'] - 2 \geqq 2^{N_1+N_2} \sum_{p \subset p'} (\operatorname{tr} U[\dot{p}] - 2) \tag{5.4}$$

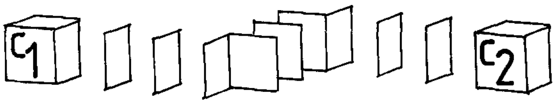

Fig. 2. A pair of monopoles in the time 0 plane Σ and a string of magnetic flux joining them. The monopoles are located at cubes c_1, c_2 where $\rho[c_i] = -1$, and the string consists of a sequence of plaquettes p for which $\sigma(p) = -1$. Considered as elements of the dual lattice of Σ these plaquettes form a path joining points c_1 and c_2.
The same figure applies to monopoles of arbitrary size $8^N a^3$ and string of magnetic flux of cross section $4^N a^2$ joining them. In this case the cubes and plaquettes are of the lattice Λ_N

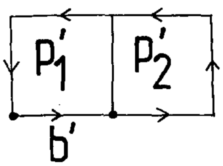

Fig. 3. Illustration to Eq. (5.5)

Specializing to a plaquette p' in Λ_N we see that making $\sigma_N[p'] = -1$ costs at least an amount of $2\beta \cdot 4^{-N}$ of action. The factor 4^{-N} represents a bound on the possible savings achieved by spreading the flux over an area of $4^{-N}a^2$.

To prove inequality (5.4) we divide the rectangle p' into two rectangles p'_1 and p'_2 of $2^{N_1-1} \times 2^{N_2}$ plaquettes each (for $N_1 \geqq N_2$). One has then

$$U[\dot{p}'] = V U_2 V^{-1} U_1 \tag{5.5}$$

where $U_i = U[\dot{p}'_i]$ and $V = U[b']$. $\dot{p}'_i$ is the boundary of p'_i with a choice of initial point as indicated by dots in Fig. 3; the path b' joins these initial points.

We introduce 4-dimensional unit vectors $s_j = (\mathbf{s}_j, t_j)$ by

$$U_j = t_j \mathbb{1} + i\boldsymbol{\sigma} \cdot \mathbf{s}_j \tag{5.6}$$

$\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are Pauli matrices. In this notation

$$\mathrm{tr}\, U_j = 2t_j \tag{5.7a}$$

$$\mathrm{tr}\, U[\dot{p}'] = 2(t_1 t_2 - R\mathbf{s}_1 \cdot \mathbf{s}_2) \tag{5.7b}$$

$R = R(V)$ is a 3-rotation given by (the fundamental formula of spinor calculus) $V\sigma^k V^* = \sigma^l R(V)_l{}^k$.

It follows that

$$\mathrm{tr}\, U[\dot{p}'] \geqq 2(t_1 t_2 - (1-t_1^2)^{1/2}(1-t_2^2)^{1/2})$$

where $-1 \leqq t_j \leqq 1$. For fixed value of $t = t_1 + t_2$, the minimum of the r.h.s. is attained at $t_1 = t_2 = \frac{1}{2}t$. This gives

$$\mathrm{tr}\, U[\dot{p}'] \geqq t^2 - 2 = (t-2)^2 + 4(t-2) + 2$$

Therefore

$$\tfrac{1}{2}(\mathrm{tr}\, U[\dot{p}'] - 2) \geqq 2(t-2)$$
$$= (\mathrm{tr}\, U[\dot{p}'_1] - 2) + (\mathrm{tr}\, U[\dot{p}'_2] - 2)$$

Iterating the procedure gives inequality (5.4). q.e.d.

From inequality (5.4) one can derive conclusions concerning confinement of the larger monopoles.

Let T be a set of $|T|$ plaquettes in Λ_N. By using chess-board estimates [13] in the same way as in Sect. 8 of our first paper [2] one deduces from inequality (5.4) that

$$\langle \prod_{p' \in T} \theta(-\mathrm{tr}\, U[\dot{p}']) \rangle \leqq D_N(\beta)^{|T|} \quad \text{with}$$

$$D_N(\beta) \leqq C_{\varepsilon,N} \exp - \beta(\tfrac{1}{12}4^{-N} - \varepsilon) \to 0 \text{ as } \beta - \infty. \tag{5.8}$$

$\varepsilon > 0$ may be taken arbitrarily small. The constant $C_{\varepsilon,N}$ may depend on ε, N but not on β. (In inequality (1.9) we chose $\varepsilon = \frac{1}{12} - \frac{1}{13}$.)

Let us now look at our (standard) model with the eyes of a quantum field theorist. The Hilbert space of physical states consists of wave functions $\Psi(U)$ which depend on arguments $U[b]$ associated with links b in the time 0 plane Σ. Among them is the wave function Ω of a vacuum state (eigenstate of the transfer matrix with eigenvalue 1) which is given by a path integral formula, (1.15) of [2]. Let us ask for the probability $P_N[c_1, c_2]$ of finding in this vacuum state a pair of (virtual) monopoles of size $8^N a^3$ located at cubes c_1 resp. c_2 of $\Lambda_N \cap \Sigma$ and linked by a string of magnetic flux (cp. Fig. 2). This probability is less than or equal to the probability of finding a string from c_1 to c_2. From the path integral formula for Ω it follows then that

$$P_N[c_1, c_2] \leqq \sum_T \langle \prod_{p' \in T} \theta(-\mathrm{tr}\, U[\dot{p}']) \rangle \tag{5.9}$$

Summation is over all possible strings T. If c_1 and c_2 are a distance L apart in units of $2^N a$ (= lattice spacing in Λ_N) then the number $|T|$ of plaquettes in T obeys $|T| \geqq L$. The number of strings of length $|T|$ is bounded by $e^{\kappa|T|}$, κ a constant. It follows therefore by combining inequalities (5.8) and (5.9) that the probability $P_N[c_1, c_2]$ decreases exponentially with the distance L between the monopoles if β is sufficiently large. We may regard such an exponential falloff as a defining feature of monopole confinement. How large β has to be may depend on N because of the N-dependence of $C_{\varepsilon,N}$ and the factor 4^{-N} multiplying β in inequality (5.8).

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