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A Method of Three-Jet Analysis in e^+e^- Annihilation

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Abstract. For three-jet events in e^+e^- annihilation, a procedure is described to determine all three jet axes by minimizing the sum of squares of transverse momenta. Computations with this procedure show that at high energies the result is quite insensitive to missing particles, such as neurals.

Experimental evidence for two-jet structure in hadron production by e^+e^- annihilation was first found at SPEAR by Hanson et al. [1]. They discovered this important phenomenon by minimizing the sum of squares of transverse momenta, as suggested previously by Bjorken and Brodsky [2].

In this paper we describe a simple procedure to analyze the three-jet events in the same spirit as that of Hanson et al. [1, 3]. This procedure has the following desirable features:

1. All three jet axes are determined.

2. It is not necessary to have the momenta of all produced particles. For example, this procedure can be used when there is no detection of neutral particles. Of course the loss of information leads to a larger error.

3. All measured momenta can be used; in other words, there is no need to introduce a cutoff for low momenta.

4. Computer time is moderate[1].

Let $\mathbf{p}_j, j=1\ldots N$, be the momenta of the N observed particles. Let the rectangular components of $\mathbf{p}_j$ be called $p_{j\alpha}$, $\alpha=1, 2, 3$. Define[2] a 3×3 symmetric real matrix M by

$$M_{\alpha\beta}=\sum_j p_{j\alpha}p_{j\beta}. \tag{1}$$

Let λ_1, λ_2, λ_3 be the eigenvalues of M, which are explicitly,

$$\lambda_k=\sum_i(\mathbf{p}_i\cdot\hat{n}_k)^2 \qquad k=1,2,3$$

where $\hat{n}_k$ are the unit length eigenvectors. Then let [4]

$$Q_k=\lambda_k\Big/\sum_j \mathbf{p}_j^2. \tag{2}$$

These normalized eigenvalues Q_k satisfy

$$Q_1+Q_2+Q_3=1, \tag{3}$$

and can be arranged so that

$$0\leqq Q_1\leqq Q_2\leqq Q_3. \tag{4}$$

Colinear events are characterized by $Q_2 \ll Q_3$, and similarly coplanar events by $Q_1 \ll Q_2$. In terms of the Q's, the sphericity S is

$$S=\tfrac{3}{2}(Q_1+Q_2). \tag{5}$$

A triangular plot, with the coordinate variables chosen to be S and $\frac{\sqrt{3}}{2}(Q_2-Q_1)$, can be used to separate two-jet, three-jet and nonplanar events as shown in Fig. 1.

We turn our attention to three-jet events, which are necessarily coplanar. Since $\hat{n}_k$ is the normalized eigenvector of M corresponding to λ_k, then these $\hat{n}_k$ form an orthonormal coordinate system. Let P be the plane

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[1] We have developed this three jet analysis program for the TASSO-Collaboration at DESY. At $E_{\mathrm{cm}}=30\,\mathrm{GeV}$, with IBM 370/168 computer, the C.P.U. time is about 0.5 s for multiplicity 15 and about 1.5 s for multiplicity 21

[2] There is a slight difference between this matrix and those of [1] and [2]

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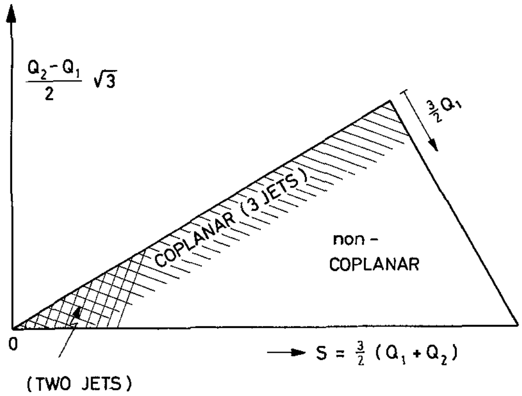

Fig. 1. Triangular plot to separate two jet, three jet and noncoplanar events (see text)

formed by $\hat{n}_2$ and $\hat{n}_3$ with $\hat{n}_1$ normal[3] to P and $\mathbf{q}_j$ be the projection of $\mathbf{p}_j$ onto this plane P. Choose coordinate system in P so that the rectangular and polar coordinates of $\mathbf{q}_j$ are respectively (q_{j1}, q_{j2}) and (q_j, θ_j). Here the axis for $\theta=0°$ is not necessarily either $\hat{n}_2$ or $\hat{n}_3$. Relabel the N momenta such that

$$0 \leqq \theta_1 \leqq \theta_2 \leqq \theta_3 \leqq \ldots \leqq \theta_N < 2\pi. \tag{6}$$

The main idea of the present analysis is as follows. For every way of partitioning these $\mathbf{q}_j$ into three non overlapping sets and for every choice of three jet axes, a sum of the squares of transverse momenta can be calculated. We minimize this sum.

The program is implemented as follows. First, choose three integers N_1, N_2, and N_3 such that

$$1 \leqq N_1 < N_2 < N_3 \leqq N \tag{7}$$

$$\theta_{N_2-1} - \theta_{N_1} < \pi \tag{8}$$

$$\theta_{N_3-1} - \theta_{N_2} < \pi \tag{9}$$

and[4]

$$2\pi + \theta_{N_1-1} - \theta_{N_3} < \pi. \tag{10}$$

The reason for the condition (8), (9), and (10) is the desire to have each jet contain only particles in one direction, not simultaneously in both directions. In this way, the N momenta are partitioned into three sets, S_1, S_2, and S_3.

$S_1 = \{N_1, N_1+1, \ldots, N_2-1\}$

$S_2 = \{N_2, N_2+1, \ldots, N_3-1\}$

and

$S_3 = \{N_3, N_3+1, \ldots, N, 1, 2, \ldots, N_1-1\}$.

The second step is to define three 2×2 matrices $M^{(1)}$, $M^{(2)}$, and $M^{(3)}$ by

$$M^{(\tau)}_{\alpha\beta} = \sum_{j \text{ in } S_\tau} q_{j\alpha} q_{j\beta} \tag{11}$$

for $\alpha, \beta = 1, 2$ and $\tau = 1, 2$, and 3. For each of these three 2×2 matrices, let $\Lambda^{(\tau)}$ be the larger eigenvalue and $\hat{m}^{(\tau)}$ the corresponding normalized eigenvector. Here $\Lambda^{(\tau)}$ is given explicitly by

$$\Lambda^{(\tau)} = \tfrac{1}{2}\{M^{(\tau)}_{11} + M^{(\tau)}_{22} + [(M^{(\tau)}_{11} - M^{(\tau)}_{22})^2 + 4(M^{(\tau)}_{12})^2]^{1/2}\}. \tag{12}$$

We impose the requirement that the signs of $\hat{m}^{(\tau)}$ can be chosen so that

$$\mathbf{q}_j \cdot \hat{m}^{(\tau)} > 0. \tag{13}$$

for j in S_τ. More explicitly, if $\phi^{(\tau)}$ is the polar angular coordinate for $\hat{m}^{(\tau)}$, then (13) is

$$\phi^{(1)} - \theta_{N_1} < \pi/2; \quad \theta_{N_2-1} - \phi^{(1)} < \pi/2 \tag{14}$$

$$\phi^{(2)} - \theta_{N_2} < \pi/2; \quad \theta_{N_3-1} - \phi^{(2)} < \pi/2 \tag{15}$$

and[5]

$$\phi^{(3)} - \theta_{N_3} < \pi/2; \quad \theta_{N_1-1} - \phi^{(3)} + 2\pi < \pi/2. \tag{16}$$

These requirements (14)–(16) are seen to be more stringent than (8)–(10). Finally, let

$$\Lambda(N_1, N_2, N_3) = \Lambda^{(1)} + \Lambda^{(2)} + \Lambda^{(3)} \tag{17}$$

and we maximize $\Lambda(N_1, N_2, N_3)$ over those values N_1, N_2, and N_3 where (14), (15) and (16) are satisfied. This maximizing partition gives the three jets and corresponding $\hat{m}^{(1)}$, $\hat{m}^{(2)}$, and $\hat{m}^{(3)}$ yield the directions of the jet axes.

This procedure is especially useful for events with only charged particles detected. Once the directions of the three jet axes are determined, from the known total energy $E_{\text{cm}}(=2\times\text{beam energy})$ of e^+e^- annihilation events, one can reconstruct the total energy of each jet with the approximation that the invariant mass of the jet is zero.

Let

$$\begin{aligned} \alpha_1 &= \sin(\phi^{(3)} - \phi^{(2)}) \\ \alpha_2 &= \sin(\phi^{(1)} - \phi^{(3)}) \\ \text{and} \\ \alpha_3 &= \sin(\phi^{(2)} - \phi^{(1)}) \end{aligned} \tag{18}$$

[3] Note that λ_1 is a measure of the flatness of an event and λ_3 is a measure of the sphericity. Similarly $\hat{n}_1$ is the vector $\hat{n}$ such that

$$\frac{\sum_i (\mathbf{p}_i \cdot \hat{n})^2}{\sum_i \mathbf{p}_i^2} \quad \text{is minimized}$$

and $\hat{n}_3$ is the vector $\hat{n}$ such that

$$\frac{\sum_i (\mathbf{p}_i \cdot \hat{n})^2}{\sum_i \mathbf{p}_i^2} \quad \text{is maximized}$$

[4] In case $N_1 = 1$, we define $\theta_0 = \theta_N - 2\pi$

[5] This $\phi^{(3)}$ may be larger than 2π. If so, it is more convenient to use $\phi^{(3)} - 2\pi$

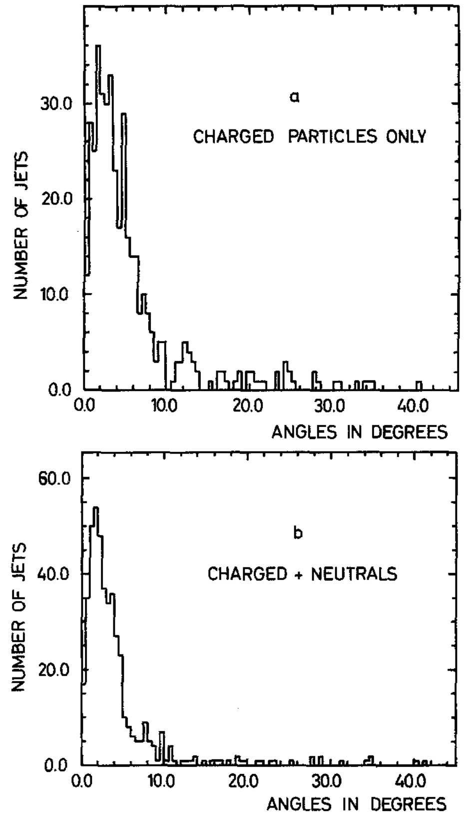

Fig. 2a and b. The distribution, at $E_{cm}=30$ GeV, of the difference of angle between reconstructed jet axis and the corresponding generated jet axis **a** using momenta of charged particles only, and **b** using momenta of both charged and neutral particles

where the ϕ's are for the particular partition N_1, N_2, N_3 that maximizes $\Lambda(N_1, N_2, N_3)$. From the directions of three jet axes, one obtains the total energy $E^{(\tau)}$ of each jet by

$$E^{(\tau)}=E_{cm}\alpha_\tau/(\alpha_1+\alpha_2+\alpha_3). \tag{19}$$

The procedure has been tested as follows. Three jets are generated by the method of Feynman and Field [5] in three directions, referred to as generated axes. The angular distribution of the three jet axes follows from the decay of heavy quarkonium into three gluons [6]. The present procedure is applied to reconstruct the jet axes. In Fig. 2 we show the distribution of the difference of angles between the reconstructed and the corresponding generated jet axes[6] at center of mass energy $E_{cm}=30$ GeV. This result is seen to be quite satisfactory both for (a) charged particles only and (b) charged and neutral particles. Note that, in the method of Feynman and Field, the generated jet axis does not coincide with the direction of the sum of momenta[7].

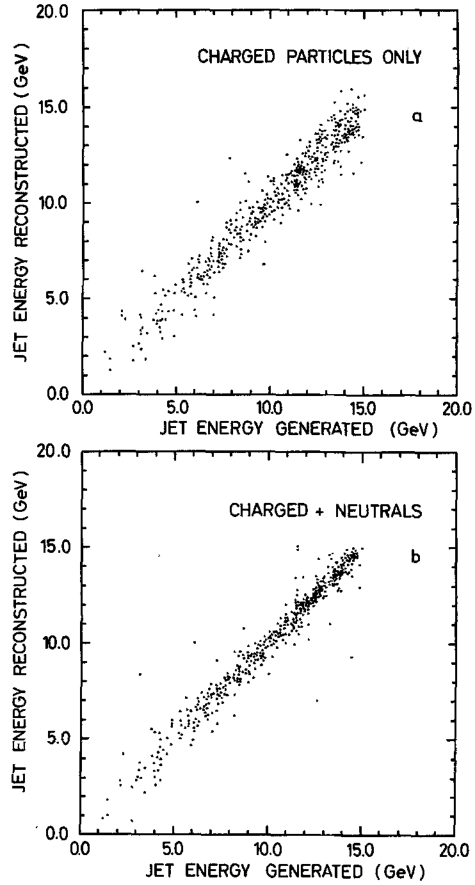

Fig. 3a and b. Comparison, at $E_{cm}=30$ GeV, of the reconstructed energy of each jet with the corresponding generated jet energy **a** using momenta of charged particles only, and **b** using momenta of both charged and neutral particles

Figure 3a and b are the two dimensional plots with reconstructed jet energies $E^{(\tau)}$ versus the generated jet energies used in the Feynman and Field procedure. These figures show the results are quite satisfactory even when the neutral particles are not observed.

One may ask the question: when is a planar event a three-jet event? The situation is quite similar to the previous case of two jets at SPEAR[1]. The question there was: how can a two-jet event be identified? In the two-jet case, the possible considerations are energy dependence of sphericity and its distribution compared with phase space. These considerations are equally applicable to the three-jet events. It is desirable to study the energy dependence of the quantity tri-jettiness J_3 and its distribution compared with phase space. Here we define tri-jettiness to be[8]

$$J_3=\left(\frac{1}{N-3}\right)\sum_{\tau=1,2,3}\left\{\sum_{j\text{ in }S_\tau}\frac{(\mathbf{q}_j\times\hat{m}^{(\tau)})^2}{\Delta_\tau^2}\right\} \tag{20}$$

[6] We associate the reconstructed jets and generated jets by taking the combinations with the smallest angle between them

[7] Our Monte Carlo program gives FWHM of about 3° from the distribution of angles between the generated jet axis and the direction of the sum of momenta at $\sqrt{s}=30$ GeV

[8] An alternative possibility is to use

$$J_3'=\left(\frac{1}{2N-5}\right)\sum_{\tau=1,2,3}\left\{\sum_{j\text{ in }S_\tau}\frac{(\mathbf{p}_j\times\hat{m}^{(\tau)})^2}{\Delta_\tau^2}\right\}$$

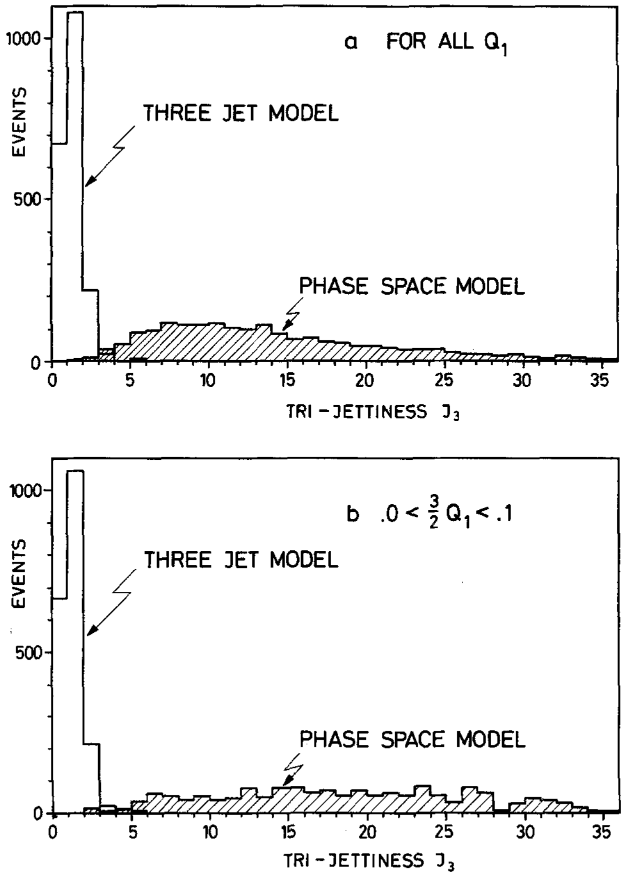

Fig. 4a and b. Tri-jettiness distributions at $E_{cm} = 30$ GeV for three-jet model and for phase space model. **a** Events with all values of Q_1. **b** Events with $0 \leqq \frac{3}{2} Q_1 \leqq 0.1$. In each Figure the curve for the three-jet model and that for the phase space model are normalized to the same number of events. Neutral and charged particles are included in all distributions. In the phase space model, we have assumed a total average multiplicity of 15.3 and π^0 fraction to be 0.54 as used in [3]

where $\Delta_\tau^2 = (1/2)\langle p_T^2 \rangle_\tau$. $\langle p_T^2 \rangle_\tau$ is the average transverse momentum squared of a jet with energy $E^{(\tau)}$ and may be assumed to be equal to that for two-jet events. To a first approximation this can be taken as a constant ($\sim 0.1\,\mathrm{GeV}^2/c^2$), independent of the energy of the jet. Note that $N-3$ is the number of degrees of freedom where N is the number of observed particles.

In Fig. 4 we show event distributions of the tri-jettiness J_3 defined by Eq. (20) at $E_{cm} = 30$ GeV for both heavy quarkonium decay into three gluon jets and phase space model. Figure 4a gives events with all values of Q_1 as defined by Eq. (2) and Fig. 4b gives events with $0 \leqq \frac{3}{2} Q_1 \leqq 0.1$, i.e. coplanar events which correspond to the shaded strip in Fig. 1. This cut contains almost all three-jet events but contains only less than 15% of the phase space events. In each of the Figs. 4a and b, the curves for the three-jet model and that for the phase space model are normalized to the same number of events. We conclude that the J_3 distributions are quite different for the three-jet model and for the phase space model.

Because of the work of Sterman and Weinberg [7], it is desirable, for the purposes of comparison with perturbation calculations in QCD, to use variables that are free of infrared divergences as the quark masses go to zero. Examples of such variables are spherocity, thrust, and acoplanarity [8]. While these variables have the desired infrared properties, the axes defined this way turn out to have to deal with the annoying property of being nonanalytic functions of the momenta [9]. In view of the similarity of the results in the upsilon region from sphericity and thrust [4], it is possible that the jet axes found by the present procedure of minimizing the sum of squares of transverse momenta, can be used for comparison with QCD.

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9 For example, to calculate Thrust T, one defines

$$T = 2 \max \frac{\tilde{\sum\limits_i} p_{\parallel}^i}{\sum\limits_i |\mathbf{p}_i|}$$

$\tilde{\sum\limits_i}$ = sum over particles in a hemisphere.

The function T has a number of local maxima. To find the global maximum, the computation is very tedious for high multiplicity events. We want to thank P. Söding for pointing out the triplicity method using thrust analysis for reconstruction of three jets by Brandt and Dahmen [9]. The triplicity method is considerably more involved than our method. For multiplicity 21, there are a couple billion combinations to deal with while we are working with a couple thousand combinations. However, S. Brandt has informed us that the method of triplicity has been simplified and the number of combinations is now reduced.