Nuclear Physics B180 [FS2] (1981) 23–60

HIGH-TEMPERATURE EXPANSIONS FOR THE FREE ENERGY OF VORTICES AND THE STRING TENSION IN LATTICE GAUGE THEORIES

Gernot MÜNSTER[1]

II. Institut für Theoretische Physik der Universität Hamburg, Germany

Received 10 July 1980

We derive high-temperature cluster expansions for the free energy of vortices in SU(2) and Z_2 lattice gauge theories in 3 and 4 dimensions. The expected behaviour of the vortex free energy is verified. It obeys an area law behaviour. The coefficient of the area is shown to be equal to the string tension between static quarks. We calculate its expansion up to 12th order. For SU(2) in 4 dimensions the result is compared with Monte Carlo calculations of Creutz and is in good agreement at strong and intermediate coupling.

1. Introduction and summary of results

Gauge field theories have been formulated on a lattice [1, 2] in order to study the strong coupling region and to apply non-perturbative methods. On the other hand, lattice regularization might be a first step in a constructive approach to continuum gauge field theory. In the last years many results about the phases of lattice gauge theories have been obtained. Certain gauge field configurations, called vortices, turned out to be relevant for the phase structure [3–10]. The behaviour of the free energy of vortices turns out to be an important quantity for the characterization of different phases [8, 9, 11].

In this paper we study the vortex free energy in strong coupling pure lattice gauge theories by the method of cluster expansions. We consider models with gauge groups SU(2) and Z_2 on a hypercubical euclidean lattice Λ in $\nu = 3$ and 4 dimensions. The gauge field variables $U(b) \in SU(2)$ and $\sigma(b) \in Z_2$ are attached to the links b of the lattice. For plaquettes P, with boundary ∂P consisting of four links, one defines

$$U(\dot{P}) = \prod_{b \in \partial P} U(b), \qquad \text{(path-ordered product)},$$

$$\sigma(\dot{P}) = \prod_{b \in \partial P} \sigma(b). \tag{1.1}$$

[1] Supported by Studienstiftung des Deutschen Volkes.

Expectation values are evaluated with the help of the path measure

$$d\mu = Z^{-1} \exp L \prod_b dU(b), \tag{1.2}$$

where L is the euclidean action, dU is the Haar measure on the gauge group and

$$Z = \int \prod_b dU(b) \exp L \tag{1.3}$$

is the partition function.

For the definition of the free energy of vortices per unit extension one takes for the lattice Λ a finite box with specified boundary conditions. Either Λ is considered as a vortex container with some fixed boundary conditions $U(b)$, $b \in \partial\Lambda$ as in ref. [8] or Λ is considered as a torus with periodic boundary conditions for the gauge field as in ref. [9]. By singular gauge transformations characterized by non-trivial elements $\gamma \in \Gamma$ of the center of the gauge group, one changes the boundary conditions such that a quantum of vortex flux is created. In the case of a torus this leads to the twisted boundary conditions of 't Hooft [9]. The corresponding change in the free energy of the system divided by the extension of vortex in the limit of an infinite extension is then called the vortex free energy per unit extension. For a more detailed discussion of this quantity we refer the reader to ref. [11]. There its use for the characterization of the Higgs phase has been investigated.

In this paper we shall be concerned with the behaviour of the vortex free energy at strong coupling, i.e., high temperature in the language of statistical mechanics. There the system is in the confinement phase [12]. Usually the confining phase is characterized by the area law behaviour of the Wilson loop expectation value. Alternatively it may be characterized by the behaviour of the vortex free energy. The vortex free energy per unit extension is supposed to decrease exponentially with the thickness of the vortex container [8, 9].

We shall verify this behaviour in the high-temperature region and thus confirm the above characterization of the confinement phase. We derive high-temperature cluster expansions for the vortex free energy and obtain an exponential decrease with the cross-section area of the container. The coefficient of the area is shown to be equal to the string tension between static quarks. We calculate the expansion for the string tension up to 12th order. For the case of SU(2) in 4 dimensions the result is compared with Monte Carlo computations done by Creutz [13]. The results are in good agreement at strong and intermediate coupling.

The string tension for SU(2) in 3 dimensions has been calculated previously by Duncan and Vaidya [14]. For Z_2 the expansion has been calculated independently by Kimura [15]. In a recent paper he gives the result up to 14th order. For 3 dimensions the series seems to indicate the existence of a surface roughening temperature [33] for the three-dimensional Ising model.

2. Cluster expansions

2.1. LATTICE GAUGE THEORIES AS POLYMER SYSTEMS

Pure lattice gauge theories can be transformed into Gruber-Kunz polymer systems [17] by standard methods [12, 18]. Let the action of a pure lattice gauge theory with gauge group G be

$$L = \sum_{\mathrm{P}} L_{\mathrm{P}} = \sum_{\mathrm{P}} \beta d_\chi^{-1} \chi\big(U(\dot{\mathrm{P}})\big), \tag{2.1}$$

where χ is the real part of the character of a faithful representation of G with dimension $d_\chi = \chi(1)$. The Peter-Weyl theorem enables us to write down a Fourier expansion of $\exp L_{\mathrm{P}}$:

$$\exp L_{\mathrm{P}} = \sum_{\nu \in \hat{\mathrm{G}}} c_\nu(\beta) \chi_\nu\big(U(\dot{\mathrm{P}})\big), \tag{2.2}$$

where $\hat{\mathrm{G}}$ is the set of all non-equivalent irreducible unitary representations of the compact group G. The coefficients c_ν are determined through the orthogonality relations of characters by

$$c_\nu(\beta) = \int_{\mathrm{G}} \mathrm{d}U \chi_\nu(U^{-1}) \exp L_{\mathrm{P}}(U). \tag{2.3}$$

Example 1: Z_2 theory

$$L_{\mathrm{P}} = \beta \sigma(\dot{\mathrm{P}}). \tag{2.4}$$

Z_2 has two inequivalent irreducible representations: the trivial one and the defining representation. We denote the character of the defining representation by $\nu = \frac{1}{2}$, because it is the restriction of the spin-$\frac{1}{2}$ representation of SU(2) to its center Z_2. The Fourier expansion is

$$\exp L_{\mathrm{P}} = \cosh \beta + \sinh \beta \cdot \sigma(\dot{\mathrm{P}}). \tag{2.5}$$

Example 2: SU(2) theory

$$L_{\mathrm{P}} = \tfrac{1}{2} \beta \operatorname{tr} U(\dot{\mathrm{P}}) \tag{2.6}$$

The representations of SU(2) are labelled by $j = 0, \frac{1}{2}, 1, \ldots$. With the help of the Weyl integration formula [19] for class functions on SU(2), one finds

$$\begin{aligned} c_j(\beta) &= (2\pi)^{-1} \int_0^{2\pi} \mathrm{d}\varphi \big(\sin \tfrac{1}{2}\varphi\big)^2 2 \exp\big(\beta \cos \tfrac{1}{2}\varphi\big) \big(\sin \tfrac{1}{2}\varphi\big)^{-1} \sin\big(j + \tfrac{1}{2}\big)\varphi \\ &= 2(2j+1)\beta^{-1} I_{2j+1}(\beta), \end{aligned} \tag{2.7}$$

with modified Bessel functions I_n. It will be convenient to write

$$\exp L_{\mathrm{P}} = c_0(\beta)\Big\{1 + \sum_{j\neq 0} (c_0(\beta))^{-1} c_j(\beta)\chi_j\big(U(\dot{\mathrm{P}})\big)\Big\}$$

$$=: c_0(\beta)\{1 + f_{\mathrm{P}}\}\,. \tag{2.8}$$

Since we are interested in expectation values or ratios of partition functions the factors $c_0(\beta)$ cancel out and we are allowed to write

$$\exp L_{\mathrm{P}} = 1 + f_{\mathrm{P}} = 1 + \sum_{j\neq 0} d_j a_j(\beta)\chi_j\big(U(\dot{\mathrm{P}})\big)\,,$$

$$d_j a_j(\beta) = (c_0(\beta))^{-1} c_j(\beta)\,, \qquad d_j = d_{\chi_j}\,, \tag{2.9}$$

from now on. For Z_2 this reads

$$\exp L_{\mathrm{P}} = 1 + (\tanh\beta)\sigma(\dot{\mathrm{P}})\,, \qquad d_{\frac{1}{2}} = 1\,. \tag{2.10}$$

For SU(2) we have

$$\exp L_{\mathrm{P}} = 1 + \sum_{j\neq 0} (2j+1)(I_1(\beta))^{-1} I_{2j+1}(\beta)\chi_j\big(U(\dot{\mathrm{P}})\big)\,,$$

$$d_j = 2j + 1\,. \tag{2.11}$$

Expanding the product

$$\exp L = \prod_{\mathrm{P}} \exp L_{\mathrm{P}} = \prod_{\mathrm{P}} (1 + f_{\mathrm{P}})\,, \tag{2.12}$$

we get

$$\exp L = \sum_{\mathcal{P}} \prod_{\mathrm{P}\in\mathcal{P}} f_{\mathrm{P}}\,, \tag{2.13}$$

where the sum is over all possible sets $\mathcal{P} = \{\mathrm{P}_1, \mathrm{P}_2, \dots\}$ of plaquettes in the lattice Λ. If one defines plaquettes to be connected if they share a link, one can decompose an arbitrary set $\mathcal{P}$ into connected components X_i:

$$\mathcal{P} = \bigcup_i \mathrm{X}_i\,, \qquad \mathrm{X}_i, \mathrm{X}_j \text{ disconnected if } i \neq j\,. \tag{2.14}$$

We write $\mathcal{P} = \Sigma_i X_i$ in this case. Calculating the partition function

$$Z = \int \prod_{b} dU(b) \exp L = \sum_{\mathcal{P}} \int \prod_{b} dU(b) \prod_{P \in \mathcal{P}} f_P, \tag{2.15}$$

the contributions of the components X_i of $\mathcal{P}$ factorize into products of

$$\Phi(X_i) = \int \prod_{b} dU(b) \prod_{P \in X_i} f_P, \tag{2.16}$$

and the result is

$$Z = \sum_{\mathcal{P} = \Sigma X_i} \prod_{i} \Phi(X_i). \tag{2.17}$$

Since f_P does not contain the trivial representation, $\Phi(X_i)$ is zero if X_i contains a free link in the interior of Λ. A link is called free if it belongs to one plaquette of X_i only. The only contributions come from those connected sets X_i of plaquettes, which have no free links or whose free links are contained in the boundary $\partial\Lambda$ of Λ. Let us call such sets of plaquettes polymers. $\Phi(X_i)$ is the activity of the polymer X_i. Write

$$\Phi(\mathfrak{P}) = \prod_{X \in \mathfrak{P}} \Phi(X) \tag{2.18}$$

if $\mathfrak{P}$ is a set of polymers which are mutually disconnected. The partition function is then

$$Z = \sum_{\mathfrak{P}} \Phi(\mathfrak{P}). \tag{2.19}$$

This is the partition function of a general polymer system on a lattice, as studied by Gruber and Kunz [17].

2.2. CLUSTER EXPANSION FOR GENERAL NON-TRANSLATION-INVARIANT POLYMER SYSTEMS ON ARBITRARY FINITE LATTICES

Usually cluster expansions for the free energy or correlation functions of lattice systems are derived by studying occurrence factors of certain graphs on a lattice and making use of translation invariance. In this case it suffices to keep only contributions proportional to the lattice volume $|\Lambda|$ and by doing this one arrives at the consideration of connected graphs.

In the case of the vortex free energy we are not allowed to follow this line of approach. One reason for this is the fact that the graphs under consideration cover a whole cross section of the lattice and their occurrence factors are not proportional to $|\Lambda|$. The other reason is that in the Mack-Petkova approach [8] one is

interested in systems in a vortex container, which are not translation invariant but depend crucially on the boundary conditions. So we have to use a cluster expansion which is derived without relying on translation invariance. This can be done with the help of moment-cumulant transformations.

2.2.1. Moment-cumulant transformations [20]. Let I be a set, which will be a discrete one in later applications. By $(\alpha,\dots,\beta)_n$ we denote a combination of n elements of I with repetitions. A moment $\langle\ \rangle$ is a mapping which assigns to each combination $(\alpha,\dots,\beta)$ a number $\langle\alpha,\dots,\beta\rangle$ and $\langle\phi\rangle = 0$ for the empty combination. It may be looked at as a collection of symmetric functions of variables in I. A multiplication of moments $\langle\ \rangle'$ and $\langle\ \rangle''$ is defined by

$$\langle\ \rangle = \langle\ \rangle' \bigcirc \langle\ \rangle'',$$

$$\langle\alpha,\dots,\beta\rangle = \sum_{P_2} \langle\alpha,\dots,\gamma\rangle'\langle\delta,\dots,\varepsilon\rangle'', \tag{2.20}$$

where the sum goes over all partitions P_2 of $(\alpha,\dots,\beta)$ into two combinations $(\alpha,\dots,\gamma)$ and $(\delta,\dots,\varepsilon)$. The elements $\alpha,\dots,\beta$ are to be considered as distinguishable even if they occur several times.

Example:

$$\langle\alpha,\beta,\gamma\rangle = \langle\alpha\rangle'\langle\beta,\gamma\rangle'' + \langle\beta\rangle'\langle\alpha,\gamma\rangle'' + \langle\gamma\rangle'\langle\alpha,\beta\rangle''$$
$$+ \langle\alpha,\beta\rangle'\langle\gamma\rangle'' + \langle\beta,\gamma\rangle'\langle\alpha\rangle'' + \langle\alpha,\gamma\rangle'\langle\beta\rangle'',$$

$$\langle\alpha,\alpha,\alpha\rangle = 3\langle\alpha\rangle'\langle\alpha,\alpha\rangle'' + 3\langle\alpha,\alpha\rangle'\langle\alpha\rangle''.$$

The identity 1 of this multiplication is not itself a moment:

$$1(\alpha,\dots,\beta)_n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{2.21}$$

The $\bigcirc$-exponential of a moment [ ] is now defined by

$$\exp_{\bigcirc}[\ ] = 1 + \sum_{n=1}^{\infty} (n!)^{-1}[\ ]^n =: 1 + \langle\ \rangle. \tag{2.22}$$

One finds

$$\langle\alpha,\dots,\zeta\rangle = \sum_{P} [\alpha,\dots,\beta][\gamma,\dots,\delta]\dots[\mu,\dots,\nu], \tag{2.23}$$

where the sum runs over all partitions of $(\alpha,\dots,\zeta)$ into several combinations $(\alpha,\dots,\beta),(\gamma,\dots,\delta),\dots$. $[\ ]$ is called the Ursell function or cumulant of $\langle\ \rangle$.

Examples:

$$\langle\alpha\rangle=[\alpha],$$

$$\langle\alpha,\beta\rangle=[\alpha,\beta]+[\alpha][\beta],$$

$$\langle\alpha,\beta,\gamma\rangle=[\alpha,\beta,\gamma]+[\alpha,\beta][\gamma]+[\beta,\gamma][\alpha]+[\alpha,\gamma][\beta]+[\alpha][\beta][\gamma].$$

The inverse of the cumulant-moment transformation $[\ ]\to\langle\ \rangle$ is given by

$$[\ ]=\ln_{\circ}(1+\langle\ \rangle)\overset{\text{def}}{\Leftrightarrow}\exp_{\circ}[\ ]=1+\langle\ \rangle,$$

$$[\alpha,\dots,\zeta]=\sum_{\mathrm{P}}(-1)^{k-1}(k-1)!\underbrace{\langle\alpha,\dots,\beta\rangle\langle\gamma,\dots,\delta\rangle\dots\langle\mu,\dots,\nu\rangle}_{k\text{ factors}}. \tag{2.24}$$

Now let $\phi_\alpha,\alpha\in\mathrm{I}$ be real or complex variables indexed by elements of I, and define the generating functional of a moment $\langle\ \rangle$ by

$$F(\{\phi_\alpha\})=\sum_{n=1}^{\infty}\sum_{\alpha_1,\dots,\alpha_n}(n!)^{-1}\langle\alpha_1,\dots,\alpha_n\rangle\phi_{\alpha_1}\cdot\dots\cdot\phi_{\alpha_n}$$

$$=\sum_{(\alpha,\dots,\beta)}\left(\prod_i n_i!\right)^{-1}\langle\alpha,\dots,\beta\rangle\phi_\alpha\cdot\dots\cdot\phi_\beta. \tag{2.25}$$

The n_i are the multiplicities of the elements of $(\alpha,\dots,\beta)$. For the cumulant $[\ ]$ of $\langle\ \rangle$ define a generating functional $f(\{\phi_\alpha\})$ in the same way. The main theorem of the moment-cumulant formalism states that

$$1+F(\{\phi_\alpha\})=\exp f(\{\phi_\alpha\}). \tag{2.26}$$

It is proven by differentiating both sides several times with respect to the ϕ_α.

A useful rule for calculating cumulants is the inside-outside rule: replace every moment $\langle\gamma,\dots,\delta\rangle$ in the expansion (2.24) of $[\alpha,\dots,\zeta]$ by $(\langle\gamma,\dots,\delta,\sigma\rangle-\langle\gamma,\dots,\delta\rangle\langle\sigma\rangle)$ and you get the expansion of $[\alpha,\dots,\zeta,\sigma]$ into moments.

2.2.2. Cluster expansions. We now apply the formalism of moment-cumulant transformations to a general polymer system and derive a cluster expansion. For this purpose we choose I to be the set of all polymers X on Λ. We consider the activities $\phi_{\mathrm{X}}:=\Phi(\mathrm{X})$ as variables. With the definition

$$\langle\mathrm{X}_1,\dots,\mathrm{X}_n\rangle=\begin{cases}1, & \text{if every pair } \mathrm{X}_i,\mathrm{X}_j \text{ is not connected},\\ 0, & \text{otherwise},\end{cases}$$

$$\langle\phi\rangle=0, \tag{2.27}$$

we recognize the generating functional of the moment $\langle\ \rangle$ as the partition function of the polymer system:

$$Z(\{\Phi(X)\}) = 1 + \sum_{n=1}^{\infty} \sum_{X_1,\ldots,X_n} (n!)^{-1} \langle X_1,\ldots,X_n\rangle \Phi(X_1)\cdot\ldots\cdot\Phi(X_n). \quad (2.28)$$

The factor of $n!$ is cancelled by taking all permutations of $X_1,\ldots,X_n$ together, which appear separately in $\Sigma_{X_1,\ldots,X_n}$.

The main theorem then leads us to the expansion of $\ln Z$:

$$\ln Z(\{\Phi(X)\}) = \sum_{(X_1,\ldots,X_n)} \left(\prod_i n_i!\right)^{-1} [X_1,\ldots,X_n]\Phi(X_1)\cdot\ldots\cdot\Phi(X_n). \quad (2.29)$$

By induction it is easy to prove with the help of the inside-outside rule that $[X_1,\ldots,X_n] = 0$ unless $X_1 \cup \ldots \cup X_n$ is connected. A connected set of different polymers X_i, $i = 1,\ldots,k$ with multiplicities n_i will be called a cluster and denoted by

$$C = (X_1^{n_1}, X_2^{n_2}, \ldots, X_k^{n_k}). \quad (2.30)$$

We define

$$a(C) = \left(\prod_i n_i!\right)^{-1} \left[\underbrace{X_1,\ldots,X_1}_{n_1}, \underbrace{X_2,\ldots,X_2}_{n_2}, \ldots, \underbrace{X_k,\ldots,X_k}_{n_k}\right] =: \left(\prod_i n_i!\right)^{-1} \hat{a}(C) \quad (2.31)$$

and arrive at the cluster expansion for $\ln Z$ for an arbitrary polymer system:

$$\ln Z = \sum_C a(C) \prod_{X_i \in C} \Phi(X_i)^{n_i}. \quad (2.32)$$

A simplification of the cluster expansion can be obtained as follows. If a polymer X can be decomposed into two polymers X_1 and X_2 which are connected only through a single link, the activity factorizes:

$$\Phi(X) = \Phi(X_1)\Phi(X_2). \quad (2.33)$$

In the expansion of $\ln Z$ the contribution of such a polymer X appears twice. First X is counted as a cluster consisting of a single polymer. Besides that one has a cluster consisting of the polymers X_1 and X_2. Both contributions cancel against each other. This result is generally true for all polymers X, which can be decomposed into polymers X_i, such that each X_i is connected to the remaining part of X

only through a single link. Thus one need not consider clusters of this type which we call reducible.

2.2.3. Relation to the cluster expansion for translation-invariant systems. If a polymer system has translational invariance the cluster expansion for $\ln Z$ can be derived in an alternative way. Because the free energy density $F = (\beta|\Lambda|)^{-1} \ln Z$ is an intensive quantity, we can obtain $\ln Z$ as the term proportional to $|\Lambda|$ in the expansion of

$$Z = \exp(-\beta|\Lambda|F) = 1 - \beta|\Lambda|F + \dots . \tag{2.34}$$

A set $\mathfrak{P}$ of polymers gives a certain contribution $\Phi(\mathfrak{P})$ to the expansion of Z. Let us group together all those sets $\mathfrak{P}'$ which differ from $\mathfrak{P}$ by translations of the polymers. They all give the same contribution $\Phi(\mathfrak{P})$. If $\mathfrak{P} = (X_1, \dots, X_n)$ let us denote the number of related sets $\mathfrak{P}'$ by $g(X_1, \dots, X_n)$. For single polymers we have $g(X) = |\Lambda|$. For two polymers we find the relation

$$g(X_1)g(X_2) = g(X_1, X_2) + b(X_1, X_2), \tag{2.35}$$

where $b(X_1, X_2)$ is the number of ways to put X_1 and X_2 on the lattice (only translations allowed), such that they are connected with each other, i.e., they form a cluster. For the terms of order $|\Lambda|$, which are denoted by a hat, this yields

$$0 = \hat{g}(X_1, X_2) + \hat{b}(X_1, X_2) = \hat{g}(X_1, X_2) + b(X_1, X_2). \tag{2.36}$$

In a similar way one finds for the case of several polymers

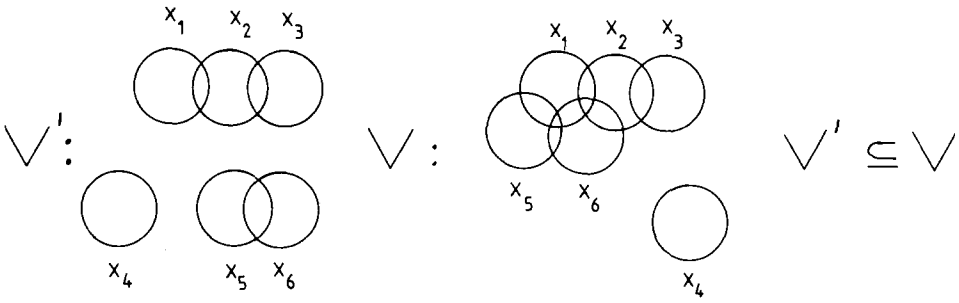
$$0 = \hat{g}(X_1, \dots, X_n) + \text{terms where some polymers are connected}. \tag{2.37}$$

The remaining terms can be reduced step by step to the occurrence numbers of clusters. Carrying through this reduction one finds the following result:

$$\hat{g}(X_1, \dots, X_n) = \sum_{C=(X_1,\dots,X_n)} \hat{a}(C). \tag{2.38}$$

The sum runs over all clusters which one can build out of $X_1, \dots, X_n$ by translating them independently over the lattice. The coefficients $\hat{a}(C)$ are here determined in the following way. We consider the Venn diagrams, i.e., set diagrams of $X_1, \dots, X_n$. They are diagrams which show, how the X_i overlap. All possible Venn diagrams V of $X_1, \dots, X_n$ form a semi-ordered set, where one sets $V' \subseteq V$, if every connected component of V' is completely contained in a component of V and the connectedness conditions within it are preserved.

Example:



Then $\hat{a}(C)$ is given by

$$\hat{a}(C) = \sum_{\mathfrak{K}} (-1)^{|\mathfrak{K}|}. \tag{2.39}$$

The sum goes over all chains $\mathfrak{K}: V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_m$, $|\mathfrak{K}| := m$ where V_0 consists of isolated polymers and V_m is the Venn diagram of the cluster C. It is possible to show that this $\hat{a}(C)$ coincides with the one of subsect. 2.2.2. Inserting the combinatorical factors $n_i!$ for the multiplicities one arrives again at the cluster expansion.

2.2.4. Improvement of the convergence by partial resummation. Even if the lattice Λ is finite, the cluster expansion for $\ln Z$ may diverge because it includes a summation over all multiplicities n_i. It is possible to improve the convergence by doing these summations explicitly. If $C = (X_1^{n_1}, \ldots, X_k^{n_k})$ is a certain cluster, call $S = (X_1, \ldots, X_k)$ its skeleton cluster. For the skeleton clusters define partition functions

$$Z(S) = \sum_{\mathfrak{D} \subseteq S} \prod_{X \in \mathfrak{D}} \Phi(X). \tag{2.40}$$

The sum includes all sets $\mathfrak{D}$ of polymers $X \in S$, which are mutually disconnected. Again the cluster expansion leads to

$$\ln Z(S) = \sum_{C \subseteq S} a(C) \prod_{X \in C} \Phi(X). \tag{2.41}$$

The clusters $C \subseteq S$ are of the form $(X_1^{n_1}, \ldots, X_k^{n_k})$, $X_i \in S$, $n_i \geqslant 0$. Define

$$\Psi(S) = \sum_{C' \subseteq S} a(C') \prod_{X \in C'} \Phi(X), \tag{2.42}$$

where only clusters C′ with $n_i > 1$, all i, are included. This resummation of multiplicities gives us

$$\ln Z = \sum_{S} \Psi(S) . \tag{2.43}$$

Next we express $\Psi(S)$ in terms of logarithms of partition functions of sub-skeleton clusters. The set of skeleton clusters is a partially ordered set through

$$A = (X_1, \ldots, X_k) \subseteq (X_1, \ldots, X_m) = B \Leftrightarrow (X_i \in A \Rightarrow X_i \in B) . \tag{2.44}$$

Through successive application of

$$\Psi(S) = \ln Z(S) - \sum_{A \subsetneq S} \Psi(A) , \tag{2.45}$$

one finds

$$\Psi(S) = \sum_{S' \subseteq S} \sigma_{SS'} \ln Z(S') = \sum_{S' \subseteq S} \sigma_{SS'} \ln \Big(\sum_{\mathfrak{P} \subseteq S'} \prod_{X \in \mathfrak{P}} \Phi(X) \Big) . \tag{2.46}$$

The coefficients $\sigma_{SS'}$ are defined by

$$\sigma_{SS'} = \sum_{\mathcal{K}(S, S')} (-1)^{|\mathcal{K}|} . \tag{2.47}$$

Summation is over all chains $\mathcal{K}(S, S')$: $S' = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_m = S$ and $|\mathcal{K}| = m$. Formula (2.46) is a Möbius inversion [21] of (2.43). In (2.46) the sum over multiplicities is replaced by logarithms explicitly. The convergence problems in (2.32) which are due to the Taylor expansions of functions like $\ln(1 + x)$ are thus eliminated.

3. Cluster expansion for the free energy of vortices

In this section we shall apply the cluster expansion method to study the free energy of vortices in the high-temperature region. We consider pure lattice gauge theories

(i) on a vortex container with boundary conditions U and U_γ as in ref. [11, (3.8)];

(ii) on a torus with periodic and antiperiodic (twisted) boundary conditions as in ref. [11, (4.30)].

As gauge groups we take SU(2) and Z_2. Unless stated otherwise the lattice spacing a is set equal to 1.

3.1. LEADING TERM

We insert the cluster expansions for $\ln Z$ into the following formula (ref. [11], (3.10)) for the vortex free energy in case (i):

$$f_\Lambda(U) = -l^{-1}\left[\ln Z_\Lambda(U_\gamma) - \ln Z_\Lambda(U)\right], \tag{3.1}$$

where $l = d_1$ in $\nu = 3$ and $l = d_1 d_4$ in $\nu = 4$ dimensions is the extension of the vortex and $d_2 d_3$ is the vortex container cross section. We obtain

$$f_\Lambda(U) = -l^{-1}\sum_{\mathrm{C}} a(\mathrm{C})\left\{\prod_{\mathrm{X}_i\in\mathrm{C}} \Phi_\gamma(\mathrm{X}_i)^{n_i} - \prod_{\mathrm{X}_i\in\mathrm{C}} \Phi(\mathrm{X}_i)^{n_i}\right\}. \tag{3.2}$$

The activities of polymers in general depend on the boundary conditions, which is marked by the index γ for boundary conditions U_γ. If the intersection of a polymer X with $\partial\Lambda$ is completely contained in a simply connected part of $\partial\Lambda$, the activities $\Phi(\mathrm{X})$ and $\Phi_\gamma(\mathrm{X})$ are equal. This is due to the fact that U_γ differs from U by a gauge transformation on every simply connected part of $\partial\Lambda$. Therefore a cluster C gives no contribution to $f_\Lambda(U)$ unless it contains at least one polymer, whose intersection with $\partial\Lambda$ winds around Λ in a topologically non-trivial way. The minimal clusters of this type are planes Ξ_{x_1}: $x_1 = \text{const}$ in $\nu = 3$ dimensions and planes Ξ_{x_1,x_4}: $x_1 = \text{const}$, $x_4 = \text{const}$ in $\nu = 4$ dimensions, if the vortex flux is chosen in the x_1, $x_1 - x_4$ direction, respectively, as in (3.1). These clusters contain $|\Xi| = d_2 d_3$ plaquettes and we have

$$\Phi_\gamma(\Xi) \neq \Phi(\Xi). \tag{3.3}$$

The leading term in the expansion comes from these minimal surfaces

$$f_\Lambda(U) = l^{-1}\sum_{x_1,(x_4)}\left[\Phi\left(\Xi_{x_1,(x_4)}\right) - \Phi_\gamma\left(\Xi_{x_1,(x_4)}\right)\right] + \ldots. \tag{3.4}$$

We are interested in the behaviour of the vortex free energy per unit extension

$$f(d_2, d_3) = \lim_{l\to\infty} f_\Lambda \tag{3.5}$$

in the limit of a large cross section $|\Xi|$. For the evaluation of the above activities it is then sufficient to take into account only the term $d_{1/2}a_{1/2}(\beta)\chi_{1/2}(U(\dot{\mathrm{P}}))$ in f_P, because the other terms are suppressed by a factor of

$$(d_j a_j / d_{1/2} a_{1/2})^{|\Xi|}$$

in the activity.

Successive application of the generalized orthogonality relation

$$\int \mathrm{d}U \chi(UV_1)\chi(U^{-1}V_2) = d_\chi^{-1}\chi(V_1V_2), \tag{3.6}$$

allows integration over all variables $U(b)$, $b \in \Xi$, and the result is

$$\Phi(\Xi) = (d_{1/2})^{-|\Xi|+1}(d_{1/2}a_{1/2}(\beta))^{|\Xi|}\chi_{1/2}(U(\partial\Xi)). \tag{3.7}$$

This yields the leading term

$$\begin{aligned} f_\Lambda(U) &= 2\,d_{1/2}(a_{1/2}(\beta))^{|\Xi|}l^{-1}\sum_{x_1,(x_4)}\chi_{1/2}\Big(U\big(\partial\Xi_{x_1,(x_4)}\big)\Big) \\ &= C\exp(-\alpha_0|\Xi|), \end{aligned} \tag{3.8}$$

with

$$\alpha_0 = -\ln(a_{1/2}(\beta)) \tag{3.9}$$

and

$$\begin{aligned} C &= 2d_{1/2}l^{-1}\sum_{x_1,(x_4)}\chi_{1/2}\Big(U\big(\partial\Xi_{x_1,(x_4)}\big)\Big) \leqslant 2d_{1/2}^2, \\ l &= d_1, (d_1d_4). \end{aligned} \tag{3.10}$$

For Z_2 we find

$$\alpha_0 = -\ln\tanh\beta; \tag{3.11}$$

for SU(2) we find

$$\alpha_0 = -\ln(I_2(\beta)/I_1(\beta)). \tag{3.12}$$

In case (ii) the leading contribution comes from the closed planes Ξ_{x_1}, Ξ_{x_1,x_4} which generate non-trivial homology classes on the torus. The activities are calculated as in case (i) and the result is

$$\Phi(\Xi) = (d_{1/2})^{-|\Xi|}(d_{1/2}a_{1/2}(\beta))^{|\Xi|} = a_{1/2}(\beta)^{|\Xi|}, \tag{3.13}$$

$$f_\Lambda = f(d_2, d_3) = 2a_{1/2}(\beta)^{|\Xi|} = 2\exp(-\alpha_0|\Xi|), \tag{3.14}$$

with α_0 as in (3.9).

According to Mack and Petkova [8] and to 't Hooft [9] such a behaviour of the vortex free energy is characteristic for a confinement phase. The leading term of the cluster expansion thus confirms this characterization of the confinement phase which is known to occur at high temperatures β^{-1}.

3.2. COMPUTATION OF CORRECTIONS TO THE COEFFICIENT α

The contribution of further clusters to the free energy of vortices yields corrections to the coefficient α_0, as will be shown in this subsection. For definiteness we consider the case of a torus with periodic and twisted boundary conditions. The results for a vortex container with fixed boundary conditions are similar. We consider the limit of a large cross section $d_2 d_3$. In particular we assume that d_2 and d_3 are larger than the order n of the expansion in β we consider. Therefore to order β^n we are allowed to neglect contributions of order $\beta^{|\Xi|}$, β^{d_2} or β^{d_3}.

As explained in subsect. 3.1 the contributing clusters C each consist of a large polymer Ξ and some additional polymers X_i attached to Ξ. Ξ covers a whole cross section of the lattice like the leading polymer. Clusters containing several large polymers like Ξ are of order $\beta^{|\Xi|}$ and will be neglected in the following. Furthermore, it suffices to take into account only the character $\chi_{1/2}$ on most of Ξ. Higher characters only appear to order $\beta^{|\Xi|}$.

The large polymer Ξ is obtained from some plane $\Xi_0 = \Xi_{x_1,(x_4)}$ by addition of several decorations. One gets a decoration in the following way. Cut out a hole of Ξ_0 by removing a connected set of plaquettes. Take a rigid configuration of plaquettes as decoration and fit it into the hole, such that an admissible large polymer originates. This is illustrated in fig. 1. If we consider some order β^n in the

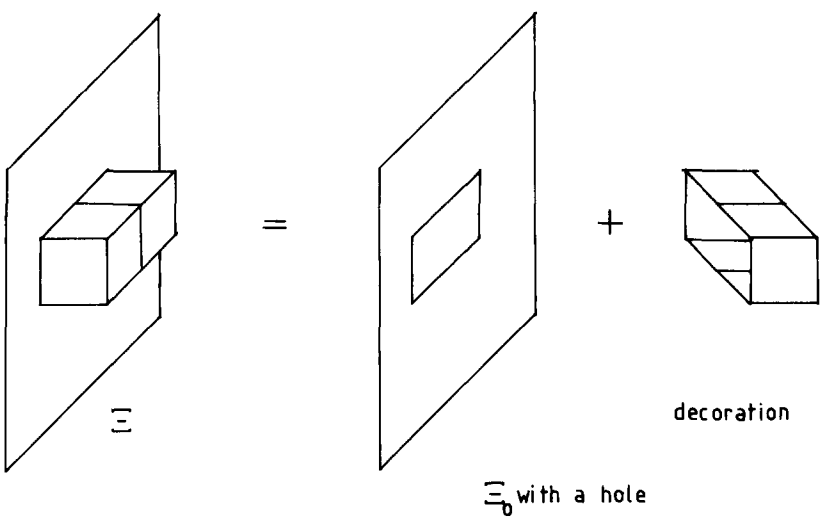


Fig. 1. Decomposition of a large polymer Ξ into a plane Ξ_0 with a hole and a decoration.

expansion, the diameter of Ξ is assumed to be much larger than n (see above). In this case the decorations are much smaller than Ξ itself.

The other polymers X_i of the cluster C are each attached to Ξ. In order to study how the clusters contribute to f_Λ, we shall perform a second moment-cumulant transformation. For this purpose we write down the expansion of f_Λ in terms of two new kinds of polymers X_i, Y_i and take the logarithm. Consider some polymer attached to Ξ. If it does not touch any of the decorations at Ξ, we call it a polymer of type X. If it is connected with some decoration, we regard them as a rigid unit. Polymers of type Y are such decorations with or without some other polymers attached to them. The cluster C under consideration may be viewed at as composed of polymers X_i and Y_i touching a plane Ξ_0. The product of the activities of all polymers in C can be expressed in the form

$$\Phi(\Xi_0)\prod_i \Phi(X_i)\prod_k \Phi(Y_k), \qquad \Phi(\Xi_0)=\exp(-\alpha_0|\Xi|), \tag{3.15}$$

with well-defined activities $\Phi(Y_k)$. In analogy with (2.28) we write

$$f_\Lambda = 2\Phi(\Xi_0)\left\{1+\sum_{m>0}\sum_{W_1,\ldots,W_m}(m!)^{-1}\langle W_1,\ldots,W_m\rangle\prod_{i=1}^{m}\Phi(W_i)\right., \tag{3.16}$$

where W_i is any polymer of type X or Y. The moment function $\langle W_1,\ldots,W_m\rangle$ is defined to be zero unless the W_i form an admissible cluster C, in which case its value is set equal to the combinatorial coefficient a(C). The moment-cumulant transformation yields

$$\ln\left(\tfrac{1}{2}f_\Lambda\right)=\ln\Phi(\Xi_0)+\sum_{(W_1^{n_1},\ldots,W_k^{n_k})}[W_1^{n_1},\ldots,W_k^{n_k}]\prod_i (n_i!)^{-1}\Phi(W_i)^{n_i}. \tag{3.17}$$

The cumulant function [ ] has the property that it is zero unless the W_i form a connected set. We shall call such a rigid object of W_i's a supercluster. Because of translation invariance along the plane Ξ_0 the contribution of each supercluster is multiplied by $|\Xi_0|$. Therefore we get the important result

$$\ln\left(\tfrac{1}{2}f_\Lambda\right)=-\alpha_0|\Xi_0|-\Delta\alpha|\Xi_0|=:-\alpha|\Xi_0|,$$

$$f_\Lambda=f(d_2,d_3)=2\exp(-\alpha|\Xi_0|), \qquad |\Xi_0|=d_2d_3. \tag{3.18}$$

$\Delta\alpha$ is independent of the size of Ξ_0 to the order n of the expansion, which we consider. The largest possible superclusters to order β^n have a diameter smaller than n. They appear in the expansion of $\Delta\alpha$, because the diameter of Ξ_0 is larger than n. On the other hand, in higher orders of the expansion $\Delta\alpha$ depends on the size of Ξ_0. This is due to the fact that superclusters of an extension larger than Ξ_0 cannot contribute then.

In subsect. 3.3 we shall see that the cluster expansion converges for small β. Thus, the supposed limiting behaviour as $|\Xi_0| \to \infty$ of the vortex free energy at high temperatures is confirmed to all orders of β. The limit

$$\alpha_\infty = -\lim_{d_2, d_3 \to \infty} (d_2 d_3)^{-1} \ln f(d_2, d_3) \tag{3.19}$$

exists and we have for finite lattices Λ

$$\alpha = \alpha_\infty + O(\beta^d), \qquad d = \min(d_2, d_3). \tag{3.20}$$

Now that we know that f_Λ exponentiates in the form (3.18), we can go back to the original expansion in terms of clusters. To compute α_∞ we only consider terms proportional to $|\Xi_0|$:

$$f_\Lambda = 2 \exp(-\alpha_0 |\Xi_0|)\{1 - \Delta\alpha \cdot |\Xi_0| + \dots\}. \tag{3.21}$$

This gives us the correction $\Delta\alpha$. As an example take a single polymer Ξ, which is a plane with two separate cubes as decorations. The cubes are not allowed to overlap. See fig. 2. In $\nu = 3$ dimensions the number of possible arrangements is $2|\Xi_0|(|\Xi_0| - 5)/2 + |\Xi_0|(|\Xi_0| - 1)$. The term proportional to $|\Xi_0|$ is $\kappa = -6$. With the activity $\Phi(\Xi) = \Phi(\Xi_0) \cdot a_{1/2}(\beta)^8 =: \Phi(\Xi_0) \cdot \hat{\Phi}(\Xi)$ we get a contribution $-6\, a_{1/2}(\beta)^8$ to $-\Delta\alpha$. In the appendix we have listed all clusters contributing to $\Delta\alpha$ up to order β^{12}. The activities of the polymers and the coefficients κ are given for gauge groups Z_2 and SU(2) in $\nu = 3$ and 4 dimensions. Summing these contributions we get the following results for α_∞ [32].

For gauge group SU(2) we write $u = a_{1/2}, v = a_1, w = a_{3/2}$.

$\nu = 3$ dimensions:

$$\begin{aligned}\alpha_\infty = &-\ln u - 2u^4 + 4u^6 - 6u^4 v - 10u^8 + 4u^{10} + 24u^8 v - 36u^6 v^2 - 6v^5 \\ &- \tfrac{602}{3} u^{12} + 204u^{10} v - 36u^8 v^2 - 24u^6 v^3 - 24u^4 v^4 + 18v^6 \\ &- 8u^9 w - 12v^5 w u^{-1} + \cdots ;\end{aligned} \tag{3.22}$$

$\nu = 4$ dimensions:

$$\begin{aligned}\alpha_\infty = &-\ln u - 4u^4 + 8u^6 - 12u^4 v - 56u^8 + 120u^{10} - 168u^8 v - 72u^6 v^2 - 12v^5 \\ &- \tfrac{4732}{3} u^{12} + 888u^{10} v - 396u^8 v^2 - 48u^6 v^3 - 48u^4 v^4 + 36v^6 \\ &- 48u^9 w - 24v^5 w u^{-1} \dots \\ = &-\ln u - 4u^4 - \tfrac{176}{8} u^8 - \tfrac{10936}{405} u^{10} - \tfrac{1532044}{1215} u^{12} - \cdots .\end{aligned} \tag{3.23}$$

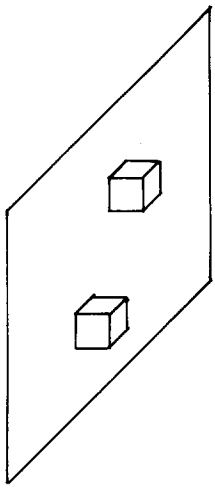


Fig. 2. Polymer Ξ, which is a plane with two separate cubes as decorations. Illustration to the example following (3.21).

For gauge group Z_2 with $x = \tanh\beta$:
$\nu = 3$ dimensions:

$$\alpha_\infty = -\ln x - 2x^4 - 2x^6 - 10x^8 - 16x^{10} - \tfrac{242}{3}x^{12} - \cdots ; \tag{3.24}$$

$\nu = 4$ dimensions:

$$\alpha_\infty = -\ln x - 4x^4 - 4x^6 - 56x^8 - 144x^{10} - \tfrac{3616}{3}x^{12} - \cdots . \tag{3.25}$$

We shall discuss our results in later subsections.

3.3. EQUALITY OF α AND THE STRING TENSION

With the help of cluster expansions, Osterwalder and Seiler [12] proved that the Wilson criterion for confinement of static quarks in pure lattice gauge theories is fulfilled at high temperatures. They proved the convergence of the cluster expansion for

$$\mathfrak{W}(\mathcal{C}) = \langle \chi_{1/2}(U(\mathcal{C})) \rangle , \qquad \mathcal{C} = \partial\Xi_0, \text{ a closed loop}, \tag{3.26}$$

at high temperatures and proved the area law

$$\mathfrak{W}(\mathcal{C}) \leqslant K_1 \exp(-K_2|\Xi_0|) \tag{3.27}$$

for suitable constants $K_1, K_2 > 0$.

In this subsection we study the high-temperature expansion of $\mathfrak{W}(\mathcal{C})$ for $\mathcal{C}$ a rectangular path parallel to the x_2, x_3 axes. We shall demonstrate that

$$\ln \mathfrak{W}(\mathcal{C}) = -\alpha' |\Xi_0| + \tau |\partial \Xi_0| + c, \qquad \alpha' > 0. \tag{3.28}$$

To order β^n of the expansion α', τ and c are independent of the size of Ξ_0, if the diameter of Ξ_0 is larger than n. α' is the string tension. It gives the slope of the linear potential $V(r) = \alpha' \cdot r + \ldots$ between far separated static quarks. We shall see that α' is equal to the coefficient α which appeared in the area law for the vortex free energy. For simplicity we consider a plane Ξ_0 of size $d_2 \times d_3$ parallel to the x_2-x_3 plane. We write all formulae for gauge group SU(2). The analysis is valid also for other groups. We have

$$\begin{aligned} \langle \chi_{1/2}(U(\mathcal{C})) \rangle &= Z^{-1} \int \prod_{\mathrm{b}} \mathrm{d}U(\mathrm{b}) \chi_{1/2}(U(\mathcal{C})) \exp L \\ &=: Z^{-1} Z(\mathcal{C}). \end{aligned} \tag{3.29}$$

$Z(\mathcal{C})$ can be expanded like Z in terms of polymers. See sect. 2 and ref. [12]. There are two types of polymers, namely closed polymers as in the expansion of Z and polymers Ξ which possess $\partial \Xi_0$ as their boundary. Each graph contributing to $Z(\mathcal{C})$ consists of one polymer Ξ with boundary $\mathcal{C} = \partial \Xi_0$ and several closed polymers. All polymers are mutually disconnected.

Now we define two new types of polymers X_i, Y_i in a similar way as in subsect. 3.2. Polymers Y_i consist of

(i) a decoration fitted into some hole in Ξ_0;

(ii) closed polymers which do not touch the decoration (i), but touch Ξ_0 such that the intersection is contained completely in the hole to which the decoration is attached. For an example see fig. 3. Polymers X_i are usual closed polymers. The graphs in the expansion of $Z(\mathcal{C})$ can be thought of as built of polymers X_i and Y_i, such that they are mutually disconnected and the X_i do not touch Ξ_0. The expansion is of the form

$$Z(\mathcal{C}) = d_{1/2} \Phi(\Xi_0) \left\{ 1 + \sum_{m>0} \sum_{\mathrm{W}_1, \ldots, \mathrm{W}_m} (m!)^{-1} h(\mathrm{W}_1, \ldots, \mathrm{W}_m) \prod_{i=1}^{m} \Phi(\mathrm{W}_i) \right\}, \tag{3.30}$$

where W_i is any polymer of type X or Y and $\Phi(\mathrm{W}_i)$ are well-defined activities. The function h equals 1, if the W_i form an admissible graph, otherwise it is zero. On the other hand we have as in sect. 2

$$Z = 1 + \sum_{m>0} \sum_{\mathrm{W}_1, \ldots, \mathrm{W}_m} (m!)^{-1} e(\mathrm{W}_1, \ldots, \mathrm{W}_m) \prod_{i=1}^{m} \Phi(\mathrm{W}_i), \tag{3.31}$$

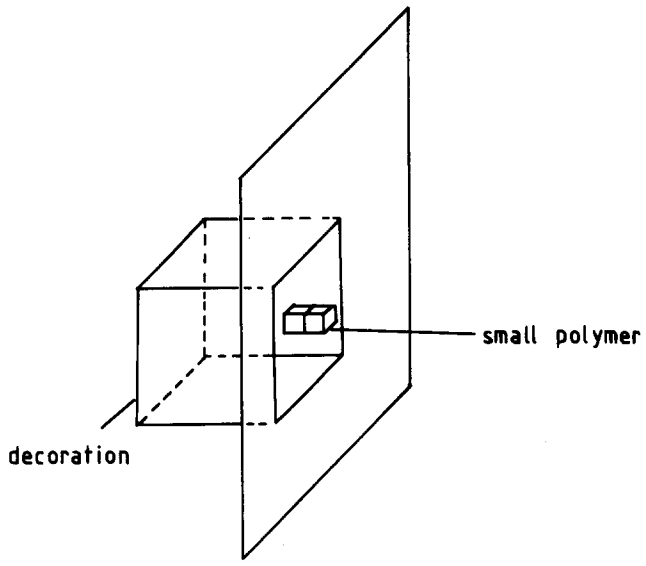


Fig. 3. Polymer of type Y in the cluster expansion for the Wilson loop expectation value. See the discussion before (3.30).

where the function e equals 1, if all W_i are of type X and are mutually disconnected, otherwise $e = 0$. With the help of the moment-cumulant transformation we get for the logarithms

$$\ln\bigl(Z(\mathcal{C})/d_{1/2}\Phi(\Xi_0)\bigr) = \sum_{(W_1^{n_1},\ldots,W_k^{n_k})} \hat{h}(W_1^{n_1},\ldots,W_k^{n_k}) \prod_{i=1}^{k} (n_i!)^{-1}\Phi(W_i)^{n_i}, \tag{3.32}$$

$$\ln Z = \sum_{(W_1^{n_1},\ldots,W_k^{n_k})} \hat{e}(W_1^{n_1},\ldots,W_k^{n_k}) \prod_{i=1}^{k} (n_i!)^{-1}\Phi(W_i)^{n_i}. \tag{3.33}$$

The hat denotes the corresponding cumulants. This yields

$$\ln\bigl(Z(\mathcal{C})/d_{1/2}\Phi(\Xi_0)Z\bigr)$$

$$= \sum_{(X_i^{n_i})} \prod_i (n_i!)^{-1}\Phi(X_i)^{n_i}\Bigl\{ -\hat{e}(X_i^{n_i}) + \sum_{(Y_k^{n_k})} \hat{h}(X_i^{n_i}, Y_k^{n_k}) \prod_k (n_k!)^{-1}\Phi(Y_k)^{n_k}\Bigr\}. \tag{3.34}$$

$\hat{h}$ has the property of being zero unless the occurring polymers form a connected configuration. Furthermore, $\hat{h}(X_i^{n_i}) = \hat{e}(X_i^{n_i})$ if the polymers X_i are not connected with Ξ_0. This is due to the fact that in this case the moments h and e coincide for subsets of $\{X_i^{n_i}\}$. Therefore, we end up with an expansion in terms of graphs which consist of connected objects attached to Ξ_0. If we had perfect translation invariance we would get the result that $\ln \mathfrak{W}(\mathcal{C})$ is proportional to $|\Xi_0|$. But in our case the graphs are not allowed to move freely near the boundary of Ξ_0. We have to subtract the contribution of graphs which are forbidden by this boundary effect. On the other hand there are graphs which only exist at the boundary lines or corners of Ξ_0. These effects lead to terms proportional to $|\partial\Xi_0|$ and to constant terms. In this way we get the result (3.28). α', τ and c do not depend on the size of Ξ_0 to order β^n of the expansion, if n is smaller than the diameter of Ξ_0. Compare the discussion in subsect. 3.2.

Now consider a lattice Λ with periodic boundary conditions and let Ξ_0 be a cross section $\Xi_{x_1,(x_4)}$ of Λ as in subsect. 3.2. The expansion of $\ln \mathfrak{W}(\mathcal{C})$ is as above, except that now the boundary effects are different. But the coefficient α' of the area $|\Xi_0|$ is seen to be the same. In the range of small β where the cluster expansion converges the limit

$$\alpha'_\infty = -\lim_{|\Xi_0|\to\infty} |\Xi_0|^{-1} \ln \mathfrak{W}(\mathcal{C}), \tag{3.35}$$

which is the string tension, exists.

Our next aim is to show that the coefficient α in the area law for the vortex free energy is equal to the string tension α', at least in the range of validity of the expansion. The partition functions $Z_\Lambda =: Z_+$ and $Z_{\Lambda,\gamma} =: Z_-$ with periodic and twisted boundary conditions respectively, can be represented as

$$Z_+ = Z_1 + Z_2, \qquad Z_- = Z_1 - Z_2,$$

with

$$Z_1 = \sum_{\mathfrak{D}_1} \prod_{X\in\mathfrak{D}_1} \Phi(X), \qquad Z_2 = \sum_{\mathfrak{D}_2} \prod_{X\in\mathfrak{D}_2} \Phi(X). \tag{3.36}$$

$\mathfrak{D}_1$ is any set of mutually disconnected polymers X, such that $\prod_{X\in\mathfrak{D}_1}\Phi(X)$ is the same for periodic and twisted boundary conditions, while in the case of $\mathfrak{D}_2$ the corresponding expression changes its sign by a change of the boundary conditions*. Z_1 is of order 1, while Z_2 is of order $\beta^{|\Xi_0|}$, as we have seen earlier. We get

$$\begin{aligned} \ln(Z_-/Z_+) &= \ln((1 - Z_2/Z_1)/(1 + Z_2/Z_1)) \\ &= -2(Z_2/Z_1)(1 + O(\beta^{|\Xi_0|})) \\ &= -2(Z_2/Z_+)(1 + O(\beta^{|\Xi_0|})). \end{aligned} \tag{3.37}$$

* Here we consider polymers with a definite assignment of characters to their plaquettes.

Corrections of order $\beta^{|\Xi_0|}$ will be neglected from now on. Then Z_2 is of the form

$$Z_2 = \sum_{x_1,(x_4)} \sum_i \Phi\left(\Xi^{(i)}_{x_1,(x_4)}\right) Z\left(\Lambda \setminus \bar{\Xi}^{(i)}_{x_1,(x_4)}\right). \tag{3.38}$$

$\Xi^{(i)}_{x_1,(x_4)}$ are large polymers which descend from $\Xi_{x_1,(x_4)}$ by decorations. $Z(\Lambda \setminus \bar{\Xi}^{(i)}_{x_1,(x_4)})$ is the partition function which one gets by taking into account only those polymers that do not touch $\Xi^{(i)}_{x_1,(x_4)}$. This yields, with $l = d_1, (d_1 d_4)$,

$$f_\Lambda = l^{-1} \sum_{x_1,(x_4)} \sum_i 2\Phi\left(\Xi^{(i)}_{x_1,(x_4)}\right) Z\left(\Lambda \setminus \bar{\Xi}^{(i)}_{x_1,(x_4)}\right) Z_+^{-1}. \tag{3.39}$$

Translation invariance in x_1, x_4 leads to

$$f_\Lambda = 2 \sum_i \Phi(\Xi^{(i)}) Z(\Lambda \setminus \bar{\Xi}^{(i)}) Z_+^{-1}. \tag{3.40}$$

Here the large polymers $\Xi^{(i)}$ descend from some fixed plane Ξ_0. Up to boundary effects (see the earlier discussion) this expression is equal to the expansion of $\mathcal{W}(\partial \Xi_0)$ as given by Osterwalder and Seiler [12]. From this we conclude that the coefficient α in the area law (3.18) for f_Λ is equal to the string tension α' to order β^n of the high-temperature expansion, as long as n is smaller than the diameter of Ξ_0. The proof of the convergence of the cluster expansion for $\mathcal{W}(\partial \Xi_0)$ [12] applies in the same way for f_Λ in (3.40). So we get as final result

$$\alpha_\infty = \alpha'_\infty \tag{3.41}$$

within the range of convergence of the high temperature expansion. The string tension of SU(2) lattice gauge theory in $\nu = 3$ dimensions has been calculated previously by Duncan and Vaidya [14] up to order β^{14}. Our results are in agreement with theirs.

3.4. DISCUSSION OF RESULTS

As we discussed in ref. [11] the free energy of vortices per unit extension might serve as a convenient quantity for the characterization of different phases in lattice gauge theories. Usually the confinement phase is defined by an area law behaviour of the Wilson loop expectation value. On the other hand, according to Mack and Petkova [8] and to 't Hooft [9] the confinement phase of pure lattice gauge theories is distinguishable by the fact that the free energy of vortices per unit extension falls off exponentially with the thickness of the vortex container. The high-temperature expansion, which we performed in this work, confirms this criterion. In the high-temperature region, where confinement is known to occur [12], the vortex free energy indeed falls off exponentially with the cross section of the container.

Furthermore, we obtained the result that in the high-temperature expansion the coefficient α in this area law is equal to the string tension up to small corrections which vanish in the limit of an infinite cross section. Such a relation was predicted by 't Hooft [9] on the basis of more general arguments.

The numerical results for the expansion of α will be discussed in the two following subsections.

3.4.1. String tension of SU(2) lattice gauge theory in $\nu = 4$ dimensions. Here the lattice spacing a is not set equal to 1. As we mentioned earlier the string tension of SU(2) lattice gauge theory in $\nu = 3$ dimensions has been calculated previously up to order β^{14} in the high-temperature expansion by Duncan and Vaidya [14]. They discuss the series extensively and we shall not elaborate on this further.

More interesting is the case of SU(2) in $\nu = 4$ dimensions. This is the number of space-time dimensions of the physical world and it is believed that the SU(2) gauge theory reflects the general properties of a gluon field theory with gauge group SU(3). There are some indications that $\nu = 4$ is the critical dimension for non-abelian lattice gauge theories [22, 13]. This means that the strong coupling confinement phase extends down to zero coupling. The point $g = 0$ would then be a critical point of the theory.

One would like to know if the theory possesses a continuum limit with persisting confinement property. Due to asymptotic freedom [23] the continuum limit of SU(2) or SU(3) lattice gauge theories is supposed to be related to the weak coupling behaviour [24, 10]. It is expected to be a limit, where the lattice spacing a and at the same time the bare coupling g go to zero in a way determined by renormalization group equations. If asymptotic freedom in the form predicted by perturbation theory holds in the physical theory, and confinement persists in the continuum limit where the bare coupling parameter $\beta \to \infty$, then the string tension should behave like [13]

$$\alpha \approx a^{-2} \exp -\frac{6\pi^2}{11}(\beta - \tilde{\beta}), \tag{3.42}$$

at large β, where $\tilde{\beta}$ is some constant.

The connection between strong and weak coupling regions was studied by several authors [25, 13, 14, 26, 27]. Of special interest is the intermediate coupling region where a transition from strong coupling to weak coupling behaviour of, e.g., the string tension should take place. Monte Carlo calculations for SU(2) in $\nu = 4$ dimensions [13] indicate a rather sharp changeover. In Mack's theory of quark confinement [27], which uses the assumption that confinement of static quarks is due to condensation of vortices [3–9], the transition signals the appearance of a non-empty range of distances $a < d < d_c(\beta)$ where perturbation theory applies. For values of β above it vortices need a certain thickness $d_c(\beta) > a$ in order to condense.

In fig. 4 we have plotted $a^2\alpha$ according to (3.23) and the points given in ref. [13]. The high-temperature series is in very good agreement with the Monte Carlo data in the intermediate coupling region between $\beta = 1.6$ and $\beta = 2.25$. At $\beta = 2.25$ the curve matches nicely the weak coupling fit of Creutz. Above $\beta = 2.3$ the data of Creutz follow the weak coupling asymptotic freedom behaviour (3.42) with $\tilde{\beta} \approx 2$. In the weak coupling region α becomes exponentially small compared to the strong coupling region. There the correct behaviour is not reproduced by the high-temperature series (3.23) which in fact goes through zero near $\beta = 2.4$.

In conclusion we see that high-temperature expansions yield accurate values of the string tension at strong and intermediate coupling and their prediction joins smoothly to the expected behaviour at weak coupling. Furthermore, fitting the weak coupling curve to the high-temperature series as in fig. 4 allows an estimate of $\tilde{\beta} = 2$. This constant is important because it relates the string tension to a renormalization scale in the following way. If one uses a renormalization scheme, where the string tension α is held fixed while the lattice spacing a and the coupling g go to zero, one gets

$$g^2 \sim -\frac{12\pi^2}{11 \ln am}, \qquad \text{for small } a. \tag{3.43}$$

The constant mass m is a renormalization scale. It is related to the string tension by

$$m = \alpha^{1/2} \exp -\tfrac{3}{11}\pi^2\tilde{\beta} \approx 4.6 \cdot 10^{-3}\alpha^{1/2}. \tag{3.44}$$

In a recent paper A. and P. Hasenfratz [28] discuss the relation of m to the renormalization scale parameter Λ of perturbative gauge theory.

The question of an extrapolation of the cluster expansion series might arise. We would like to add some remarks on this point. First one should notice that the series (3.22)–(3.25) are not necessarily diverging in the intermediate coupling region. The fact that the Wilson loop expectation value on a finite lattice is represented by a convergent expansion supports this point of view. The series (3.22)–(3.25) are not power series in β. Instead by using as expansion parameters the Fourier coefficients $0 \leqslant a_j < 1$ one is in effect doing a partial resummation of the usual high-temperature series in β. These coefficients are much smaller than β in the intermediate coupling region. The series has a finite convergence region, but it is not possible to extract its extension from the first few terms. Second the application of extrapolation methods (Padé, Borel, etc.) requires some properties which are not known to be true in our case. Furthermore knowledge of the asymptotic behaviour of the numerical coefficients is necessary. Without this information any use of extrapolation methods is not justified*.

A partial resummation by Möbius inversion (see subsect. 2.2.4.) has been performed, but gives only very small changes.

* I thank J. Zinn-Justin for a remark on this point.

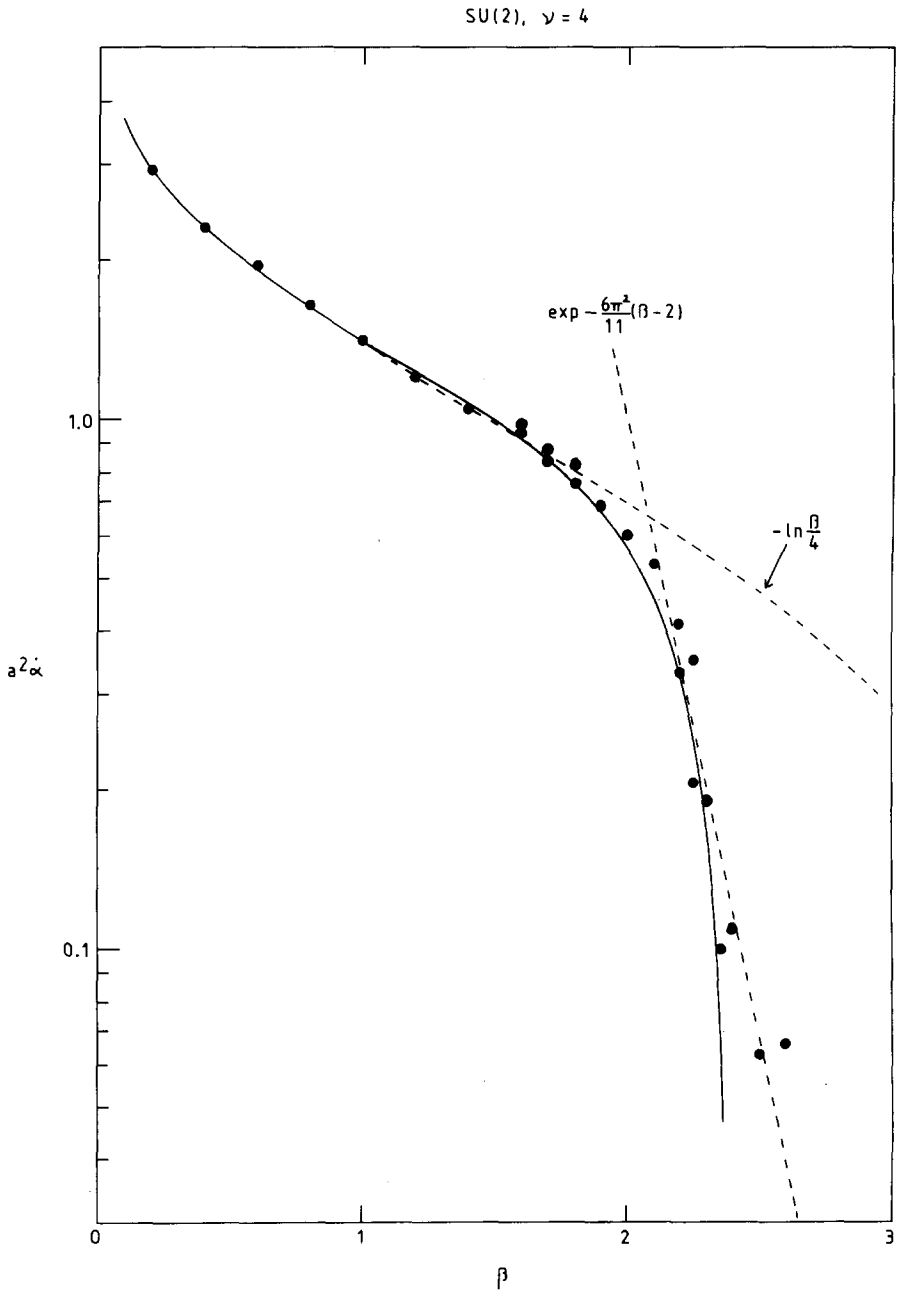


Fig. 4. The string tension α times the lattice spacing squared as a function of $\beta = 4/g^2$ for SU(2) lattice gauge theory in $\nu = 4$ dimensions. The solid line represents the result of cluster expansions up to 12th order according to (3.23). The dots are results of a Monte Carlo calculation of Creutz [13]. The dashed lines are the lowest-order high-temperature curve and the weak coupling fit of ref. [13].

In order to illustrate the contributions of the different orders in the expansion of $a^2\alpha$ we plotted the series up to orders 8, 10 and 12 in fig. 5.

3.4.2 String tension of Z_2 lattice gauge theory. When this paper was in preparation we received a preprint by Kimura [15], who calculated the high-temperature series for the string tension of Z_2 lattice gauge theory in $\nu = 3$, 4 and 5 dimensions up to order x^{14} in $x = \tanh\beta$. The 14th order terms are $-114x^{14}$ in $\nu = 3$ and $-4196x^{14}$ in $\nu = 4$ dimensions. The results are discussed extensively in ref. [15] and we shall add a few remarks only.

The Z_2 lattice gauge theory is known to undergo a phase transition at $\beta_c = 0.76$ in $\nu = 3$ and $\beta_c = \frac{1}{2}\ln(1 + \sqrt{2}) \approx 0.44$ in $\nu = 4$ dimensions [29]. For small values of β it is in the confinement phase. At large values of β it is in a Higgs phase, where the string tension vanishes. The phase transition at β_c is of second order in $\nu = 3$ [29] and is supposed to be of first order in $\nu = 4$ dimensions [30]. Therefore the

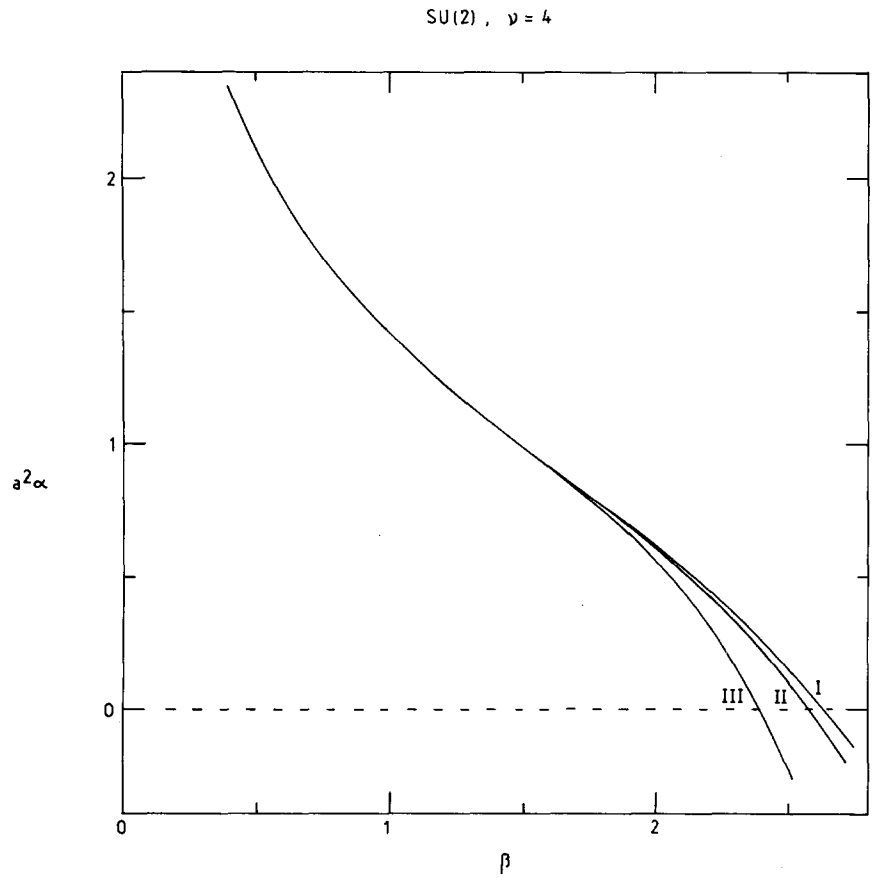


Fig. 5. Results of cluster expansions up to 8th (curve I), 10th (curve II) and 12th order (curve III) for $a^2\alpha$ for SU(2) lattice gauge theory in $\nu = 4$ dimensions. Plotted are the corresponding polynomials in u, v and w. See (3.23).

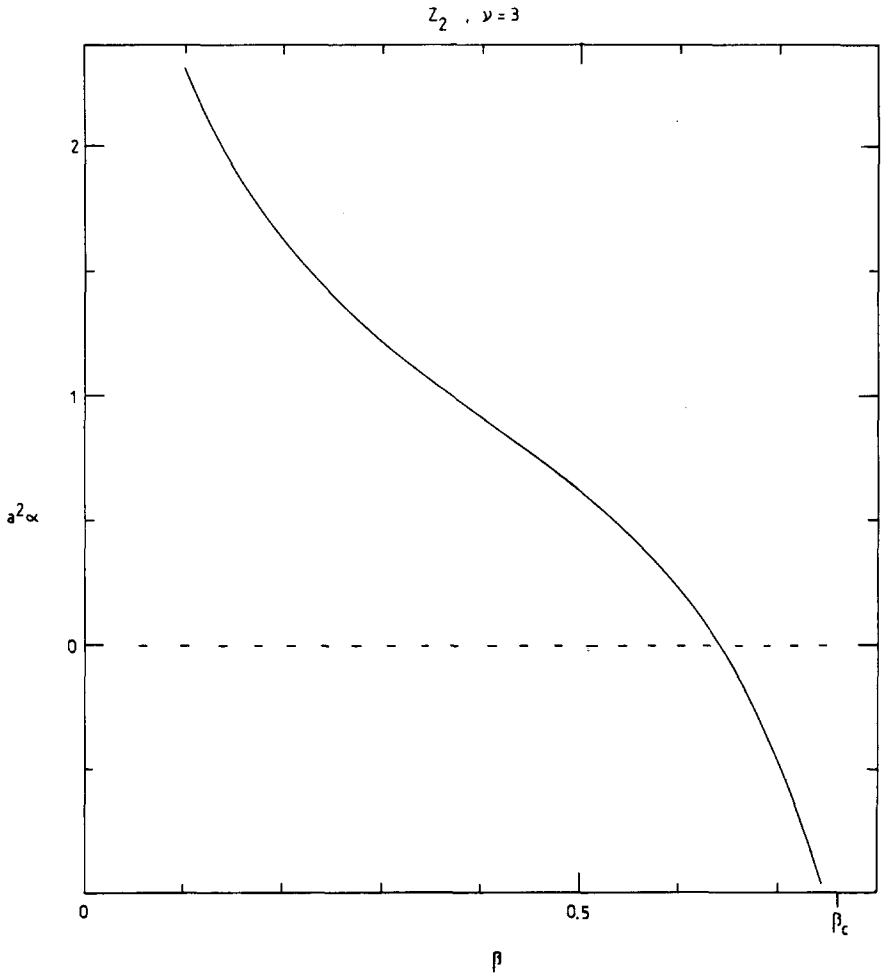


Fig. 6. Results of high-temperature expansions up to order x^{14} ($x = \tanh\beta$) for $a^2\alpha$ for Z_2 lattice gauge theory in $\nu = 3$ dimensions, including the 14th order terms of Kimura [15]. The critical couplings β_c are indicated.

string tension should go to zero at β_c for $\nu = 3$, whereas in $\nu = 4$ dimensions it might have a discontinuity at β_c. In this case the value of α just below β_c is not zero. This value plays an important role in Mack's theory of confinement [27]. It can be estimated with the help of the high-temperature series (see fig. 7). For $\beta > \beta_c$ the series still gives non-vanishing values for α while the true value is zero. This is due to the first-order nature of the phase transition.

In the case of $\nu = 3$ dimensions the situation is different. The string tension already vanishes at some value $\beta < \beta_c$ in the high temperature expansion (3.24). By duality [31, 29] the three-dimensional Z_2 gauge theory is transformed into the three-dimensional Ising model. The string tension α at inverse temperature β in the gauge model is by duality equal to the surface tension of the Ising model at inverse

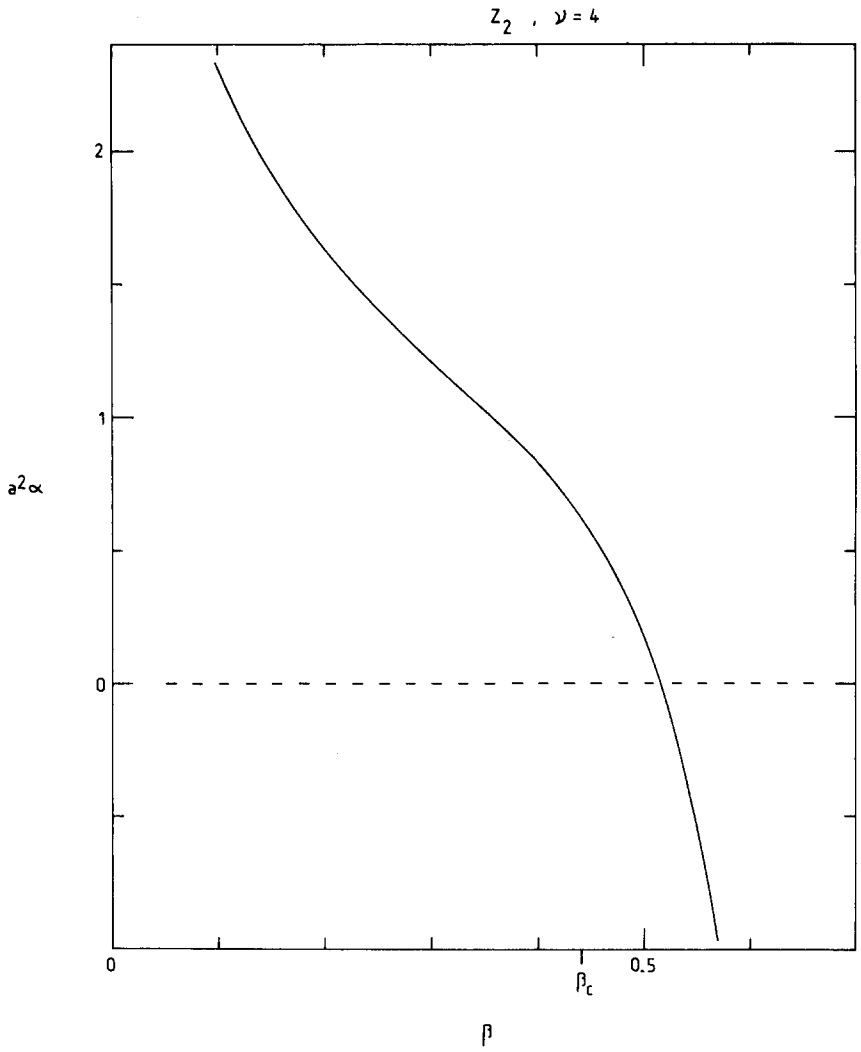


Fig. 7. As fig. 6, but for $\nu = 4$ dimensions.

temperature $\beta^* = -\frac{1}{2} \ln \tanh \beta$ [16]. Therefore, the series (3.24) is also a low-temperature series for the surface tension of the Ising model. Its vanishing well-below the critical temperature T_c, while the true surface tension is supposed to vanish at T_c, indicates the existence of a "roughening temperature" $T_R < T_c$ for the three-dimensional Ising model [33].

In figs. 6 and 7 the high-temperature series for α in $\nu = 3$ and $\nu = 4$ dimensions are plotted including the 14th order terms of Kimura [15].

I would like to thank Prof. G. Mack very much for constant support. He discussed the subject with me many times.

NOTE ADDED

Drouffe sent to me an unpublished paper (Numerical analysis of transitions in 3-dimensional lattice gauge systems, Stony Brook preprint ITP-SB-78-35, 1978), in which he calculates high-temperature series for the string tension of several 3-dimensional models up to 16th order, including the case of SU(2) and Z_2. My results are in agreement with his. The 14th order terms of Kimura [15] and Drouffe do not agree.

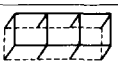
Appendix

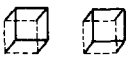
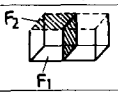
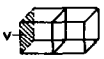
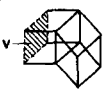
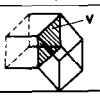
GRAPHS IN THE EXPANSION OF α

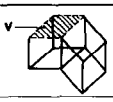
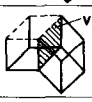
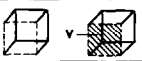
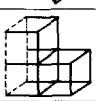
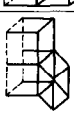
On the following pages we show the graphs which contribute to the string tension α in SU(2) or Z_2 lattice gauge theories in $\nu = 3$ or 4 dimensions up to order β^{12}.

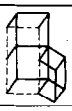
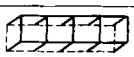
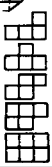
First the clusters C consisting of a single polymer Ξ_i are listed. The pictures show the corresponding decorations to the plane Ξ_0. The dashed lines indicate, where the decorations are fitted to Ξ_0. Lines which are not parallel are meant to be orthogonal. The table gives the activities $\hat{\Phi}(\Xi_i) = \Phi(\Xi_i) \cdot \Phi(\Xi_0)^{-1}$ in terms of the variables u, v, w for SU(2) [see (3.22)] and in terms of $x = \tanh\beta$ for Z_2. Furthermore it gives the counting factors κ for $\nu = 3$ and 4 dimensions. See the explanation following (3.21).

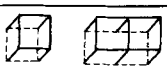
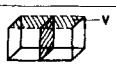
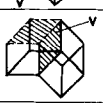
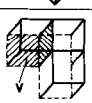
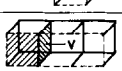
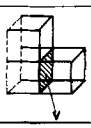
Next follow the clusters C consisting of a large polymer Ξ_i and one and two small polymers X_j, respectively. The combinatorical factors a(C) are displayed.

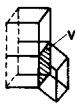
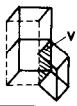
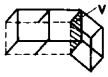
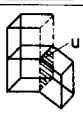
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=Z_2 ($\hat{\Phi}=x^k$) k	κ ν=3	κ ν=4
1		u^4	4	2	4
2	 F1 F2: v u / u v / w v	$3u^4v$ $3v^5$ $6v^5wu^{-1}$	/	2	4
3		u^6	6	4	8
4		u^8	8	2	4
5		u^8	8	/	8
6		u^8	8	/	32
7		u^8	8	4	8
8		u^8	8	8	16

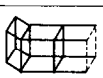
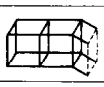
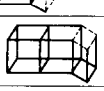
i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\Phi=x^k)$ k	χ $V=3$	$V=4$
9		u^8	8	2	4
10		u^8	8	-6	-16
11		$3u^6v^2$	—	4	8
12		F_1 F_2 u v v u $3u^6v^2$ $3u^4v^4$	—	8	16
13		$3u^8v$	—	2	4
14		$3u^8v$	—	2	4
15		$3u^8v$	—	—	8
16		$3u^8v$	—	—	8

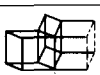
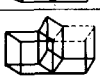
i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\Phi=x^k)$ k	χ $V=3$	$V=4$
17		$3u^8v$	—	—	32
18		$3u^8v$	—	—	32
19		$3u^8v$	—	-12	-32
20		$8u^{10}$: $8u^9w$:	10 —	1	2
21		$8u^{10}$: $8u^9w$:	10 —	—	4
22		u^{10}	10	8	16
23		u^{10}	10	—	32

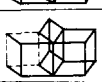
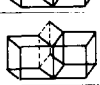
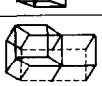
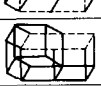
i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\hat{\Phi}=x^k)$ k	χ V=3	χ V=4
24		u^{10}	10	/	64
25		u^{10}	10	/	32
26		u^{10}	10	/	16
27		u^{10}	10	/	32
28		u^{10}	10	/	32
29		u^{10}	10	/	16
30	 similar: 	u^{10}	10	4 16 8 8 16 4	8 32 16 16 32 8

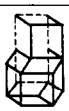
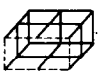
i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\hat{\Phi}=x^k)$ k	χ V=3	χ V=4
31		u^{10}	10	-40	-112
32		$6u^6v^3$	/	4	8
33		$9u^8v^2$	/	2	4
34		$9u^8v^2$	/	/	8
35		$9u^8v^2$	/	/	32
36		$3u^8v^2$	/	16	32
37		$3u^8v^2$	/	8	16
38		$3u^{10}v$	/	8	16

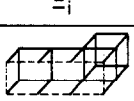
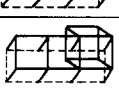
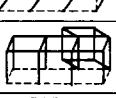
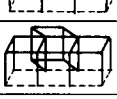
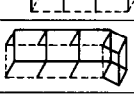
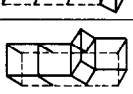
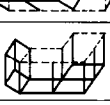
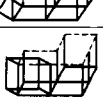
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=z_2 ($\Phi=x^k$) k	χ v=3	χ v=4
39		$3u^{10}v$	/	/	32
40		$3u^{10}v$	/	/	64
41		$3u^{10}v$	/	/	32
42		$4u^{12}$	12	8	16
43		$4u^{12}$	12	/	32
44		$3u^{10}v$	/	/	32

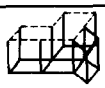
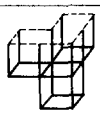
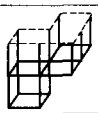
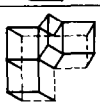
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=z_2 ($\Phi=x^k$) k	χ v=3	χ v=4
45		$3u^{10}v$	/	/	32
46		$3u^{10}v$	/	/	16
47		u^{12}	12	2	4
48		u^{12}	12	/	8
49		u^{12}	12	/	8
50		u^{12}	12	/	16
51		u^{12}	12	/	32
52		u^{12}	12	/	64
53		u^{12}	12	/	32

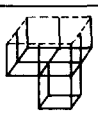
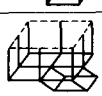
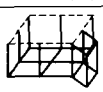
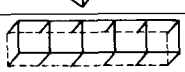
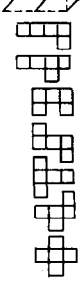
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=z_2 ($\hat{\Phi}=x^k$) k	$\varkappa$ V=3	$\varkappa$ V=4
54		u^{12}	12	8	16
55		u^{12}	12	/	32
56		u^{12}	12	/	64
57		u^{12}	12	/	32
58		u^{12}	12	/	32
59		u^{12}	12	/	32
60		u^{12}	12	/	32
61		u^{12}	12	/	32
62		u^{12}	12	/	64

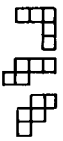
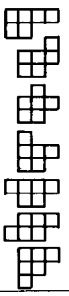
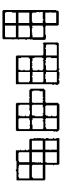
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=z_2 ($\hat{\Phi}=x^k$) k	$\varkappa$ V=3	$\varkappa$ V=4
63		u^{12}	12	/	32
64		u^{12}	12	/	32
65		u^{12}	12	/	32
66		u^{12}	12	/	8
67		u^{12}	12	/	16
68		u^{12}	12	/	32
69		u^{12}	12	/	32
70		u^{12}	12	/	64
71		u^{12}	12	/	32

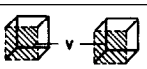
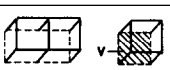
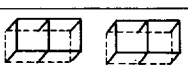
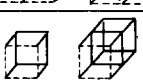
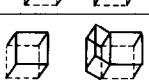
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=Z_2 ($\hat{\Phi}=x^k$) k	χ V=3	χ V=4
72		u^{12}	12	/	64
73		u^{12}	12	/	64
74		u^{12}	12	/	64
75		u^{12}	12	4	8
76		u^{12}	12	/	16
77		u^{12}	12	/	32
78		u^{12}	12	4	8
79		u^{12}	12	/	16

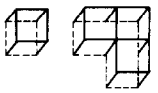
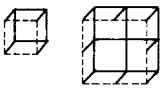
i	Ξ_i	G=SU(2) $\hat{\Phi}$	G=Z_2 ($\hat{\Phi}=x^k$) k	χ V=3	χ V=4
80		u^{12}	12	8	16
81		u^{12}	12	/	32
82		u^{12}	12	/	64
83		u^{12}	12	/	32
84		u^{12}	12	/	32
85		u^{12}	12	/	32
86		u^{12}	12	/	64
87		u^{12}	12	/	64

i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\hat{\Phi}=x^k)$ k	χ $\nu=3$	χ $\nu=4$
88		u^{12}	12	/	64
89		u^{12}	12	/	64
90		u^{12}	12	8	16
91		u^{12}	12	/	32
92		u^{12}	12	16	32
93		u^{12}	12	/	64
94		u^{12}	12	/	64

i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\hat{\Phi}=x^k)$ k	χ $\nu=3$	χ $\nu=4$
95		u^{12}	12	8	16
96		u^{12}	12	/	32
97		u^{12}	12	/	64
98	 similar: 	u^{12}	12	4 16 16 8 8 8 16 2	8 32 32 16 16 16 32 4

i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\Phi=X^k)$ k	χ $v=3$	$v=4$
98 (cont.)				8	16
				16	32
				8	16
99		u^{12}	12	16	32
				16	32
				8	16
				16	32
				8	16
				8	16
				8	16
100		u^{12}	12	16	32
				16	32
				8	16
				4	8

i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$G=Z_2$ $(\Phi=X^k)$ k	χ $v=3$	$v=4$
101		u^{12}	12	4	8
				8	16
102		u^{12}	12	2	4
103		$9u^8v^2$	/	-6	-16
104		$3u^{10}v$	/	-40	-112
105		u^{12}	12	-60	-176
106		u^{12}	12	-12	-32
107		u^{12}	12	/	-64
108 a)		u^{12}	12	/	-288
b)		$4u^{12}$	12	/	16

i	Ξ_i	$G=SU(2)$ $\hat{\Phi}$	$\hat{G}=Z_2$ $\hat{\Phi}=x^k$ k	χ v=3	χ v=4
109		u^{12}	12	-56	-160
110		u^{12}	12	-104	-304
111		u^{12}	12	-32	-96
112		u^{12}	12	$\frac{92}{3}$	$\frac{328}{3}$

Cluster $C=(\Xi_i, X_j)$, $a(c)=-1$

Ξ_i	$\hat{\Phi}(\Xi_i)$ SU(2)	Z_2	X_j	$\Phi(X_j)$ SU(2)	Z_2	χ v=3	χ v=4
Ξ_o	1	1	X_1 = 	$4u^6$ $+9v^6$	x^6	2	4
Ξ_o	1	1	X_2 = 	$4u^{10}$	x^{10}	7	14
Ξ_o	1	1	X_3 = 	$4u^{10}$	x^{10}	/	44
Ξ_o	1		X_4 = 	$12u^{10}v$	/	7	14
Ξ_o	1		X_5 = 	$12u^{10}v$	/	/	44
Ξ_o	1	1	X_6 = 	$4u^{12}$	x^{12}	/	64
Ξ_o	1	1	X_7 = 	$4u^{12}$	x^{12}	/	16
Ξ_o	1	1	X_8 =  (bothcubes touch Ξ_o)	$16u^{12}$	x^{12}	8	32
Ξ_1	u^4	x^4	X_1	$4u^6$	x^6	2	44
Ξ_2	$3u^4v$	/	X_1	$4u^6$		2	44
Ξ_3	u^6	x^6	X_1	$4u^6$	x^6	8	160
Ξ_*	$4u^6$	x^6	X_1	$4u^6$	x	/	32

$\Xi_* = \Xi_o \cup X_1$, with $\Xi_o \cap X_1$ = single link

Cluster $C = (\Xi_0, X = X_1, X' = X_1)$

			$r = 3$	$r = 4$
(a)	$X \cap X' = \phi$	$a(C) = 1$	− 14	− 48
(b)	$X \cap X' =$ single link			
	$X \cap X' \cap \Xi_0 = \phi$	$a(C) = 2$	4	8
(c)	$X \cap X'$-single link			
	$X \cap X' \cap \Xi_0 \neq \phi$	$a(C) = 2$	4	24
(d)	$X \cap X' =$ plaquette			
	$X \cap X' \cap \Xi_0 = \phi$	$a(C) = 1$	2	12
(e)	$X \cap X' =$ plaquette			
	$X \cap X' \cap \Xi_0 \neq \phi$	$a(C) = 2$	5	46
(f)	$X = X'$	$a(C) = 1$	2	4

References

[1] K. Wilson, Phys. Rev. D10 (1974) 2445
[2] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395
[3] G. 't Hooft, Nucl. Phys. B138 (1978) 1
[4] J. Glimm and A. Jaffe, Nucl. Phys. B149 (1979) 49
[5] T. Yoneya, Nucl. Phys. B144 (1978) 195
[6] D. Förster, Phys. Lett. 76B (1978) 597
[7] G. Mack and V.B. Petkova, Ann. of Phys. 123 (1979) 442
[8] G. Mack and V.B. Petkova, Ann. of Phys. 125 (1980) 117
[9] G. 't Hooft, Nucl. Phys. B153 (1979) 141
[10] G. Mack, Properties of lattice gauge theory models at low temperatures, DESY 80/03 (Jan. 1980), to appear *in* Recent developments in gauge theories, ed. G. 't Hooft et al. (Plenum, New York, 1980)
[11] G. Münster, Z. Phys. C6 (1980) 175
[12] K. Osterwalder and E. Seiler, Ann. of Phys. 110 (1978) 440
[13] M. Creutz, Phys. Rev. D21 (1980) 2308
[14] A. Duncan and H. Vaidya, Phys. Rev. D20 (1979) 903
[15] N. Kimura, Prog. Theor. Phys. 64 (1980) 310
[16] J. Bricmont, J. Lebowitz and Ch. Pfister, On the surface tension of lattice systems, Rutgers university preprint, New Brunswick (1979)
[17] C. Gruber and H. Kunz, Comm. Math. Phys. 22 (1971) 133
[18] J. Drouffe and C. Itzykson, Phys. Reports 38 (1978) 133
[19] D. Zelobenko, Compact Lie groups and their representations, (Am. Math. Soc., Providence, Rhode Island, 1973) p. 212
[20] D. Ruelle, Statistical mechanics: rigorous results, (Benjamin, New York, 1969) sect. 4;
G. Rushbrooke, G. Baker and P. Wood, *in* Phase transitions and critical phenomena, vol. 3, ed. C. Domb, M.S. Green, (Academic Press, New York, 1974)
[21] C. Domb, *in* Phase transitions and critical phenomena, vol. 3, see ref. [20]
[22] A. Migdal, JETP (Sov. Phys.) 42 (1976) 413; 743
L. Kadanoff, Rev. Mod. Phys. 49 (1977) 267
[23] G. 't Hooft, 1972, unpublished;
H. Politzer, Phys. Rev. Lett. 30 (1973) 1346;
D. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343
[24] J. Kogut, Rev. Mod. Phys. 51 (1979) 659
[25] J. Kogut, R. Pearson and J. Shigemitsu, Phys. Rev. Lett. 43 (1979) 484
[26] C. Callan, R. Dashen and D. Gross, Phys. Rev. Lett. 44 (1980) 435
[27] G. Mack, Phys. Rev. Lett. 45 (1980) 1378

[28] A. and P. Hasenfratz, Phys. Lett. 93B (1980) 165
[29] R. Balian, J. Drouffe and C. Itzykson, Phys. Rev. D11 (1975) 2098
[30] M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Lett. 42 (1979) 1390
[31] F. Wegner, J. Math. Phys. 12 (1971) 2259
[32] G. Münster, Phys. Lett. 95B (1980) 59
[33] J.D. Weeks, G.H. Gilmer and H.J. Leamy, Phys. Rev. Lett. 31 (1973) 549