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ANOMALIES OF THE FREE LOOP WAVE EQUATION IN THE WKB APPROXIMATION

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We derive a well-defined, reparametrization invariant expression for the next to leading term in the small $\hbar$ expansion of the euclidean loop Green functional $\psi(\mathscr{C})$. To this order in $\hbar$, we then verify that $\psi(\mathscr{C})$ satisfies a renormalized loop wave equation, which involves a number of local, but non-harmonic anomalous terms. Also, we find that the quantum fluctuations of the string give rise, in $3+1$ dimensions, to a correction of the static quark potential by an attractive Coulomb potential of universal strength $\alpha_{\text{string}} = \frac{1}{12}\pi$.

1. Introduction

It is conceivable [1] that a reasonable zeroth approximation to the Wilson loop expectation $\psi(\mathscr{C})$ in a non-abelian pure gauge theory can be obtained by assuming

$$L(\sigma)\psi(\mathscr{C}) = 0\,, \qquad 0 \leqslant \sigma \leqslant 2\pi\,, \tag{1}$$

$$L(\sigma) \overset{\text{def}}{=} -\frac{\delta^2}{\delta x_\mu(\sigma)\delta x_\mu(\sigma)} + M^4 x'(\sigma)^2\,, \qquad x'_\mu \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\sigma} x_\mu\,. \tag{2}$$

Here, $\mathscr{C}$ denotes a closed, infinitely differentiable and double point free curve in $\mathbb{R}^d$, $x_\mu(\sigma)$ is a parametrization of $\mathscr{C}$ and M a constant mass parameter. As it stands, the loop wave equation (1) does not have a well-defined meaning, because the second variation

$$\frac{\delta^2 F}{\delta x_\mu(\sigma)\delta x_\nu(\lambda)}$$

of a well-behaved functional F is a distribution in σ and λ, which will in general be singular at $\sigma = \lambda$. The conventional treatment [2] of this difficulty starts from the observation that $L(\sigma)$ has the form of a bunch of harmonic oscillator hamiltonians. Normal ordering with respect to a suitable basis of creation and annihilation operators then yields a well-defined wave operator $:L(\sigma):$. The trouble with this procedure is that the wave equation becomes incompatible with the requirement that $\psi(\mathscr{C})$ is reparametrization invariant:

$$R(\sigma)\psi(\mathscr{C}) = 0\,, \qquad R(\sigma) \overset{\text{def}}{=} x'_\mu(\sigma)\frac{\delta}{\delta x_\mu(\sigma)}\,. \tag{3}$$

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In fact, we have

$$[R(\sigma), :L(\lambda):] = -\delta'(\sigma-\lambda)(:L(\sigma): + :L(\lambda):) + \frac{dM^2}{6\pi}(\delta'''(\sigma-\lambda)+\delta'(\sigma-\lambda))\,, \tag{4}$$

so that any functional ψ satisfying (3) and

$$:L(\sigma):\,\psi(\mathscr{C})=0$$

must vanish. In the present context, we therefore find the conventional renormalization of eq. (1) unacceptable.

In this article, we derive the leading terms of the small $\hbar$ expansion of a functional integral representation [3] for $\psi(\mathscr{C})$. Letting $\hbar\to 0$ is equivalent to taking $M\to\infty$, while keeping $\mathscr{C}$ fixed. In this limit, one obtains an asymptotic expansion of the form

$$\psi(\mathscr{C}) \sim M^p\, e^{-M^2 A(\mathscr{C})} \sum_{\nu=0}^{\infty} (M^2)^{-\nu}\psi_\nu(\mathscr{C})\,, \tag{5}$$

where $A(\mathscr{C})$ is the minimal area enclosed by $\mathscr{C}$. The overall exponential factor $\exp(-M^2A(\mathscr{C}))$ has elsewhere been shown [4] to follow from the loop wave equation (1) and the assumption that $\psi(\mathscr{C})$ decays smoothly for uniformly large loops. We here derive a finite and reparametrization invariant, but not very explicit formula for $\psi_0(\mathscr{C})$. Still, the expressions are explicit enough to investigate how the wave equation (1) is renormalized to this order of $1/M^2$ and to compute the quantum corrections to the classical linear quark potential.

At first sight, one might attempt to determine $\psi_0(\mathscr{C})$ by inserting the expansion (5) into the loop wave equation (1), thus obtaining an equation for ψ_0:

$$\left\{2\frac{\delta A}{\delta x_\mu(\sigma)}\frac{\delta}{\delta x_\mu(\sigma)} + \frac{\delta^2 A}{\delta x_\mu(\sigma)\delta x_\mu(\sigma)}\right\}\psi_0 = 0\,. \tag{6}$$

The difficulty, however, is that the second variation of A is singular at coinciding arguments:

$$\frac{\delta^2 A}{\delta x_\mu(\sigma)\delta x_\mu(\lambda)} \overset{\sigma\to\lambda}{=} -\frac{d-2}{\pi}\frac{|x'(\sigma)|\,|x'(\lambda)|}{(\Delta s)^2} - \frac{R}{2\pi}x'^2\ln\tfrac{1}{2}\Delta s + \text{finite terms}\,. \tag{7}$$

Here, Δs is the length of the piece of $\mathscr{C}$ between $x_\mu(\sigma)$ and $x_\mu(\lambda)$, and R denotes the curvature scalar at $x_\mu(\lambda)$ of the minimal surface spanned by $\mathscr{C}$. A subtraction of the loop wave equation (1) is therefore unavoidable, once one assumes that $\psi(\mathscr{C})$ has a smooth asymptotic expansion for $M\to\infty$ (or, equivalently, for fixed M and uniformly large loops $\mathscr{C}$). In particular, we cannot use the unrenormalized eq. (6) to compute ψ_0.

Our paper is organized as follows. In sect. 2 we review the functional integral representation for $\psi(\mathscr{C})$ proposed by Eguchi [3]. The large M expansion then amounts to evaluating the integral by the saddle point method. We thus study surface fluctuations around the minimal surface spanned by $\mathscr{C}$ (sect. 3) and perform a gaussian integral to obtain $\psi_0(\mathscr{C})$ (sect. 4). In sect. 5 we then show that ψ_0 satisfies a renormalized version of eq. (6), which can be seen to stem from a renormalized, *local* loop wave equation replacing eq. (1). Finally, ψ_0 is exactly evaluated for a large rectangular loop, giving quantum corrections to the static quark potential (sect. 6), and conclusions are drawn in sect. 7. A number of appendices are included, dealing with the rather involved technicalities that arise, when computing determinants of differential operators. In appendix F we also discuss an anomalous loop heat equation, which is not directly relevant to our investigation, but may be interesting for readers familiar with another approach [17] to the quantized string theory.

2. Functional integral representation for the loop Green function

We first define a kind of loop heat kernel $K_\Gamma[x]$ as an average over all fields $\phi_\mu(z)$, where z ranges over some simply connected, bounded region Γ in $\mathbb{R}^2$:

$$K_\Gamma[x] = \int_{\phi_\mu|_{\partial\Gamma} = x_\mu} \mathscr{D}\phi_\mu \exp - S\,. \tag{8}$$

The action S is chosen to be [7]

$$S = \tfrac{1}{2}M^4 \int_\Gamma \mathrm{d}^2 z\, |g|\,, \qquad |g| = \det(g_{ab})\,, \tag{9}$$

where

$$g_{ab} = \partial_a\phi \cdot \partial_b\phi\,, \qquad a, b = 1, 2\,, \tag{10}$$

is the natural metric on the surface in $\mathbb{R}^d$ represented by $\phi_\mu(z)$. The boundary condition $\phi_\mu|_{\partial\Gamma} = x_\mu$ in eq. (8) means that only those fields ϕ_μ which map $\partial\Gamma$ onto $\mathscr{C}$ (with no further specification) should be integrated over.

The stationary points of S are minimal surfaces in a special parametrization, where $|g| = \text{constant}$. Another outstanding property of the action is its invariance under symplectic coordinate transformations in the z-plane, i.e., those transformations, which have unit jacobian. Since, by a theorem of Moser [5], any region Γ can be mapped onto any other Γ' by a symplectic transformation provided only that the two have the same area, it follows* that $K_\Gamma[x]$ does not depend on the shape of Γ, but only on the area a of Γ. It has then been shown [3] that formally

$$2M^4 \frac{\partial}{\partial a} K_\Gamma[x] = \frac{1}{x'(\sigma)^2} \frac{\delta^2}{\delta x_\mu(\sigma)\delta x_\mu(\sigma)} K_\Gamma[x]\,. \tag{11}$$

* We here assume that $\mathscr{D}\phi_\mu$ is invariant under symplectic coordinate transformations. This will be justified below in the WKB approximation.

The Laplace transform

$$\psi(\mathscr{C}) = \int_0^\infty \mathrm{d}a \; \mathrm{e}^{-a/2} K_\Gamma[x] \tag{12}$$

therefore provides a formal solution of the loop wave equation (1), which, by analogy with the one-particle case [3], can be expected to be something like a Green function for loops.

$K_\Gamma[x]$ is a reparametrization invariant functional of $x_\mu(\sigma)$. Superficially, this follows from the fact that the boundary condition on the fields ϕ_μ integrated over in eq. (8) does not refer to a parametrization $x_\mu(\sigma)$ of $\mathscr{C}$, but merely requires that the boundary of the surface described by ϕ_μ is $\mathscr{C}$. The real question is, however, whether or not this boundary condition makes the integral (8) well-defined (disregarding ultraviolet divergences). That this is indeed the case, will be verified later within the WKB approximation. In particular, an ill-defined average over all parametrizations of $\mathscr{C}$ effectively gets absorbed into the group volume factor stemming from the symplectic transformations, which leave Γ invariant. This group volume factor on the other hand, can be easily extracted from $K_\Gamma[x]$ by the Fadeev–Popov method.

As $M \to \infty$, we see that the integral (8) is dominated by the stationary points of S. The $1/M^2$ expansion of K_Γ is therefore obtained by expanding (8) around the minimal surface enclosed by $\mathscr{C}$. Let $\varphi_\mu(z)$ denote a solution of the equations of motion implied by $\delta S = 0$*:

$$\partial_a(|g| g^{ab} \partial_b \varphi_\mu) = 0\,, \tag{13}$$

subject to the boundary condition

$$\varphi_\mu(z(\sigma)) = x_\mu(\sigma), \text{ for some parametrization } z(\sigma) \text{ of } \partial\Gamma\,. \tag{14}$$

$\varphi_\mu(z)$ is not uniquely determined by the loop $\mathscr{C}$, but only up to a symplectic coordinate transformation mapping Γ onto itself**. Eq. (13) implies that $|g| =$ constant and, since

$$A(\mathscr{C}) = \int_\Gamma \mathrm{d}^2 z \; |g|^{1/2}\,,$$

we have

$$|g|^{1/2} = A/a\,. \tag{15}$$

To compute $\psi_0(\mathscr{C})$ we thus have to study the gaussian fluctuations of ϕ_μ around φ_μ.

* In what follows, g_{ab} denotes the metric (10) of the minimal surface described by φ_μ. g^{ab} is the inverse of g_{ab} and indices are raised and lowered with these tensors.

** If $\mathscr{C}$ bounds several minimal surfaces, we choose the one with least area. The critical case, where there is more than one surface with least area, will not be considered.

3. Properties of the fluctuation operator

Consider a fluctuation

$$\phi_\mu = \varphi_\mu + \varepsilon\eta_\mu + \varepsilon^2\xi_\mu + \mathrm{O}(\varepsilon^3) \tag{16}$$

around φ_m, which leaves the boundary $\mathscr{C}$ fixed, i.e.

$$\eta_\mu \propto x'_\mu , \qquad \xi_\mu - \frac{\eta^2}{2x'^2} x''_\mu \propto x'_\mu \qquad (\text{along } \partial\Gamma) . \tag{17}$$

Then, up to $\mathrm{O}(\varepsilon^2)$ we have

$$M^{-4}S = \frac{A^2}{2a} + \varepsilon^2 \frac{A}{2a} \int_\Gamma \mathrm{d}^2 z \; |g|^{1/2} \eta_\mu \Delta_{\mu\nu} \eta_\nu , \tag{18}$$

where

$$\Delta_{\mu\nu} = -|g|^{-1/2} \partial_a |g|^{1/2} M^{ab}_{\mu\nu} \partial_b , \tag{19a}$$

$$M^{ab}_{\mu\nu} = \delta_{\mu\nu} g^{ab} + 2(\partial^a \varphi_\mu)(\partial^b \varphi_\nu) - (\partial^a \varphi_\nu)(\partial^b \varphi_\mu) - g^{ab} (\partial^c \varphi_\mu)(\partial_c \varphi_\nu) . \tag{19b}$$

Here, we have introduced various factors $|g|^{1/2}$ for later convenience in order to have a fluctuation operator $\Delta_{\mu\nu}$, which is invariant under *general* coordinate transformations in the z-plane.

A close inspection of $\Delta_{\mu\nu}$ reveals that it splits into a longitudinal and a transversal part

$$\Delta_{\mu\nu} = \Delta^{\mathrm{L}}_{\mu\nu} + \Delta^{\mathrm{T}}_{\mu\nu} , \tag{20a}$$

$$\Delta^{\mathrm{L}}_{\mu\nu} = Q_{\mu\rho} \Delta_{\rho\sigma} Q_{\sigma\nu} , \qquad \Delta^{\mathrm{T}}_{\mu\nu} = (\delta_{\mu\rho} - Q_{\mu\rho}) \Delta_{\rho\sigma} (\delta_{\sigma\nu} - Q_{\sigma\nu}) , \tag{20b}$$

with $Q_{\mu\nu}$ the projector onto the tangent plane of the background minimal surface:

$$Q_{\mu\nu} = (\partial_a \varphi_\mu) g^{ab} (\partial_b \varphi_\nu) . \tag{21}$$

More explicitly, for a longitudinal mode

$$\eta_\mu = \omega^a \partial_a \varphi_\mu \tag{22}$$

we have

$$\Delta_{\mu\nu} \eta_\nu = -\partial^a [|g|^{-1/2} \partial_b (|g|^{1/2} \omega^b)] \partial_a \varphi_\mu , \tag{23}$$

which is again longitudinal.

From eq. (23) we see that all modes of the form

$$\eta_\mu = |g|^{-1/2} (\partial_a \lambda) \epsilon_{ab} \partial_b \varphi_\mu , \qquad \epsilon_{ab} = -\epsilon_{ba} , \qquad \epsilon_{12} = 1 , \tag{24}$$

are annihilated by $\Delta_{\mu\nu}$. These zero modes are a reflection of the invariance of the action S under symplectic coordinate transformations. Namely, if ϕ_μ [eq. (16)] is related to φ_μ by an infinitesimal area preserving coordinate transformation, they have the same action and consequently $\Delta_{\mu\nu}\eta_\nu = 0$. Note that for the zero modes (24), the boundary condition (17) reduces to Dirichlet boundary conditions:

$$\lambda = 0 , \qquad \text{along } \partial\Gamma . \tag{25}$$

In the space of longitudinal modes, $\Delta^{\mathrm{L}}_{\mu\nu}$ has no other zero modes. For the non-zero eigenvalues, we assert that they are precisely the same as those of the laplacian

$$L = -|g|^{-1/2}\partial_a |g|^{1/2} g^{ab}\partial_b \tag{26}$$

with Neumann boundary conditions imposed:

$$Lv = Ev , \qquad n^a\partial_a v = 0 , \qquad \text{along } \partial\Gamma , \tag{27a}$$

$$n^a = g^{ab}|g|^{1/2}\epsilon_{bc}\dot{z}^c , \qquad \dot{z}^c = \frac{\mathrm{d}}{\mathrm{d}s} z^c . \tag{27b}$$

Here, ds denotes the arc length element:

$$\mathrm{d}s = (g_{ab}\,\mathrm{d}z^a\,\mathrm{d}z^b)^{1/2} = |x'|\,\mathrm{d}\sigma . \tag{28}$$

Indeed, defining

$$\eta_\mu = (\partial^a v)\partial_a\varphi_\mu ,$$

we see from eq. (23) that

$$\Delta_{\mu\nu}\eta_\nu = E\eta_\mu , \qquad \eta_\mu \propto x'_\mu , \qquad \text{along } \partial\Gamma . \tag{29}$$

Conversely,

$$v = \frac{1}{E}|g|^{-1/2}\partial_a[|g|^{1/2}g^{ab}(\partial_b\varphi_\mu)\eta_\mu]$$

solves eq. (27) if η_μ is a longitudinal solution of eq. (29).

We finally mention that in case of a planar curve $\mathscr{C}$, Δ^{T} reduces to $d-2$ copies of the laplacian L. In any case, Δ^{T} is an elliptic differential operator in the space of transverse fluctuations with Dirichlet boundary conditions and has a discrete spectrum of (in general) positive eigenvalues.

4. Computation of $\psi_0(\mathscr{C})$

The gaussian approximation to the integral (8) is now obtained easily, the only complication being that a symplectic group volume factor must carefully be extracted à la Fadeev–Popov. A most convenient "gauge" fixing condition is

$$\epsilon_{ab}(\partial_a\varphi_\mu)\partial_b\eta_\mu = 0 \tag{30}$$

which is similar to the background gauge condition used in gauge theories (see, e.g., ref. [13]). The outcome then is

$$K_\Gamma[x]=\exp -\tfrac{1}{2}\left\{M^4\frac{A^2}{a}+W\right\},$$

$$W=\mathrm{Tr}\ln\frac{M^2|g|}{2\pi}\Delta^{\mathrm{T}}+\mathrm{Tr}\ln\frac{M^2|g|}{2\pi}L-\mathrm{Tr}\ln\frac{M^2|g|}{2\pi}\tilde{L}. \tag{31}$$

The first trace here is to be taken over transversal modes only. The boundary conditions for Δ^{T} and L (the contribution of the longitudinal modes) are Dirichlet and Neumann, respectively [cf. eq. (27)]. The trivial zero mode, v = constant, of L is to be omitted in the trace, of course. The determinant of $\tilde{L}$ is the Fadeev-Popov determinant. As a differential operator, $\tilde{L}$ is equal to L [eq. (26)], but the boundary conditions imposed are Dirichlet rather than Neumann.

Of course, W is UV divergent and must be regularized. The divergent parts of W can then be subtracted by adding appropriate counter terms to the action S. To regularize the determinants in eq. (31), we use the Pauli-Villars method*. We thus introduce a set of large regulator masses M_j and fixed numbers ε_j ($j=1,\ldots,\nu$) such that

$$\sum_{j=1}^{\nu}\varepsilon_j=-1\,,\qquad \sum_{j=1}^{\nu}\varepsilon_j M_j^{2p}=0\,,\qquad (p=1,\ldots,\nu-1)\,. \tag{32}$$

When the contribution of the corresponding Pauli-Villars ghosts is taken into account, we obtain a regularized expression for W:

$$W_{\mathrm{reg}}=W_{\mathrm{reg}}(\Delta^{\mathrm{T}})+W_{\mathrm{reg}}(L)-W_{\mathrm{reg}}(\tilde{L})\,, \tag{33}$$

where, for example,

$$W_{\mathrm{reg}}(\Delta^{\mathrm{T}})=\mathrm{Tr}\ln\left\{\frac{M^2|g|}{2\pi}\Delta^{\mathrm{T}}\prod_{j=1}^{\nu}\left[\frac{M^2|g|}{2\pi}(\Delta^{\mathrm{T}}+M_j^2)\right]^{\varepsilon_j}\right\}$$

$$=\mathrm{Tr}\left\{\ln\Delta^{\mathrm{T}}+\sum_{j=1}^{\nu}\varepsilon_j\ln(\Delta^{\mathrm{T}}+M_j^2)\right\}. \tag{34}$$

W_{reg} is finite provided $\nu\geq 2$. To extract the divergent part of W_{reg} as $M_j\to\infty$, we proceed as follows: first note that

$$W_{\mathrm{reg}}(\Delta^{\mathrm{T}})=-\int_0^\infty\frac{\mathrm{d}t}{t}\left(1+\sum_{j=1}^{\nu}\varepsilon_j\,\mathrm{e}^{-tM_j^2}\right)\mathrm{Tr}\,\mathrm{e}^{-t\Delta^{\mathrm{T}}}.$$

As $t\to 0$, we have

$$\mathrm{Tr}\,\mathrm{e}^{-t\Delta^{\mathrm{T}}}=\sum_{l=0}^{2}t^{-l/2}\alpha_{l/2}(\Delta^{\mathrm{T}})+\mathrm{O}(\sqrt{t})\,, \tag{35}$$

* We use a version of this method familiar from instanton calculations [18].

and consequently, as $M_j \to \infty$,

$$W_{\mathrm{reg}}(\Delta^{\mathrm{T}}) = -\alpha_1(\Delta^{\mathrm{T}}) \sum_{j=1}^{\nu} \varepsilon_j M_j^2 \ln M_j^2 + 2\sqrt{\pi}\alpha_{1/2}(\Delta^{\mathrm{T}}) \sum_{j=1}^{\nu} \varepsilon_j M_j$$
$$+ \alpha_0(\Delta^{\mathrm{T}}) \sum_{j=1}^{\nu} \varepsilon_j \ln M_j^2 + W_{\mathrm{PV}}(\Delta^{\mathrm{T}}) . \tag{36}$$

Explicitly, the finite part W_{PV} is*

$$W_{\mathrm{PV}}(\Delta^{\mathrm{T}}) = \alpha_0 \Gamma'(1) - \int_0^{\infty} \frac{\mathrm{d}t}{t} \{\mathrm{Tr}\, \mathrm{e}^{-t\Delta^{\mathrm{T}}} - \alpha_1 t^{-1} - \alpha_{1/2} t^{-1/2} - \alpha_0 \theta(1-t)\} . \tag{37}$$

The divergent parts of W_{reg} are thus proportional to the Seeley coefficients $\alpha_{l/2}$, which in turn can be calculated perturbatively. A table of the Seeley coefficients of Δ^{T}, L and $\tilde{L}$ is contained in appendix B.

Summarizing, we have

$$W_{\mathrm{reg}} = -\frac{d-2}{4\pi} A(\mathscr{C}) \sum_{j=1}^{\nu} \varepsilon_j M_j^2 \ln M_j^2 - \tfrac{1}{4} d L(\mathscr{C}) \sum_{j=1}^{\nu} \varepsilon_j M_j$$
$$+ \left(\frac{d-8}{6} - \frac{\varkappa(\mathscr{C})}{4\pi} \right) \sum_{j=1}^{\nu} \varepsilon_j \ln M_j^2 + W_{\mathrm{PV}}(\Delta^{\mathrm{T}}) + W_{\mathrm{PV}}(L) - W_{\mathrm{PV}}(\tilde{L}) . \tag{38}$$

Here, $L(\mathscr{C})$ denotes the length of $\mathscr{C}$, and $\varkappa$ is the integral of the curvature of the minimal surface:

$$\varkappa(\mathscr{C}) = \int_{\Gamma} \mathrm{d}^2 z \, |g|^{1/2} R , \qquad R\text{: curvature scalar**}. \tag{39}$$

Eq. (38) can be simplified noting that (appendix C)

$$W_{\mathrm{PV}}(L) - W_{\mathrm{PV}}(\tilde{L}) = \ln A(\mathscr{C}) - \frac{\varkappa(\mathscr{C})}{4\pi} + C , \tag{40}$$

where C is a constant. $W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$ is perhaps also calculable. In any case, it is a well-defined functional of the background field φ_μ and the parameter region Γ. Suppose that $\tilde{\varphi}_\mu(z)$, $z \in \tilde{\Gamma}$, is another solution of the field equation (13) describing the same minimal surface bounded by $\mathscr{C}$ as φ_μ does. Then, there exists a mapping $f: \Gamma \to \tilde{\Gamma}$ with constant jacobian such that $\varphi_\mu(z) = \tilde{\varphi}_\mu(f(z))$. Since Δ^{T} is invariant under general coordinate transformations, it follows that the spectra and hence the determinants of $\Delta^{\mathrm{T}}(\varphi)$ and $\Delta^{\mathrm{T}}(\tilde{\varphi})$ are identical. We therefore can consider $W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$ to be a functional of $\mathscr{C}$ alone, which, as such, is independent of Γ.

* One can show that $W_{\mathrm{PV}} = -\zeta'(0)$, where $\zeta(s)$ is the zeta function of Δ^{T}.

** R is twice the gaussian curvature.

To obtain $\psi(\mathscr{C})$ from $K_\Gamma[x]$ we finally have to integrate over a [cf. eq. (12)]. Since $W_{\rm reg}$ is independent of a, this is easily done:

$$\psi(\mathscr{C}) = CM\,\mathrm{e}^{-M^2A(\mathscr{C})}\exp\left\{\frac{d-2}{8\pi}A(\mathscr{C})\sum_{j=1}^{\nu}\varepsilon_j M_j^2\ln M_j^2\right.$$
$$\left.+\tfrac{1}{8}dL(\mathscr{C})\sum_{j=1}^{\nu}\varepsilon_j M_j + \frac{1}{8\pi}\varkappa(\mathscr{C})\left(\sum_{j=1}^{\nu}\varepsilon_j\ln M_j^2+1\right)-\tfrac{1}{2}W_{\rm PV}(\Delta^{\rm T})\right\}. \quad (41)$$

The most divergent term can thus be absorbed in a renormalization of M^2. The term proportional to $L(\mathscr{C})$ can be removed by a wave-function renormalization

$$\psi(\mathscr{C}) \to \mathrm{e}^{-mL(\mathscr{C})}\psi(\mathscr{C}) \quad (42)$$

(such a renormalization is also needed to define the Wilson loop expectation value in perturbation theory [6]). To remove the logarithmic divergence, we have to add a counter term

$$\Delta S = \left[\frac{1}{8\pi}\left(\sum_{j=1}^{\nu}\varepsilon_j\ln M_j^2+1\right)-\gamma\right]\int_\Gamma \mathrm{d}^2z\,|g|^{1/2}R\,, \quad (43)$$

to the action, where g_{ab} and R are the metric and curvature scalar of the surface $\phi_\mu(z)$ to be integrated over in eq. (8). A new dimensionless coupling constant γ appears here, which is not too surprising, since the integral (8) is superficially non-renormalizable.

To sum up, we obtain the following renormalized WKB approximation to the loop Green function:

$$\psi_{\rm ren}(\mathscr{C}) = CM\,\mathrm{e}^{-M^2A(\mathscr{C})}\exp\{\gamma\varkappa(\mathscr{C})-\tfrac{1}{2}W_{\rm PV}(\Delta^{\rm T})\}\,. \quad (44)$$

A finite wave-function renormalization factor [eq. (42)] has been omitted here, since it does not influence the renormalized wave equation (cf. subsect. 5.3) in any essential way. Also, if $\psi(\mathscr{C})$ is identified with the Wilson loop expectation value in QCD, such a factor merely amounts to a finite quark mass renormalization. We emphasize that although we were unable to produce an explicit formula for $W_{\rm PV}(\Delta^{\rm T})$, $\psi_{\rm ren}(\mathscr{C})$ has been shown to be a well-defined and reparametrization invariant functional of $x_\mu(\sigma)$. We finally mention that eq. (44) could also be derived *formally* starting from

$$\psi(\mathscr{C}) = \int_{\phi_\mu|_{\partial\Gamma}=x_\mu}\mathscr{D}\phi_\mu\,\exp -M^2\mathscr{A}[\phi]\,,$$

where

$$\mathscr{A}[\phi] = \int_\Gamma \mathrm{d}^2x\,|g|^{1/2}$$

is the area of the surface described by $\phi_\mu(z)$. Indeed, noting that

$$\mathscr{A}[\phi]=\mathscr{A}[\varphi]+\varepsilon^2\int_\Gamma \mathrm{d}^2z\ |g|^{1/2}\eta_\mu\Delta^\mathrm{T}_{\mu\nu}\eta_\nu+\mathrm{O}(\varepsilon^3)$$

for fluctuations (16) around a minimal surface, one obtains eq. (44) by a saddle-point integration as before, except that one must be rather cavalier about various meaningless measure factors, which appear, when the Fadeev–Popov procedure is applied to deal with the longitudinal modes.

5. The renormalized wave equation

In this section, we investigate to what extent the renormalized amplitude ψ_0 given by eq. (44) satisfies the loop wave equation (6). This involves two steps: first we must find a manageable expression for the second variation of $A(\mathscr{C})$ and secondly we must compute

$$p_\mu\frac{\delta}{\delta x_\mu}W_{\mathrm{PV}}(\Delta^\mathrm{T})\qquad\text{and}\qquad p_\mu\frac{\delta}{\delta x_\mu}\varkappa(\mathscr{C})\,.$$

For notational convenience we use the abbreviation

$$p_\mu=\frac{\delta A}{\delta x_\mu}\,. \tag{45}$$

5.1. SECOND VARIATION OF THE MINIMAL AREA $A(\mathscr{C})$

In terms of the field $\varphi_\mu(z)$ [eqs. (13) and (14)] describing the minimal surface Σ spanned by $\mathscr{C}$, we have

$$p_\mu=|g|^{-1/2}(\partial_a\varphi_\mu)\epsilon_{ab}(\partial_b\varphi_\nu)x'_\nu\,. \tag{46}$$

$p_\mu(\sigma)$ is thus a vector in the tangent plane of Σ at $x_\mu(\sigma)$. Furthermore,

$$p\cdot x'=0\,,\qquad p^2=x'^2\,, \tag{47}$$

so that

$$\delta p_\mu(\sigma)=0$$

for any longitudinal variation $\delta x_\mu(\lambda)$ of $\mathscr{C}$, which vanishes, when λ is near σ. To compute

$$\frac{\delta^2 A}{\delta x_\mu(\sigma)\delta x_\nu(\lambda)}$$

for $\sigma\neq\lambda$, it is therefore sufficient to consider transversal variations of $\mathscr{C}$ only:

$$(x'\cdot\delta x)(\lambda)=(p\cdot\delta x)(\lambda)=0\,,\qquad \delta x_\mu(\lambda)=0\,,\qquad \text{for }\lambda\text{ near }\sigma\,. \tag{48}$$

For such curve deformations, we have

$$\delta p_\mu(\sigma) = |g|^{-1/2}\epsilon_{ab}\{(\partial_a\delta\varphi_\mu)(z)(\partial_b\varphi_\nu)(z) + (\partial_a\varphi_\mu)(z)(\partial_b\delta\varphi_\nu)(z)\}_{z=z(\sigma)}x'_\nu(\sigma)\,. \tag{49}$$

$\delta\varphi_\mu$ is a solution of the linearized field equation (13):

$$\Delta_{\mu\nu}\delta\varphi_\nu = 0\,, \qquad (\delta\varphi_\mu)(z(\lambda)) = \delta x_\mu(\lambda)\,. \tag{50}$$

Since δx_μ is transversal, we may assume $\delta\varphi_\mu$ to be transversal, too (cf. sect. 3). To solve the boundary value problem (50), we introduce the transverse Green function $G^{\mathrm{T}}_{\mu\nu}$ defined by

$$\Delta_{\mu\nu}G^{\mathrm{T}}_{\nu\rho}(z, w) = |g|^{-1/2}(\delta_{\mu\rho} - Q_{\mu\rho})\delta(z-w)\,, \tag{51a}$$

$$Q_{\mu\nu}G^{\mathrm{T}}_{\nu\rho} = G^{\mathrm{T}}_{\mu\nu}Q_{\nu\rho} = 0\,, \tag{51b}$$

$$G^{\mathrm{T}}_{\mu\nu}(z, w)|_{z\in\partial\Gamma} = G^{\mathrm{T}}_{\mu\nu}(z, w)|_{w\in\partial\Gamma} = 0\,. \tag{51c}$$

By Stokes' theorem we then obtain

$$\begin{aligned}\delta\varphi_\mu(w) &= \int_\Gamma \mathrm{d}^2z\ |g|^{1/2}\{\delta\varphi_\rho(z)\Delta_{\rho\nu}G^{\mathrm{T}}_{\nu\mu}(z, w) - (\Delta_{\rho\nu}\delta\varphi_\nu)(z)G^{\mathrm{T}}_{\rho\mu}(z, w)\}\\ &= \oint_{\partial\Gamma} \mathrm{d}z^a\ \epsilon_{ab}|g|^{1/2}\delta x_\rho(z)M^{bc}_{\rho\nu}\partial_c G^{\mathrm{T}}_{\nu\mu}(z, w)\\ &= -\int_0^{2\pi} \mathrm{d}\lambda\ |x'(\lambda)|\delta x_\nu(\lambda)n^a\partial_a G^{\mathrm{T}}_{\nu\mu}(z(\lambda), w)\end{aligned}$$

[n^a denotes the normal vector (27b)]. Inserting this into eq. (49) finally gives for $\sigma \neq \lambda$

$$\frac{\delta^2 A}{\delta x_\mu(\sigma)\delta x_\nu(\lambda)} = -|x'(\sigma)|n^a(\sigma)\partial_a G^{\mathrm{T}}_{\mu\nu}(z(\sigma), z(\lambda))\overleftarrow{\partial}_b n^b(\lambda)|x'(\lambda)|\,. \tag{52}$$

Of course, this is not a very explicit formula since the Green function $G^{\mathrm{T}}_{\mu\nu}$ is not generally known. However, it is good enough for our purposes; for example, the short-distance expansion (7) can be derived on the basis of eq. (52) (subsect. 5.2 and appendix D).

5.2. FIRST VARIATION OF $\varkappa$ AND $W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$

To check eq. (6) we only need to know how $\varkappa$ and $W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$ change, when $\mathscr{C}$ is deformed along the minimal surface Σ it spans. Thus, let

$$\delta x_\mu = \delta z^a\partial_a\varphi_\mu\,, \tag{53}$$

so that $\delta\varkappa$ and $\delta W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$ are the variations of $\varkappa$ and $W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$, when the background field φ_μ is kept fixed and the parameter region Γ is varied according to

$$z^a(\sigma) \to z^a(\sigma) + \delta z^a(\sigma) . \tag{54}$$

We here make use of the fact that $\varkappa$ and the eigenvalues of Δ^{T} are independent of Γ as long as φ_μ satisfies the correct boundary conditions along $\partial\Gamma$. From the definition (39) of $\varkappa$ it now follows that

$$\delta\varkappa = \delta z^a \epsilon_{ab} z'^b |g|^{1/2} R ,$$

and therefore

$$p_\mu \frac{\delta\varkappa}{\delta x_\mu} = x'^2 R . \tag{55}$$

To compute $\delta W_{\mathrm{PV}}(\Delta^{\mathrm{T}})$ is more difficult. Let E be an eigenvalue of Δ^{T}:

$$\Delta_{\mu\nu} v_\nu = E v_\mu , \qquad Q_{\mu\nu} v_\nu = 0 , \qquad v_\mu|_{\partial\Gamma} = 0 ,$$
$$\int_\Gamma \mathrm{d}^2 z \, |g|^{1/2} v^2 = 1 .$$

When $\partial\Gamma$ is varied we have

$$\Delta_{\mu\nu}\delta v_\nu = \delta E v_\mu + E \delta v_\mu , \qquad Q_{\mu\nu}\delta v_\nu = 0 ,$$
$$\delta v_\mu + \delta z^a \partial_a v_\mu = 0 , \qquad (\text{along } \partial\Gamma) .$$

Integration yields

$$\begin{aligned} \delta E &= \int_\Gamma \mathrm{d}^2 z \, |g|^{1/2} (v_\mu \Delta_{\mu\nu} \delta v_\nu - \delta v_\mu \Delta_{\mu\nu} v_\nu) \\ &= -\int_0^{2\pi} \mathrm{d}\sigma \, |x'(\sigma)| \delta z^a(\sigma) n^b(\sigma) (\partial_a v \cdot \partial_b v)(z(\sigma)) . \end{aligned}$$

It follows that

$$p_\mu \frac{\delta E}{\delta x_\mu} = -x'^2 (n^a \partial_a v)^2 . \tag{56}$$

Next we apply this formula to compute [cf. eq. (34)]

$$\begin{aligned} &p_\mu \frac{\delta W_{\mathrm{reg}}(\Delta^{\mathrm{T}})}{\delta x_\mu(\sigma)} \\ &\quad = -\left\{ |x'| n^a \partial_a \left[G^{\mathrm{T}}_{\mu\mu}(z, w) + \sum_{j=1}^{\nu} \varepsilon_j G^{\mathrm{T},M_j}_{\mu\mu}(z, w) \right] \overleftarrow{\partial}_b n^b |x'| \right\}_{z=w=z(\sigma)} , \end{aligned} \tag{57}$$

where $G^{\mathrm{T},M}_{\mu\nu}$ denotes the massive Green function of $\Delta^{\mathrm{T}}_{\mu\nu}$. We thus see that except for the Pauli–Villars regulator contribution, which subtracts the short-distance singularities of $n^a \partial_a G^{\mathrm{T}}_{\mu\mu} \overleftarrow{\partial}_b n^b$, $p_\mu (\delta/\delta x_\mu) W_{\mathrm{reg}}$ equals the second variation (52) of A at coinciding points. Now, in the limit where first $\sigma \to \lambda$ and then $M_j \to \infty$, we have (appendix D)

$$\sum_{j=1}^{\nu} \varepsilon_j n^a \partial_a G^{\mathrm{T},M_j}_{\mu\mu}(z, w) \overleftarrow{\partial}_b n^b \Big|_{\substack{z=z(\sigma)\\ w=z(\lambda)}}$$

$$= -\frac{d-2}{\pi(\Delta s)^2} - \frac{R}{2\pi}[\ln \tfrac{1}{2}\Delta s - \Gamma'(1) + \tfrac{1}{4}d] + \frac{d-2}{12\pi} \frac{(p \cdot x'')^2}{(x'^2)^3}$$

$$+ \sum_{j=1}^{\nu} \varepsilon_j \left[\frac{d-2}{4\pi} M_j^2 \ln M_j^2 - \frac{d-2}{4} \frac{p \cdot x''}{|x'|^3} M_j + \frac{R}{4\pi} \ln M_j^2 \right], \tag{58}$$

where

$$\Delta s = \int_\sigma^\lambda \mathrm{d}\omega \, |x'(\omega)| \tag{59}$$

is the length of the (shorter) arc between $x(\sigma)$ and $x(\lambda)$.

From general principles, we known that the short-distance singularities in eq. (57) must cancel. Via eq. (52), we thus infer the validity of the short-distance expansion (7). Defining a finite part

$$\mathrm{F.P.} \frac{\delta^2 A}{\delta x_\mu(\lambda) \delta x_\mu(\lambda)}$$

$$= \lim_{\sigma \to \lambda} \left\{ \frac{\delta^2 A}{\delta x_\mu(\sigma) \delta x_\mu(\lambda)} + \frac{d-2}{\pi} \frac{|x'(\sigma)|\,|x'(\lambda)|}{(\Delta s)^2} + \frac{R}{2\pi} x'(\lambda)^2 \ln \tfrac{1}{2}\Delta s \right\}, \tag{60}$$

eq. (57) becomes

$$p_\mu \frac{\delta W_{\mathrm{reg}}}{\delta x_\mu} = \mathrm{F.P.} \frac{\delta^2 A}{\delta x_\mu \delta x_\mu} - \frac{1}{2\pi}(\Gamma'(1) - \tfrac{1}{4}d) x'^2 R - \frac{d-2}{12\pi} \left(\frac{p \cdot x''}{x'^2} \right)^2$$

$$- \sum_{j=1}^{\nu} \varepsilon_j \left[\frac{d-2}{4\pi} x'^2 M_j^2 \ln M_j^2 - \frac{d-2}{4} \frac{p \cdot x''}{|x'|} M_j + \frac{x'^2 R}{4\pi} \ln M_j^2 \right].$$

Finally, noting

$$p_\mu \frac{\delta \alpha_1(\Delta^{\mathrm{T}})}{\delta x_\mu} = \frac{d-2}{4\pi} x'^2 ,$$

$$p_\mu \frac{\delta \alpha_{1/2}(\Delta^{\mathrm{T}})}{\delta x_\mu} = \frac{d-2}{8\sqrt{\pi}} \frac{p \cdot x''}{|x'|} ,$$

$$p_\mu \frac{\delta \alpha_0(\Delta^{\mathrm{T}})}{\delta x_\mu} = -\frac{1}{4\pi} x'^2 R ,$$

we see from eq. (36) that the variation $W_{PV}(\Delta^T)$ is finite indeed as $M_j \to \infty$:

$$p_\mu \frac{\delta W_{PV}}{\delta x_\mu} = \text{F.P.}\,\frac{\delta^2 A}{\delta x_\mu \delta x_\mu} - \frac{1}{2\pi}(\Gamma'(1) - \tfrac{1}{4}d)x'^2 R - \frac{d-2}{12\pi}\left(\frac{p \cdot x''}{x'^2}\right)^2 . \tag{61}$$

5.3. A LOCAL RENORMALIZED WAVE EQUATION

Summarizing the results of the preceeding subsections, we see that the WKB amplitude,

$$\psi_0(\mathscr{C}) = C \exp\{\gamma \varkappa(\mathscr{C}) - \tfrac{1}{2} W_{PV}(\Delta^T)\}, \tag{62}$$

satisfies the following renormalized wave equation, which replaces eq. (6):

$$\left\{2p_\mu \frac{\delta}{\delta x_\mu} + \text{F.P.}\,\frac{\delta^2 A}{\delta x_\mu \delta x_\mu} - \frac{d-2}{12\pi}\left(\frac{p \cdot x''}{x'^2}\right)^2 - \frac{1}{2\pi}(\Gamma'(1) - \tfrac{1}{4}d + 4\pi\gamma)x'^2 R\right\}\psi_0 = 0 . \tag{63}$$

The divergent functional laplacian acting on A has thus been replaced by its finite part and two completely new anomalous terms must be added.

Since R and p_μ depend non-locally on the curve $\mathscr{C}$, eq. (63) is not local. However, we can consider it stemming from a *local*, renormalized (up to the second order in $1/M^2$) full wave equation, which replaces the formal eq. (1). To see this, note first of all that when the local operator

$$\hat{p}_\mu(\sigma) = -\frac{1}{M^2}\frac{\delta}{\delta x_\mu(\sigma)} \tag{64}$$

is acting on a wave function $\psi(\mathscr{C})$ of the form (5), we have

$$\hat{p}_\mu(\sigma)\psi(\mathscr{C}) = p_\mu(\sigma)\psi(\mathscr{C})\left(1 + O\left(\frac{1}{M^2}\right)\right). \tag{65}$$

The anomalous terms in eq. (63) can thus be rewritten as local operators acting on the full amplitude $\psi(\mathscr{C})$, provided they are polynomials of $p_\mu, p'_\mu, \ldots$ and local functions of $x'_\mu, x''_\mu, \ldots$. This is obviously the case for $(x'^2)^{-2}(p \cdot x'')^2$ and a little algebra (appendix E) shows that along $\mathscr{C}$

$$R = -\frac{2}{(x'^2)^3}\{x'^2 x''^2 - (x' \cdot x'')^2 - 2(x'' \cdot p)^2 + p^2 p'^2 - (p \cdot p')^2\}. \tag{66}$$

We thus obtain the following reparametrization covariant, local equation:

$$\begin{aligned}\Bigg\{-\lim_{\sigma\to\lambda}\Bigg[&\frac{\delta^2}{\delta x_\mu(\sigma)\delta x_\mu(\lambda)} - M^2\frac{d-2}{\pi}\frac{|x'(\sigma)|\,|x'(\lambda)|}{(\Delta s)^2}\\ &-\frac{M^2}{2\pi}x'^2\hat{R}\ln\tfrac{1}{2}\Delta s\Bigg] + M^4 x'^2 - M^2\frac{d-2}{12\pi}\left(\frac{x''\cdot\hat{p}}{x'^2}\right)^2\\ &-\frac{M^2}{2\pi}(\Gamma'(1) - \tfrac{1}{4}d + 4\pi\gamma)x'^2\hat{R}\Bigg\}\psi(\mathscr{C}) = 0 ,\end{aligned} \tag{67}$$

where $\hat{R}$ is equal to the expression (66) with p replaced by $\hat{p}$.

The renormalized wave equation (67) has no no practical value of course, but it shows that the WKB amplitude (44) does not violate the fundamental requirement that the interactions of the string bits with each other are short ranged (this is certainly one of the basic assumptions to be made, when linking the string theory to a non-abelian gauge theory [1]). We do not expect eq. (67) to be valid to all orders of $1/M^2$. Rather, the full wave equation is likely to contain an infinite number of anomalous terms involving higher powers of $1/M^2$ and $\hat{p}_\mu$* as well as new free parameters like γ. It is conceivable then that upon summing the large M expansion, the wave equation becomes non-local within a range of order $1/M$.

6. Quantum corrections to the quark-antiquark potential

The potential $V(R)$ between two infinitely heavy quarks at a distance R due to the glue string between them can be extracted from $\psi(\mathscr{C})$ by considering a flat rectangular loop $\mathscr{C}$ with side lengths T and R and taking T to infinity:

$$\psi(\mathscr{C}) \sim \exp - TV(R)\,, \qquad (T\to\infty)\,. \tag{68}$$

From the WKB formula we first of all obtain a linear potential M^2R, but there is also a quantum correction to this classical result: we shall show below that

$$V_{\mathrm{WKB}}(R) = M^2R - \frac{d-2}{2}\frac{\pi}{12}R^{-1} + \text{constant}\,. \tag{69}$$

Thus, the quantum fluctuations of the string give rise to an attractive effective Coulomb potential in $3+1$ dimensions. The "coupling constant"

$$\alpha_{\text{string}} = \tfrac{1}{12}\pi = 0.261\ldots \tag{70}$$

is universal: it does not depend on any parameter at all. For dimensional reasons, we expect that eq. (70) is exact, i.e., the corrections of $V(R)$ due to higher orders of the $1/M^2$ expansion decrease more rapidly than R^{-1} for $R\to\infty$. We emphasize that eq. (69) is valid for large R only and is not to be compared with the perturbative quark potential, which is accurate for small R.

We now proceed to prove eq. (69). For a flat loop $\mathscr{C}$, $\varkappa(\mathscr{C})$ is zero, of course, and

$$W_{\mathrm{PV}}(\Delta^{\mathrm{T}}) = (d-2)\,W_{\mathrm{PV}}(-\Delta_\Gamma)\,. \tag{71}$$

Here, Γ has been identified with the region bounded by $\mathscr{C}$ and $\Delta_\Gamma = \partial_a\partial_a$ is the laplacian in Γ with Dirichlet boundary conditions imposed. Although $W_{\mathrm{PV}}(-\Delta_\Gamma)$ is not generally explicitly calculable, one can prove the following (appendix C): let

* Using $p^2 = x'^2$, the expression (66) can be reduced to a polynomial of only second degree in p. However, to verify the reparametrization invariance of the resulting formula, one has to use $p^2 = x'^2$ again. Since this relation is no longer valid, when p is replaced by $\hat{p}$, the alternative $\hat{R}$ would not be a reparametrization-invariant operator.

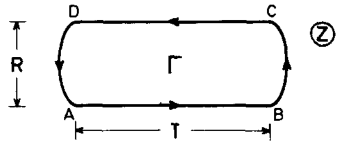

Fig. 1. A rectangular Wilson loop $\mathscr{C}$ with smoothed out corners.

$f(z)$, $z \in \Gamma$, be a holomorphic mapping of Γ onto some other region $\tilde{\Gamma}$. Then,

$$W_{\mathrm{PV}}(-\Delta_\Gamma) = W_{\mathrm{PV}}(-\Delta_{\tilde{\Gamma}}) + \frac{1}{12\pi}\int_{\partial\Gamma} \mathrm{d}\sigma \frac{\epsilon_{ab} z'^a z''^b}{z'^2} \ln |\partial_z f|^2$$

$$+ \frac{1}{12\pi}\int_\Gamma \mathrm{d}^2 z\ \partial_z \ln |\partial_z f|^2 \partial_{\bar{z}} \ln |\partial_z f|^2 , \tag{72}$$

where $z^a(\sigma)$ is any parametrization of $\partial\Gamma$. We apply this formula to the curve $\mathscr{C}$ shown in fig. 1, i.e., to a standard Wilson loop with smoothed out corners at $A = -\frac{1}{2}iR$, $B = T - \frac{1}{2}iR$, $C = T + \frac{1}{2}iR$ and $C = \frac{1}{2}iR$*. It is understood that as $T \to \infty$, the shape of the curve between B and C, and D and A, respectively, is kept fixed. Next, we choose

$$f(z) = \frac{e^{\pi z/R} - 1}{e^{\pi z/R} + 1}, \tag{73}$$

which maps the strip

$$\{z \in \mathbb{C} \mid -\tfrac{1}{2}R < \mathrm{Im}\, z < \tfrac{1}{2}R\}$$

onto the interior of the unit disk. f maps Γ, the region encircled by $\mathscr{C}$, onto a domain $\tilde{\Gamma}$, which is shown in fig. 2.

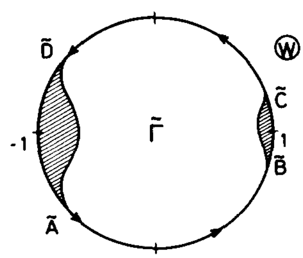

Fig. 2. Image of Γ under the holomorphic mapping (73). When $T \to \infty$, the shaded area near 1 smoothly shrinks to zero.

* A strictly rectangular loop is not appropriate, because corners give rise to additional divergences, which have not been subtracted from $W_{\mathrm{PV}}(-\Delta_\Gamma)$.

To determine the quantum correction to the potential $V(R)$ we have to evaluate $W_{\mathrm{PV}}(-\Delta_\Gamma)$ for $T\to\infty$. In this limit, $\tilde{\Gamma}$ smoothly approaches a bounded region so that, in eq. (72), $W_{\mathrm{PV}}(-\Delta_{\tilde{\Gamma}})$ does not contribute to the quark potential. Noting

$$\partial_z f=\frac{2\pi}{R}\frac{\mathrm{e}^{\pi z/R}}{(\mathrm{e}^{\pi z/R}+1)^2}\to\frac{2\pi}{R}\mathrm{e}^{-\pi z/R},\qquad (\mathrm{Re}\,z\to\infty),$$

we see furthermore that as $T\to\infty$

$$\begin{aligned}\int_{\partial\Gamma}\mathrm{d}\sigma\frac{\epsilon_{ab}z'^a z''^b}{z'^2}\ln|\partial_z f|^2&=-T\frac{2\pi}{R}\int_B^C\mathrm{d}\sigma\frac{\epsilon_{ab}z'^a z''^b}{z'^2}+\mathrm{O}(1)\\&=-T\frac{2\pi^2}{R}+\mathrm{O}(1)\end{aligned}$$

(the integral in the coefficient of T is just the total angle, about which z' rotates, when one runs along $\partial\Gamma$ from B to C, and is therefore independent of the precise shape of the curve). Finally, noting

$$|\partial_z\ln|\partial_z f|^2|^2=\frac{\pi^2}{R^2}|f|^2\to\frac{\pi^2}{R^2},\qquad (\mathrm{Re}\,z\to\infty),$$

we obtain

$$\int_\Gamma\mathrm{d}^2z\;|\partial_z\ln|\partial_z f|^2|^2=\frac{\pi^2}{R}T+\mathrm{O}(1).$$

Altogether, we have

$$W_{\mathrm{PV}}(-\Delta_\Gamma)=-\frac{\pi}{12R}T+\mathrm{O}(1),\tag{74}$$

which proves eq. (69).

7. Conclusions

The fact that the WKB approximation to the string Green functional $\psi(\mathscr{C})$ derived in this paper satisfies a renormalized local wave equation [eq. (67)], which is compatible with reparametrization invariance, is encouraging and feeds our hope that the classical Nambu–Goto string theory can ultimately be quantized without violating fundamental principles. To leading order in $\hbar$, the renormalized wave equation reduces to the formal loop wave equation (1), but in the next to leading order a few local anomalous terms appear. That these are not harmonic, i.e., not quadratic in x_μ and $\hat{p}_\mu$, shows once more that a simple-minded normal ordering subtraction of the formal loop wave operator $L(\sigma)$ cannot work. The appearance of anharmonic quantum corrections to the loop wave equation is not totally surprising from the point of view of the classical string theory either, since the classical phase space is a complicated non-linear manifold. Also, it matches with the observation [14] that random surfaces on a lattice must have some self-interaction, if the associated partition function is to be finite.

The most interesting physical result of our investigation concerns the potential between static quarks due to a relativistic (glue) string connecting them. Namely, we found that the transverse zero-point vibrations of the string produce an effective attractive Coulomb potential, which adds to the classical linear potential [cf. eq. (69)]. The strength of this Coulomb potential turned out to be universal and exactly calculable. In $3+1$ dimensions, we obtained $\alpha_{\text{string}} = 0{\cdot}261\ldots$, a value, which is about what one gets from fitting the charmonium levels with a non-relativistic potential model (Eichten et al. [15], for example, quote 0.52 as their favourite value). A detailed phenomenological discussion is premature, however, since, e.g., we do not yet know what precisely the kinetic energy associated with the collective motion of the string is, but expect it to contribute substantially to the S-P level splitting.

The meaning of the superficial non-renormalizability of the integral representation of the string Green functional $\psi(\mathscr{C})$ [eqs. (8), (12)] is not clear to us, but we can think of various possibilities. It is, of course, conceivable that the quantized string theory cannot live without an ultraviolet cutoff. On the other hand, we cannot exclude the possibility that a renormalizable expression is obtained after summing the large M expansion (or parts of it). This is what happens in the three dimensional non-linear σ-model, where the $1/n$ expansion is renormalizable, but the canonical small coupling expansion is not [16]. This non-perturbative issue can perhaps be studied within the large d (= space-time dimensionality) expansion, to which $\psi(\mathscr{C})$ appears to be accessible. A third possibility to understand the non-renormalizability of the large M expansion is finally suggested by the renormalized WKB loop wave equation (67). Namely, it might turn out that the full renormalized wave equation is effectively non-local with a range of order $1/M$; in other words, that the string fattens upon quantization. It is conceivable then at the infinitely many parameters arising from the non-renormalizability of the theory merely reflect the existence of many different self-consistent *non-local* string models.

One of us (M.L.) would like to thank the staff of the Theoretical Physics Division of the Tel-Aviv University for hospitality and interesting discussions.

Appendix A

A MORE EXPLICIT FORM OF Δ^{T}

The transverse fluctuation operator Δ^{T} acts on modes η_μ orthogonal to the minimal surface Σ described by φ_μ, i.e., modes satisfying

$$\eta_\mu \partial_a \varphi_\mu = 0\,, \qquad a = 1, 2. \tag{A.1}$$

To compute the Seeley coefficients of Δ^{T} and the short-distance behaviour of the massive Green function $G^{\mathrm{T},M}_{\mu\nu}$, we first eliminate the constraint (A.1) by introducing a suitable moving frame along Σ. With respect to the moving frame, Δ^{T} reduces to a second-order elliptic differential operator acting on *unconstrained* wave functions

$$\lambda_i(z)\,, \qquad z\in\Gamma\,, \qquad i=1,\dots,d-2\,, \qquad \lambda_i|_{\partial\Gamma}=0\,. \tag{A.2}$$

Standard perturbative techniques then apply to this form of Δ^{T}.

The moving frame along Σ is built from the vectors

$$\partial_a\varphi_\mu\,, \qquad e^i_\mu\,, \qquad (i=1,\dots,d-2) \tag{A.3}$$

satisfying

$$e^i\cdot e^j=\delta^{ij}\,, \qquad e^i\cdot\partial_a\varphi=0\,. \tag{A.4}$$

Of course, the transversal basis vectors e^i are not unique, but any two choices e^i and $\tilde{e}^i$ are related by a gauge transformation

$$\tilde{e}^i=\Lambda^i_j e^j\,, \qquad \Lambda\in \mathrm{O}(d-2)\,. \tag{A.5}$$

As z varies, the frame moves according to

$$\begin{aligned}\partial_a\partial_b\varphi_\mu&=\Gamma^c_{ab}\,\partial_c\varphi_\mu+V^i_{ab}e^i_\mu\,,\\ \partial_a e^i_\mu&=-A^{ij}_a e^j_\mu-V^i_{ab}g^{bc}\partial_c\varphi_\mu\,,\end{aligned} \tag{A.6}$$

the connection coefficients being

$$\begin{aligned}\Gamma^c_{ab}&=\tfrac{1}{2}g^{cd}\{\partial_a g_{bd}+\partial_b g_{ad}-\partial_d g_{ab}\}\,,\\ V^i_{ab}&=-\partial_a e^i\cdot\partial_b\varphi\,, \qquad A^{ij}_a=e^i\cdot\partial_a e^j\end{aligned} \tag{A.7}$$

[g_{ab} is the metric of Σ, cf. eq. (10)]. Under a gauge transformation (A.5), V^i_{ab} transforms like a vector and A^{ij}_a like a gauge field.

In order that a metric g_{ab} and a set of connection coefficients $V^i_{ab}=V^i_{ba}$ and $A^{ij}_a=-A^{ji}_a$ [g_{ab} determines Γ^c_{ab} via eq. (A.7)] stem from a minimal surface as described above, it is necessary and sufficient that

$$\epsilon_{ab}\{\partial_a\Gamma^d_{bc}-\Gamma^e_{ac}\Gamma^d_{be}+V^i_{ac}V^i_{be}g^{ed}\}=0\,, \tag{A.8}$$

$$\epsilon_{ab}\{\partial_a V^i_{bc}-\Gamma^d_{ac}V^i_{bd}+A^{ij}_aV^j_{bc}\}=0\,, \tag{A.9}$$

$$\epsilon_{ab}\{\partial_a A^{ij}_b+A^{ik}_aA^{kj}_b-V^i_{ac}V^j_{bd}g^{cd}\}=0\,, \tag{A.10}$$

$$\partial_a|g|=0\,, \qquad g^{ab}V^i_{ab}=0\,. \tag{A.11}$$

Namely, the first three relations are the integrability conditions for eq. (A.6), whereas eq. (A.11) guarantees that the surface is minimal.

Any transversal mode η_μ can be represented by

$$\eta_\mu=e^i_\mu\lambda_i\,. \tag{A.12}$$

After some algebra, using eqs. (A.8)–(A.11), one finds

$$\Delta_{\mu\nu}\eta_\nu = e^i_\mu (H\lambda)_i \,, \tag{A.13}$$

where H is given by

$$\begin{aligned} H &= -|g|^{-1/2} D_a |g|^{1/2} g^{ab} D_b + U \,, \\ D^{ij}_a &= \delta^{ij}\partial_a + A^{ij}_a \,, \qquad U^{ij} = |g|^{-1}\epsilon_{ac}\epsilon_{bd} V^i_{ab} V^j_{cd} \,. \end{aligned} \tag{A.14}$$

Obviously, H is an elliptic differential operator, which is gauge covariant and invariant under general coordinate transformations in z-space (provided the fields g_{ab}, A_a and V_{ab} are transformed appropriately). H also represents Δ^{T} in the sense that if

$$\Delta^{\mathrm{T}}\eta = E\eta \,, \qquad Q\eta = 0 \,, \qquad \int_\Gamma \mathrm{d}^2 z \; |g|^{1/2}\eta^2 = 1 \,, \tag{A.15}$$

then

$$H\lambda = E\lambda \,, \qquad \int_\Gamma \mathrm{d}^2 z \; |g|^{1/2}\lambda^2 = 1 \,, \tag{A.16}$$

where η and λ are related by the moving frame

$$\lambda_i = e^i \cdot \eta \,, \qquad \eta_\mu = e^i_\mu \lambda_i \,. \tag{A.17}$$

In particular, the Seeley coefficients of H and Δ^{T} are the same and their Green functions are simply related.

Appendix B

TABLE OF SEELEY COEFFICIENTS

Let A stand for any of the operators H (which is equivalent to Δ^{T}, cf. appendix A), L or $\tilde{L}$. We define its heat kernel by

$$\left(\frac{\partial}{\partial t} + A\right) K_t(z, w) = 0 \,, \qquad \lim_{t\to 0} K_t(z, w) = |g|^{-1/2}\delta(z - w) \tag{B.1}$$

and appropriate boundary conditions along $\partial\Gamma$, i.e., Dirichlet for H and $\tilde{L}$ and Neumann for L. For any test-function f, which is C^∞ in the closure of Γ, one has*

$$\int_\Gamma \mathrm{d}^2 z \; |g|^{1/2} f(z) K_t(z, z) \overset{t\to 0}{=} \sum_{l=0}^{2} t^{-l/2}\psi_{l/2}(f|A) + \mathrm{O}(\sqrt{t}) . \tag{B.2}$$

* For $A = H$, K_t also carries indices $i, j = 1, \ldots, d-2$. Eq. (B.1) is to be read as a matrix equation in this case, and the $\psi_{l/2}$'s are matrices, too.

The Seeley coefficients $\alpha_{l/2}(A)$ are then obtained by

$$\begin{aligned}\alpha_{l/2}(A) &= \psi_{l/2}(f=1|A)\,, \qquad (A=L,\tilde{L})\,,\\ \alpha_{l/2}(H) &= \mathrm{Tr}\;\psi_{l/2}(f=1|H)\,.\end{aligned} \tag{B.3}$$

They are the coefficients appearing in the small t expansion of the trace of the heat operator:

$$\mathrm{Tr}\,\mathrm{e}^{-tA} \overset{t\to 0}{=} \sum_{l=0}^{2} t^{-l/2}\alpha_{l/2}(A) + \mathrm{O}(\sqrt{t})\,. \tag{B.4}$$

Seeley coefficients of various operators have been calculated previously. The case without boundary is discussed in Gilkey's book [9], and the laplacians L and $\tilde{L}$ have been considered, e.g., by Kac [10] and by Stewartson and Waechter [11] for $g_{ab}=\delta_{ab}$, and by Balian and Bloch [12] for general g_{ab}. We computed the Seeley coefficients of Δ^{T}, L and $\tilde{L}$ by first choosing coordinates such that

$$\Gamma = \{z\in\mathbb{R}^2 | 0<z^a<1\}\,, \qquad g_{ab}\propto\delta_{ab}\,,$$

and perturbing about the case with constant metric (and vanishing A_a^{ij}, U^{ij} in the case of H). We do not give any details here, however, but merely quote the results:

$$\psi_1^{ij}(f|H) = \frac{\delta^{ij}}{4\pi}\int_\Gamma \mathrm{d}^2z\,|g|^{1/2}f\,, \tag{B.5}$$

$$\psi_{1/2}^{ij}(f|H) = -\frac{\delta^{ij}}{8\sqrt{\pi}}\int_{\partial\Gamma}\mathrm{d}s\,f\,, \tag{B.6}$$

$$\begin{aligned}\psi_0^{ij}(f|H) &= \frac{\delta^{ij}}{24\pi}\int_{\partial\Gamma}\mathrm{d}s\,n^a\{3\partial_a f - 2(\nabla_s\dot{z}_a)f\}\\ &\quad+\frac{1}{24\pi}\int_\Gamma \mathrm{d}^2z\,|g|^{1/2}\{\delta^{ij}R - 6|g|^{-1}\epsilon_{ac}\epsilon_{bd}V^i_{ab}V^j_{cd}\}f\,.\end{aligned} \tag{B.7}$$

The notation is as follows: $\mathrm{d}s$ is the arc length element (28), n^a denotes the normal vector (27b) and ∇_s the covariant derivative along $\partial\Gamma$:

$$\nabla_s\dot{z}_a = \frac{\mathrm{d}}{\mathrm{d}s}\dot{z}_a - \Gamma^c_{ab}\dot{z}^b\dot{z}_c\,. \tag{B.8}$$

The connection coefficients Γ^c_{ab} and V^i_{ab} are defined in appendix A. Finally, the scalar curvature R is given by

$$\begin{aligned}R &= R^a{}_{bac}g^{bc}\,,\\ R^a{}_{bcd} &= \partial_c\Gamma^a_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} - (c\leftrightarrow d)\,.\end{aligned} \tag{B.9}$$

Using the Gauss-Bonnet formula

$$\tfrac{1}{2}\int_\Gamma \mathrm{d}^2z\,|g|^{1/2}R - \int_{\partial\Gamma}\mathrm{d}s\,n^a\nabla_s\dot{z}_a = 2\pi\,, \tag{B.10}$$

and

$$|g|^{-1}\epsilon_{ac}\epsilon_{bd}V^{i}_{ab}V^{i}_{cd}=R\,, \tag{B.11}$$

which is a consequence of eq. (A.8), we obtain the Seeley coefficients of Δ^{T} (alias H):

$$\alpha_1(\Delta^{\mathrm{T}})=\frac{d-2}{4\pi}A(\mathscr{C})\,, \tag{B.12}$$

$$\alpha_{1/2}(\Delta^{\mathrm{T}})=-\frac{d-2}{8\sqrt{\pi}}L(\mathscr{C})\,, \tag{B.13}$$

$$\alpha_0(\Delta^{\mathrm{T}})=\frac{d-2}{6}-\frac{1}{4\pi}\varkappa(\mathscr{C})\,. \tag{B.14}$$

For L, the laplacian (26) with Neumann boundary conditions (27a), we find

$$\psi_1(f|L)=\frac{1}{4\pi}\int_{\Gamma}\mathrm{d}^2z\;|g|^{1/2}f, \tag{B.15}$$

$$\psi_{1/2}(f|L)=\frac{1}{8\sqrt{\pi}}\int_{\partial\Gamma}\mathrm{d}s\,f, \tag{B.16}$$

$$\psi_0(f|L)=-\frac{1}{24\pi}\int_{\partial\Gamma}\mathrm{d}s\,n^a\{3\partial_a f+2(\nabla_s\dot{z}_a)f\}+\frac{1}{24\pi}\int_{\Gamma}\mathrm{d}^2z\;|g|^{1/2}Rf, \tag{B.17}$$

$$\alpha_1(L)=\frac{1}{4\pi}A(\mathscr{C})\,, \tag{B.18}$$

$$\alpha_{1/2}(L)=\frac{1}{8\sqrt{\pi}}L(\mathscr{C})\,, \tag{B.19}$$

$$\alpha_0(L)=\tfrac{1}{6}\,. \tag{B.20}$$

If Dirichlet boundary conditions are imposed instead, only a few signs change:

$$\psi_1(f|\tilde{L})=\frac{1}{4\pi}\int_{\Gamma}\mathrm{d}^2z\;|g|^{1/2}f, \tag{B.21}$$

$$\psi_{1/2}(f|\tilde{L})=-\frac{1}{8\sqrt{\pi}}\int_{\partial\Gamma}\mathrm{d}s\,f, \tag{B.22}$$

$$\psi_0(f|\tilde{L})=\frac{1}{24\pi}\int_{\partial\Gamma}\mathrm{d}s\,n^a\{3\partial_a f-2(\nabla_s\dot{z}_a)f\}+\frac{1}{24\pi}\int_{\Gamma}\mathrm{d}^2z\;|g|^{1/2}Rf, \tag{B.23}$$

$$\alpha_{l/2}(\tilde{L})=(-1)^l\alpha_{l/2}(L)\,,\qquad (l=0,1,2)\,. \tag{B.24}$$

Appendix C

PROOF OF EQS. (40) AND (72)

Note first that L and $\tilde{L}$ are form invariant under general coordinate transformations in the z-plane, provided, of course, the metric g_{ab} is transformed appropriately. In particular, the eigenvalues of L and $\tilde{L}$ and therefore $W_{\rm PV}(L)$ and $W_{\rm PV}(\tilde{L})$ are geometric invariants, depending on the background minimal surface Σ only, but not on how Σ is parametrized. To prove eq. (40), it is convenient to choose a parametrization of Σ where

$$g_{ab}(z) = \mathrm{e}^{-2\rho(z)}\delta_{ab}\,, \qquad z \in \Gamma\,. \tag{C.1}$$

Then

$$L = -\mathrm{e}^{2\rho}\Delta\,, \tag{C.2}$$

so that $W_{\rm PV}(L)$ and $W_{\rm PV}(\tilde{L})$ are functionals of ρ and Γ. We now vary ρ keeping Γ fixed and compute the variation of $W_{\rm PV}(L)$. To this end, we first derive an expression for the variation of the eigenvalues of L. Thus, let

$$\begin{aligned} &Lv = Ev\,, \qquad \epsilon_{ab}z'^a\partial_b v = 0\,, \qquad (\text{along } \partial\Gamma)\,, \\ &\int_\Gamma \mathrm{d}^2z\, |g|^{1/2}v^2 = 1\,. \end{aligned} \tag{C.3}$$

Taking the variation of the eigenvalue equation, we obtain

$$2\delta\rho E v + L\delta v = \delta E v + E\delta v\,,$$

$$\epsilon_{ab}z'^a\partial_b\delta v = 0\,, \qquad (\text{along } \partial\Gamma)\,.$$

Then, multiplying this with $|g|^{1/2}v$ and integrating yields

$$\delta E - E\int_\Gamma \mathrm{d}^2z\, |g|^{1/2}2\delta\rho v^2 = \int_\Gamma \mathrm{d}^2z\, |g|^{1/2}(vL\delta v - \delta v L v) = 0. \tag{C.4}$$

It follows, in particular, that

$$\delta\, \mathrm{Tr}\, \mathrm{e}^{-tL} = t\frac{\partial}{\partial t}\int_\Gamma \mathrm{d}^2z\, |g(z)|^{1/2}2\delta\rho(z)K_t(z,z)\,, \tag{C.5}$$

where $K_t(z, w)$ denotes the heat kernel (B.1) of L.

As for $\Delta^{\rm T}$ [eq. (37)], $W_{\rm PV}(L)$ can be expressed in terms of the heat kernel, except that the trivial zero mode,

$$v = \text{constant} = A(\mathscr{C})^{-1/2}\,, \tag{C.6}$$

of L must properly be taken into account:

$$W_{\rm PV}(L) = (\alpha_0 - 1)\Gamma'(1) - \int_0^\infty \frac{\mathrm{d}t}{t}\{\mathrm{Tr}\,\mathrm{e}^{-tL} - \alpha_1 t^{-1} - \alpha_{1/2}t^{-1/2} - \alpha_0\theta(1-t) - \theta(t-1)\}\,. \tag{C.7}$$

Now, using eq. (C.5) and $\delta\alpha_0 = 0$ we arrive at

$$\delta W_{\rm PV}(L) = -\lim_{\varepsilon\to 0}\left\{\int_\Gamma \mathrm{d}^2 z\, |g|^{1/2} 2\delta\rho K_t(z,z)\Big|_{t=\varepsilon}^{t=\infty} - \delta\alpha_1\varepsilon^{-1} - \delta\alpha_{1/2} 2\varepsilon^{-1/2}\right\}$$

$$= 2\psi_0(\delta\rho|L) - A(\mathscr{C})^{-1}\int_\Gamma \mathrm{d}^2 z\, |g|^{1/2} 2\delta\rho\,.$$

The last term here comes from the upper end $t=\infty$ of integration, where K_t reduces to the projector onto the zero mode (C.6):

$$\lim_{t\to\infty} K_t(z,z) = A(\mathscr{C})^{-1}\,. \tag{C.8}$$

Since the Seeley coefficient function $\psi_0(f|L)$ is known [eq. (B.17)], the ρ-dependence of $W_{\rm PV}(L)$ is explicitly calculable. For the metric (C.1), we have

$$\begin{gathered} |g|^{1/2} R = 2\Delta\rho\,, \qquad \mathrm{d}s\, n^a = \epsilon_{ab}\,\mathrm{d}z^b\,, \\ \mathrm{d}s\, n^a \nabla_s \dot z_a = -\mathrm{d}\sigma \frac{\epsilon_{ab} z'^a z''^b}{z'^2} - \mathrm{d}z^a\ \epsilon_{ab}\partial_b\rho\,, \end{gathered} \tag{C.9}$$

where $z^a(\sigma)$ is any parametrization of $\partial\Gamma$. Hence,

$$\begin{aligned} \delta W_{\rm PV}(L) &= \frac{1}{6\pi}\int_\Gamma \mathrm{d}^2 z\, \delta\rho\Delta\rho + \frac{1}{4\pi}\int_{\partial\Gamma} \mathrm{d}z^a \epsilon_{ab}\partial_b\delta_\rho \\ &\quad + \frac{1}{6\pi}\int_{\partial\Gamma} \mathrm{d}z^a \epsilon_{ab}\delta\rho\,\partial_b\rho + \frac{1}{6\pi}\int_{\partial\Gamma} \mathrm{d}\sigma \frac{\epsilon_{ab} z'^a z''^b}{z'^2}\delta\rho + A(\mathscr{C})^{-1}\delta A(\mathscr{C}) \\ &= \delta\left\{-\frac{1}{12\pi}\int_\Gamma \mathrm{d}^2 z\, (\partial_a\rho)^2 - \frac{1}{4\pi}\int_\Gamma \mathrm{d}^2 z\,\Delta\rho \right. \\ &\quad \left. + \frac{1}{6\pi}\int_{\partial\Gamma} \mathrm{d}\sigma \frac{\epsilon_{ab} z'^a z''^b}{z'^2}\rho + \ln A(\mathscr{C})\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} W_{\rm PV}(L) &= -\frac{1}{12\pi}\int_\Gamma \mathrm{d}^2 z\, (\partial_a\rho)^2 - \frac{1}{4\pi}\int_\Gamma \mathrm{d}^2 z\,\Delta\rho \\ &\quad + \frac{1}{6\pi}\int_{\partial\Gamma} \mathrm{d}\sigma \frac{\epsilon_{ab} z'^a z''^b}{z'^2}\rho + \ln A(\mathscr{C}) + c(\Gamma)\,, \end{aligned} \tag{C.10}$$

where the constant $c(\Gamma)$ may depend on Γ but not on ρ.

Repeating the above calculation for $\tilde L$, we obtain on the other hand

$$\begin{aligned} W_{\rm PV}(\tilde L) &= -\frac{1}{12\pi}\int_\Gamma \mathrm{d}^2 z\, (\partial_a\rho)^2 + \frac{1}{4\pi}\int_\Gamma \mathrm{d}^2 z\,\Delta\rho \\ &\quad + \frac{1}{6\pi}\int_{\partial\Gamma} \mathrm{d}\sigma \frac{\epsilon_{ab} z'^a z''^b}{z'^2}\rho + \tilde c(\Gamma)\,, \end{aligned} \tag{C.11}$$

so that altogether

$$W_{\mathrm{PV}}(L)-W_{\mathrm{PV}}(\tilde{L})=\ln A(\mathscr{C})-\frac{1}{4\pi}\int_{\Gamma}\mathrm{d}^2z\ |g|^{1/2}R+c(\Gamma)-\tilde{c}(\Gamma)\,. \qquad \text{(C.12)}$$

This is eq. (40) except that the constant $c(\Gamma)-\tilde{c}(\Gamma)$ might still depend on Γ. It cannot, however, since all other terms in eq. (C.12) are geometric invariants and are therefore independent of what parameter region Γ is chosen.

It is now an easy matter to establish eq. (72). Let

$$f:\Gamma\to\tilde{\Gamma}, \qquad f(z)=w\,, \qquad \text{(C.13)}$$

be a holomorphic mapping. Then, $\Delta_{\tilde{\Gamma}}$ can be pulled back to an operator defined on Γ by

$$\Delta_{\tilde{\Gamma}}=4\partial_w\partial_{\bar{w}}=\frac{4}{|\partial_z f|^2}\partial_z\partial_{\bar{z}}=\mathrm{e}^{2\rho}\Delta_{\Gamma}\,. \qquad \text{(C.14)}$$

where

$$\rho=-\tfrac{1}{2}\ln|\partial_z f|^2\,. \qquad \text{(C.15)}$$

We are thus led back to the case considered above and conclude from eq. (C.11) that

$$W_{\mathrm{PV}}(-\Delta_{\tilde{\Gamma}})=W_{\mathrm{PV}}(-\Delta_{\Gamma})+\frac{1}{6\pi}\int_{\partial\Gamma}\mathrm{d}\sigma\frac{\epsilon_{ab}z'^a z''^b}{z'^2}\rho-\frac{1}{12\pi}\int_{\Gamma}\mathrm{d}^2z\ (\partial_a\rho)^2\,, \qquad \text{(C.16)}$$

where $\Delta\rho=0$ has been used and $\tilde{c}(\Gamma)$ has been identified with $W_{\mathrm{PV}}(-\Delta_{\Gamma})$ to match, when $\tilde{\Gamma}=\Gamma$. Inserting eq. (C.15) into eq. (C.16) finally yields eq. (72).

Appendix D

PROOF OF THE SHORT-DISTANCE EXPANSION (58)

We first derived eq. (58) by perturbatively expanding $G^{\mathrm{T},M}_{\mu\nu}$ around the massive Green function of the free laplacian in a square with Dirichlet boundary conditions imposed. This method is foolproof, but the actual computations turned out to be extremely tedious. We therefore resort to another rather indirect derivation, which relies on the invariance properties of Δ^{T}, power counting and some formulae obtained in the preceding appendices.

According to appendix A, Δ^{T} can be identified with the operator H [eq. (A.14)] via the moving frame e^i_{μ}. Correspondingly, the Green function $G^{\mathrm{T},M}_{\mu\nu}$ can be written in the form

$$G^{\mathrm{T},M}_{\mu\nu}(z,w)=e^i_{\mu}(z)\mathscr{G}^M_{ij}(z,w)e^j_{\nu}(w)\,, \qquad \text{(D.1)}$$

where $\mathcal{G}^M_{ij}$ is the massive Green function for H:

$$(H^{ij}+\delta^{ij}M^2)\mathcal{G}^M_{jk}(z,w)=\delta_{ik}|g|^{-1/2}\delta(z-w)\,,$$
$$\mathcal{G}^M_{ij}(z,w)=0\,,\qquad \text{for } z\in\partial\Gamma \qquad \text{or} \qquad w\in\partial\Gamma\,. \tag{D.2}$$

In what follows, we shall consider the fields g_{ab}, A^{ij}_a and U^{ij} to be independent of each other and compute the short-distance expansion of

$$C_{ij}(z,w)=\sum_{k=1}^{\nu}\varepsilon_k n^a(z)\partial_a\mathcal{G}^{M_k}_{ij}(z,w)\overleftarrow{\partial}_b n^b(w)\,,\qquad z,w\in\partial\Gamma\,, \tag{D.3}$$

in the limit where first $z\to w$ and then $M_i\to\infty$. Eq. (58) can then be obtained by contracting with e^i_μ.

We first consider the simplified case $A_a=U=0$. Then,

$$H^{ij}=\delta^{ij}\tilde{L}\,,\qquad \mathcal{G}^M_{ij}=\delta_{ij}\mathcal{G}^M\,,\qquad C_{ij}=\delta_{ij}C\,, \tag{D.4}$$

where $\mathcal{G}^M$ is the massive Green function of $\tilde{L}$. To compute C, we exploit the form invariance of C and $\tilde{L}$ under general coordinate transformations in the z-plane, which allows one to reduce the problem to a situation, where

$$\Gamma=D\overset{\text{def}}{=}\{v\in\mathbb{R}^2|\,|v|<1\}\,,$$
$$g_{ab}=e^{-2\rho}\delta_{ab}\,,\qquad \tilde{L}=-e^{2\rho}\Delta\,. \tag{D.5}$$

Our strategy to calculate C is as follows. Suppose we keep ρ fixed and distort the circle ∂D by an infinitesimal amount $\delta z(\theta)$ as shown in fig. 3. The Pauli–Villars regularized determinant of $\tilde{L}$ then varies according to

$$e^{2\rho}\epsilon_{ab}z'^a(\theta)\frac{\delta W_{\text{reg}}(\tilde{L})}{\delta z^b(\theta)}=\{n^a\partial_a\mathcal{G}(z,w)\overleftarrow{\partial}_b n^b+C(z,w)\}_{z=w=z(\theta)}\,, \tag{D.6}$$

a formula which can be established easily following the method explained in subsect. 5.2. The point now is that all terms in eq. (D.6) except C are in fact known, so that C can be determined this way.

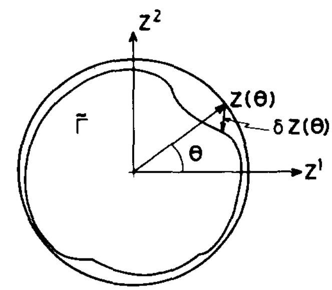

Fig. 3. A deformation $\delta z(\theta)$ of the unit circle ∂D. θ is the polar angle. The curve $\tilde{z}(\theta)=z(\theta)+\delta z(\theta)$ bounds a domain $\tilde{\Gamma}$.

We proceed to explicitly evaluate the various contributions in eq. (D.6). First note that for the special metric (D.5), the massless Green function $\mathcal{G}$ of $\tilde{L}$ is actually equal to the Green function of the ordinary laplacian:

$$\mathcal{G}(z, w) = \frac{1}{2\pi}[\ln|1 - z\bar{w}| - \ln|z - w|], \qquad z = z^1 + iz^2, \qquad \text{etc.} \tag{D.7}$$

It follows that for $z, w \in \partial D$, $z \to w$

$$\begin{aligned} n^a \partial_a \mathcal{G}(z, w)\overleftarrow{\partial}_b n^b = {} & \frac{1}{\pi(\Delta s)^2} + \frac{e^{2\rho}}{12\pi}\{1 + 2\epsilon_{ab} z'^a \partial_b \rho \\ & + 2z'^a z'^b \partial_a \partial_b \rho + (z'^a \partial_a \rho)^2\} + O(\Delta s), \end{aligned} \tag{D.8}$$

where Δs is the length (with respect to the metric g_{ab}) of the arc between z and w and the curly bracket is to be evaluated at $z = w = z(\theta)$.

Next, consider the left-hand side of eq. (D.6). From eq. (36), which is also valid when Δ^{T} is replaced by $\tilde{L}$, and the Seeley coefficients tabulated in appendix B, we obtain

$$\begin{aligned} \epsilon_{ab} z'^a \frac{\delta W_{\mathrm{reg}}(\tilde{L})}{\delta z^b} = {} & \epsilon_{ab} z'^a \frac{\delta W_{\mathrm{PV}}(\tilde{L})}{\delta z^b} \\ & + \frac{e^{-2\rho}}{4\pi} \sum_{k=1}^{\nu} \varepsilon_k M_k^2 \ln M_k^2 + \tfrac{1}{4} e^{-\rho}(1 + \epsilon_{ab} z'^a \partial_b \rho) \sum_{k=1}^{\nu} \varepsilon_k M_k . \end{aligned} \tag{D.9}$$

Eq. (C.11) with $\tilde{c}(\Gamma) = W_{\mathrm{PV}}(-\Delta_\Gamma)$ now applies, yielding

$$\begin{aligned} \epsilon_{ab} z'^a \frac{\delta W_{\mathrm{PV}}(\tilde{L})}{\delta z^b} = {} & \epsilon_{ab} z'^a \frac{\delta W_{\mathrm{PV}}(-\Delta_\Gamma)}{\delta z^b}\bigg|_{\Gamma = D} \\ & + \frac{1}{12\pi}\{(\partial_a \rho)^2 - 3\Delta\rho + 4\epsilon_{ab} z'^a \partial_b \rho + 2z'^a z'^b \partial_a \partial_b \rho\} . \end{aligned} \tag{D.10}$$

Finally, the variation of $W_{\mathrm{PV}}(-\Delta_\Gamma)$ can be computed from eq. (72). Namely, let $\tilde{\Gamma}$ be the region bounded by the deformed circle $z(\theta) + \delta z(\theta)$, $0 \leq \theta \leq 2\pi$ (cf. fig. 3), and let

$$f(z) = z + \delta f(z), \qquad \delta f(e^{i\theta}) - \delta z(\theta) \propto z'(\theta)$$

be a holomorphic mapping from D onto $\tilde{\Gamma}$. Then,

$$\begin{aligned} \delta W_{\mathrm{PV}}(-\Delta_\Gamma)|_{\Gamma = D} &= -\frac{1}{12\pi}\int_0^{2\pi} d\theta\, (\partial_z \delta f + \overline{\partial_z \delta f}) \\ &= \frac{1}{6\pi}\int_0^{2\pi} d\theta\, \epsilon_{ab} z'^a \delta z^b , \end{aligned}$$

and consequently

$$\epsilon_{ab} z'^a \frac{\delta W_{\mathrm{PV}}(-\Delta_\Gamma)}{\delta z^b}\bigg|_{\Gamma=d} = \frac{1}{6\pi}. \tag{D.11}$$

Collecting eqs. (D.8)–(D.11) and feeding them back into eq. (D.6), we obtain the short-distance expansion of $C(z, w)$:

$$\begin{aligned} C(z, w) = &-\frac{1}{\pi(\Delta s)^2} + \frac{e^{2\rho}}{12\pi}\{1 + 2\epsilon_{ab} z'^a \partial_b \rho - 3\Delta\rho + (\epsilon_{ab} z'^a \partial_b \rho)^2\} \\ &+ \frac{1}{4\pi}\sum_{k=1}^{\nu} \varepsilon_k M_k^2 \ln M_k^2 + \tfrac{1}{4} e^{\rho}(1 + \epsilon_{ab} z'^a \partial_b \rho) \sum_{k=1}^{\nu} \varepsilon_k M_k. \end{aligned} \tag{D.12}$$

A generally covariant expression reducing to (D.12) in the special case (D.5) is easily found:

$$\begin{aligned} C(z, w) = &-\frac{1}{\pi(\Delta s)^2} + \frac{1}{12\pi}(n^a \nabla_s \dot{z}_a)^2 - \frac{R}{8\pi} \\ &+ \frac{1}{4\pi}\sum_{k=1}^{\nu} \varepsilon_k M_k^2 \ln M_k^2 - \tfrac{1}{4} n^a \nabla_s \dot{z}_a \sum_{k=1}^{\nu} \varepsilon_k M_k \end{aligned} \tag{D.13}$$

(cf. appendix B for unexplained notation). This relation is thus valid for arbitrary metrics g_{ab} and domains Γ.

Let us now go back to the general case with non-zero fields A_a and U. Again, taking advantage of general covariance, we choose

$$\Gamma = Q \overset{\mathrm{def}}{=} \{v \in \mathbb{R}^2 | 0 < v^a < 1\}, \qquad 0 < w^1 < z^1 < 1, \qquad w^2 = z^2 = 0, \tag{D.14}$$

$$g_{ab} = \delta_{ab}\, e^{-2\sigma}. \tag{D.15}$$

From power counting and gauge invariance we conclude that in the short-distance limit

$$C_{ij}(z, w) = C(z, w) T(z, w)^{ij} + D(z, w) U^{ij}(w) + E(z, w) F^{ij}(w) \tag{D.16}$$

where $C(z, w)$ is given by eq. (D.13) and

$$\begin{aligned} T(z, w) &= \mathrm{P}\exp - \int_{w^1}^{z^1} dv^1\, A_1(v^1, 0), \qquad \text{P: path ordering}, \\ F &= \epsilon_{ab}(\partial_a A_b + A_a A_b). \end{aligned} \tag{D.17}$$

Furthermore, the coefficients D and E are at most logarithmically divergent as $z \to w$ and may depend on $\sigma(w)$ and the regulator masses M_k, but not on A_a, U or the derivatives of σ at w. It is not necessary to compute E, because F^{ij} is antisymmetric in the labels i, j, which ultimately will be contracted. To calculate D, we specialize to the case

$$A_a = 0\,, \qquad U^{ij} = \delta^{ij} u = \text{constant}\,, \qquad \sigma = \text{constant}\,. \tag{D.18}$$

An explicit expression for the massive Green function is then given by

$$\mathscr{G}^M_{ij}(z, w) = \delta_{ij} \int_0^\infty \mathrm{d}t \exp\left[-\mathrm{e}^{-2\sigma}(u + M^2)t\right] K_t^0(z, w)\,, \tag{D.19}$$

K_t^0 being the heat kernel of the laplacian in the square Q with Dirichlet boundary conditions:

$$K_t^0(z, w) = \frac{1}{4\pi t} \prod_{a=1,2} \sum_{m=-\infty}^{\infty} \left\{ \exp\left[-\frac{1}{4t}(z^a - w^a - 2m)^2\right] - \exp\left[-\frac{1}{4t}(z^a + w^a - 2m)^2\right]\right\}. \tag{D.20}$$

It is a trivial exercise to extract from these formulae the short-distance expansion

$$C_{ij}(z, w) = \delta_{ij}\left\{ -\frac{1}{\pi(\Delta s)^2} - \frac{u}{2\pi}\left(\ln \tfrac{1}{2}\Delta s - \Gamma'(1) - \tfrac{1}{2} \sum_{k=1}^{\nu} \varepsilon_k \ln M_k^2\right) + \frac{1}{4\pi} \sum_{k=1}^{\nu} \varepsilon_k M_k^2 \ln M_k^2 \right\}, \tag{D.21}$$

so that

$$D(z, w) = \frac{1}{2\pi}\left(\Gamma(1) - \ln \tfrac{1}{2}\Delta s + \tfrac{1}{2} \sum_{k=1}^{\nu} \varepsilon_k \ln M_k^2\right). \tag{D.22}$$

Finally, we note that for fields A_a and U stemming from a minimal surface as described in appendix A, we have

$$\begin{aligned} e^i_\mu(z) T(z, w)^{ij} e^j_\mu(w) &= d - 2 + \tfrac{1}{4} R (\Delta s)^2 + \mathrm{O}((\Delta s)^3)\,, \\ e^i_\mu U^{ij} e^j_\mu &= U^{ii} = R\,, \end{aligned} \tag{D.23}$$

so that

$$e^i_\mu(z) C_{ij}(z, w) e^j_\mu(w) = (d - 2) C(z, w) + \frac{R}{2\pi}\left(\Gamma'(1) - \tfrac{1}{2} - \ln \tfrac{1}{2}\Delta s + \tfrac{1}{2} \sum_{k=1}^{\nu} \varepsilon_k \ln M_k^2\right). \tag{D.24}$$

This relation is in fact identical with eq. (58), since

$$n^a \nabla_s \dot{z}_a = |x'|^{-3} x'' \cdot p\,, \qquad p_\mu = \frac{\delta A(\mathscr{C})}{\delta x_\mu} \tag{D.25}$$

for minimal surfaces.

Appendix E

PROOF OF EQ. (66)

Both sides of eq. (66) transform as scalars under coordinate transformations in the z-plane so that without loss, we may assume

$$g_{ab} = e^{-2\sigma}\delta_{ab}, \qquad \Gamma = Q = \{v \in \mathbb{R}^2 | 0 < v^a < 1\}.$$

The field φ_μ describing the minimal surface spanned by $\mathscr{C}$ satisfies

$$\Delta\varphi_\mu = 0, \qquad \varphi_\mu|_{\partial\Gamma} = x_\mu$$

in these coordinates. Along the piece of $\partial\Gamma$ with $v^2 = 0$, we have (for a convenient parametrization of $\mathscr{C}$)

$$x_\mu(\lambda) = \varphi_\mu(\lambda, 0),$$

$$p_\mu(\lambda) = -(\partial_2\varphi_\mu)(\lambda, 0).$$

Now,

$$R = 2\, e^{2\sigma}\Delta\sigma = -\frac{1}{x'^2}\Delta \ln(\partial_1\varphi)^2$$

$$= \frac{2}{(x'^2)^3}\{2((x'\cdot x'')^2 + 2(x'\cdot p')^2 - x'^2x''^2 - x'^2p'^2\},$$

which is equivalent to eq. (66) on account of

$$p^2 = x'^2, \qquad x'\cdot p = 0.$$

Appendix F

AN ANOMALOUS LOOP HEAT EQUATION

Recently, Polyakov [17] proposed to construct the string Green functional $\psi(\mathscr{C})$ from a loop heat kernel $\mathscr{K}[z, x]$ depending on a closed curve $z^a(\sigma)$ $(0 \leq \sigma \leq 2\pi)$ in the plane and another one, $x_\mu(\sigma)$, in space-time. $z^a(\sigma)$ is interpreted as the proper time (or, more accurately, the proper coordinates) of the string element $x_\mu(\sigma)$. $\mathscr{K}[z, x]$ is required to be invariant under simultaneous reparametrizations of z^a and x_μ, i.e.,

$$\left\{z'^a\frac{\delta}{\delta z^a} + x'_\mu\frac{\delta}{\delta x_\mu}\right\}\mathscr{K}[z, x] = 0. \tag{F.1}$$

It should also be a solution of the loop heat equation

$$\left\{\epsilon_{ab}z'^a(\sigma)\frac{\delta}{\delta z^b(\sigma)} - \frac{1}{2M^2}L(\sigma)\right\}\mathscr{K}[z, x] = 0, \tag{F.2}$$

so that a solution of the wave equation (1) can then be obtained by

$$\psi(\mathscr{C})=\int \mathscr{D}z\mathscr{K}[z,x]\,. \tag{F.3}$$

As an explicit solution of eqs. (F.1) and (F.2), Polyakov suggests to take

$$\mathscr{K}[z,x]\propto\int_{\phi_\mu(z(\sigma))=x_\mu(\sigma)} \mathscr{D}\phi_\mu \exp\left(-\tfrac{1}{2}M^2\int_\Gamma \mathrm{d}^2z\,(\partial_a\phi_\mu)^2\right), \tag{F.4}$$

where Γ is the region in the plane encircled by $z^a(\sigma)$, $0\leq\sigma\leq 2\pi$. Performing the gaussian integral over ϕ_μ, this can be rewritten as

$$\begin{gathered}\mathscr{K}[z,w]\propto[\det(-\Delta_\Gamma)]^{-d/2}\exp(-\tfrac{1}{2}M^2W)\,,\\ W=-\oint_{\partial\Gamma}\mathrm{d}z^a\,\mathrm{d}w^c\epsilon_{ab}\epsilon_{cd}x_\mu(z)\partial_bG_\Gamma(z,w)\overleftarrow{\partial}_dx_\mu(w)\,.\end{gathered} \tag{F.5}$$

Here, $G_\Gamma(z,w)$ denotes the Green function of the laplacian in Γ with Dirichlet boundary conditions and $\det(-\Delta_\Gamma)$ is the Pauli–Villars finite part* of the determinant of $-\Delta_\Gamma$ [cf. sect. 4 and eq. (72)].

As the loop wave equation itself, the heat equation (F.2) is only formally valid and must be supplemented by a subtraction prescription. Specifically, for the kernel (F.5) one finds

$$\frac{\delta^2\mathscr{K}[z,x]}{\delta x_\mu(\sigma+\varepsilon)\delta x_\mu(\sigma)}\overset{\varepsilon\to 0}{=}\left\{\frac{M^2d}{\pi\varepsilon^2}+\mathrm{O}(1)\right\}\mathscr{K}[z,x]\,. \tag{F.6}$$

Furthermore, as in sect. 5 one shows that $\mathscr{K}[z,x]$ is an exact solution of the following renormalized heat equation, which replaces eq. (F.2):

$$\begin{aligned}&\left\{\epsilon_{ab}z'^a(\sigma)\frac{\delta}{\delta z^b(\sigma)}-\frac{d}{24\pi}\left[2\frac{z'\cdot z'''}{z'^2}+\frac{z''^2}{z'^2}-4\left(\frac{z'\cdot z''}{z'^2}\right)^2\right](\sigma)\right\}\mathscr{K}[z,x]\\ &=\frac{1}{2M^2}\lim_{\varepsilon\to 0}\left[-\frac{\delta^2}{\delta x_\mu(\sigma+\varepsilon)\delta x_m(\sigma)}+\frac{M^2d}{\pi\varepsilon^2}+M^4x'(\sigma)^2\right]\mathscr{K}[z,x]\,.\end{aligned} \tag{F.7}$$

The anomaly appearing here is a function of z^a alone and is crucial to maintain the reparametrization covariance of the equation. On the other hand it sheds some doubt on whether $\psi(\mathscr{C})$ can indeed by constructed from $\mathscr{K}[z,x]$ via eq. (F.3), in particular since the singularity (F.6) is not what one expects for $\psi(\mathscr{C})$.

* The infinite parts do not show any interesting dependence on $\partial\Gamma$, being proportional to the area and circumference of Γ, respectively.

References

[1] Y. Nambu, Phys. Lett. 80B (1979) 372
[2] C. Rebbi, Phys. Reports 12 (1974) 1;
J. Scherk, Rev. Mod. Phys. 47 (1975) 123
[3] T. Eguchi, Phys. Rev. Lett. 44 (1980) 126
[4] M. Lüscher, Phys. Lett. 90B (1980) 277
[5] J. Moser, Trans. Am. Math. Soc. 120 (1965) 286
[6] V.S. Dotsenko and S.N. Vergeles, Renormalizability of phase factors in the non-abelian gauge theory, Landau Institute preprint (1980)
[7] A. Schild, Phys. Rev. D16 (1977) 1722;
Y. Nambu, *in* The quark confinement and field theory, ed. D.R. Stump and D.H. Weingarten (Wiley, New York, 1977)
[8] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and products (Academic Press, New York, 1965)
[9] P.B. Gilkey, The index theorem and the heat equation (Publish or Perish, Boston, 1974)
[10] M. Kac, Amer. Math. Monthly 73 (1966), part II, 1
[11] K. Stewartson and R.T. Waechter, Proc. Cambridge Phil. Soc. 69 (1971) 353
[12] R. Balian and C. Bloch, Ann. of Phys. 64 (1971) 271
[13] H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D12 (1975) 482
[14] D. Weingarten, Phys. Lett. 90B (1980) 280
[15] E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane and T.M. Yan, Phys. Rev. D21 (1980) 203
[16] I. Ya. Aref'eva, E.R. Nissimov and S.J. Pacheva, Comm. Math. Phys. 71 (1980) 213
[17] A.M. Polyakov, Nuc. Phys. B164 (1980) 171
[18] G. 't Hooft, Phys. Rev. D14 (1976) 3432;
B. Berg and M. Lüscher, Comm. Math. Phys. 69 (1979) 57