COMMENT ON THE $O(\alpha_s^2)$ CORRECTIONS TO JET-PRODUCTION IN e^+e^- ANNIHILATION

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Using recent results of Ellis et al. I calculate the $O(\alpha_s^2)$ corrections to thrust distribution in e^+e^- annihilation. The numerical importance of the change of the four-momentum squared which determines the strength of the running coupling constant is studied in detail.

In a recent paper Ellis et al. [1] have published analytic results for the $O(\alpha_s^2)$ corrections to the event shape in e^+e^- annihilation [2,3]. They have calculated C and D [3] distributions, where C gives a measure of the multijet structure of the event but with a large weight to planar events, while D is a measure of the deviation from planarity [‡1]. Since the two jet two loop contributions are not calculated they impose a cut $C > 1/2$ to exclude the two jet region.

Their numerical evaluation of the C distribution is not the most effective one [‡2] and they do not discuss the sensitivity of the size of the corrections after absorbing some large logarithmic terms into the running coupling constant. Furthermore their formulae are not applicable directly for an appropriate introduction of the jet fragmentation.

In ref. [1], however, all the important details are published so the above mentioned minor shortcomings can be easily eliminated. First of all with a slight modification of the pole terms (which have been subtracted from the four-jet matrix elements to regularize the infrared and mass singularities), any distribution can be calculated with high accuracy.

The same modification leads to formulae where jet fragmentation can be trivially introduced.

The purpose of the present letter is to calculate thrust distributions [5] with this method and to study the numerical importance of the change of the four-momentum squared which determines the strength of the running coupling constant. Throughout this paper I use the notation of ref. [1].

The partons for the different subprocesses to $O(\alpha_s^2)$ are labelled as:

$$\gamma^*(Q) \to q(p_1) + \bar{q}(p_2) + G(p_3)\,, \tag{1}$$

$$\gamma^*(Q) \to q(p_1) + \bar{q}(p_2) + G(p_3) + G(p_4)\,, \tag{2}$$

$$\gamma^*(Q) \to \bar{q}(p_1) + \bar{q}(p_2) + q(p_3) + q(p_4)\,. \tag{3}$$

It is convenient to introduce invariant variables

$$s_{ij} = (p_i + p_i)^2\,, \quad s_{ijk} = (p_i + p_j + p_k)^2\,,$$

$$y_{ij} = s_{ij}/Q^2\,, \quad y_{ijk} = s_{ijk}/Q^2\,, \quad (i<j<k)\,. \tag{4}$$

All the partons are massless. The four-jet matrix elements have poles in the limit when $s_{ij} \to 0$. These pole terms are explicitly given in ref. [1] with the help of eqs. (3.17), (3.19) and (3.22). The $1/\epsilon^2$ and $1/\epsilon$ singularities of the loop corrections [‡3] have been cancelled by integrating the pole term over a three dimensional subspace of the five-dimensional phase space of the four-parton final states [‡4].

‡1 Acoplanarity distributions have been calculated in ref. [4].
‡2 With one million Monte Carlo points still large fluctuations are present.

‡3 A $\overline{\mathrm{MS}}$ scheme and dimensional regularization was used, $\epsilon = (4-n)/2$.
‡4 The integration over the beam direction can be performed trivially.

Since the pole terms represent the "three jet-like" part of the four-jet matrix elements it seems to be natural to limit the region of the integral over the singularity $1/s_{ij}$ at a mass value which is approximately equal to the average invariant mass of a physical jet. In particular such a procedure is obligatory if the effects of hadronization are to be taken into account. According to the experimental data [6] the average mass value of a physical jet is $\langle m_J \rangle \approx 5$ at $s = 30$–35 GeV, therefore the cut-off parameter must have the value of $\approx$4–6 GeV. A change in the parameter m_J is reabsorbed in a corresponding change of the fragmentation properties of the partons.

Therefore the upper limit of the integrals over s_{ij} in formulae (3.17), (3.19) and (3.22) of ref. [1] has to be modified with the insertions of θ-functions $\theta(m_J^2 - s_{13})$, $\theta(m_J^2 - s_{34})$, etc. With this change of the region of the phase space integral, the pole contribution will be different. In particular eq. (3.26) of ref. [1] is replaced by:

$$\sigma_0^{-1}(\mathrm{d}\sigma^{(s)} + \mathrm{d}\sigma^{(J)}) = \frac{\alpha_s(Q^2)}{2\pi} C_F \frac{1}{Q^4} \int \mathrm{d}s_{12}\, \mathrm{d}s_{13}\, \mathrm{d}s_{23} \times \delta(1 - y_{12} - y_{13} - y_{23}) \left[T(s_{12}, s_{13}, s_{23}) \times \left\{1 + \frac{\alpha_s}{2\pi}(J_1 + J_2 + J_3)\right\} + \frac{\alpha_s}{2\pi} F(s_{12}, s_{13}, s_{23}) \right], \tag{5}$$

where $\sigma^0 = 4\pi\alpha^2/Q^2(\Sigma_i\, eq_i^2)$, $T(s_{12}, s_{13}, s_{23})$ is the three-jet matrix element

$$T(s_1, s_2, s_3) = \frac{s_3}{s_2} + \frac{s_2}{s_3} + 2\left[\frac{s_1(s_1 + s_2 + s_3)}{s_2 \cdot s_3}\right]. \tag{6}$$

$F(s_{12}, s_{13}, s_{23})$ is defined by eq. (2.21) of ref. [1], J_1, J_2, J_3 are given by the equations

$$J_1 = C_F \left[\tfrac{2}{3}\pi^2 - \ln^2 \frac{x_1}{y_{12}} - \ln^2 \frac{x_2}{y_{12}} - 2\mathrm{Li}_2\left(1 - \frac{x_1}{y_{12}}\right) - 2\mathrm{Li}_2\left(1 - \frac{x_2}{y_{12}}\right) - 1 - \tfrac{3}{2}\ln(x_1 x_2)\right], \tag{7a}$$

$$J_2 = N_C \left[\tfrac{1}{3}\pi^2 - \tfrac{1}{2}\ln^2 \frac{x_3}{y_{13}} - \tfrac{1}{2}\ln^2 \frac{x_3}{y_{23}} - \tfrac{1}{2}\ln^2 \frac{x_1}{y_{13}} - \tfrac{1}{2}\ln^2 \frac{x_2}{y_{23}} + \tfrac{1}{2}\ln^2 \frac{x_1}{y_{12}} + \tfrac{1}{2}\ln^2 \frac{x_2}{y_{12}} - \mathrm{Li}_2\left(1 - \frac{x_3}{y_{13}}\right) - \mathrm{Li}_2\left(1 - \frac{x_2}{y_{23}}\right) + \mathrm{Li}_2\left(1 - \frac{x_1}{y_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{x_2}{y_{12}}\right) - \mathrm{Li}_2\left(1 - \frac{x_1}{y_{13}}\right) - \mathrm{Li}_2\left(1 - \frac{x_2}{y_{23}}\right) + \tfrac{67}{18} - \tfrac{11}{6}\ln x_3\right], \tag{7b}$$

$$J_3 = T_R \left[\tfrac{2}{3}\ln(x_3) - \tfrac{10}{9}\right], \tag{7c}$$

where x_1, x_2, x_3 and

$$x_1 = \min(y_{12}y_{13}, m_J^2/Q^2), \quad x_2 = \min(y_{12}y_{23}, m_J^2/Q^2), \quad x_3 = \min(y_{13}y_{23}, m_J^2/Q^2), \tag{8}$$

and $C_F = 4/3$, $N_C = 3$, $T_R = n_f/2$, n_f denotes the number of flavours. The running coupling constant is defined with the $O(\alpha_s^2)$ correction

$$\alpha_s(Q)/2\pi = [b_0 \ln(Q^2/\Lambda^2) + (b_1/b_0)\ln\ln(Q^2/\Lambda^2)]^{-1}, \tag{9}$$

where

$$b_0 = \tfrac{1}{6}(11 N_C - 4 T_R), \quad b_1 = (\tfrac{17}{6}N_C^2 - \tfrac{5}{3}N_C T_R - C_F T_R).$$

If we take $x_1 = y_{12}y_{13}$, $x_2 = y_{12}y_{23}$ and $x_3 = y_{13}y_{23}$ we reproduce formula (3.26) of ref. [1]. The cross section (5)–(7) is the analogue of the Sterman–Weinberg formula for three-jet production [7]. In the limit $m_J^2 \to 0$ at the symmetric point $y_{12} = y_{13} = y_{23} = 1/3$ we obtain:

$$J_1 + J_2 + J_3 \to 2C_F[-\ln^2(3m_J^2/Q^2) - \tfrac{3}{2}\ln(m_J^2/Q^2) - \tfrac{1}{2}] + N_C[\tfrac{67}{18} - \ln^2(3m_J^2/Q^2) - \tfrac{11}{6}\ln(m_J^2/Q^2)] + T_R[\tfrac{2}{3}\ln(m_J^2/Q^2) - \tfrac{10}{9}]. \tag{10}$$

For small jet mass values these terms lead to large and negative corrections. These mass singularities are cancelled by adding the four-jet like contributions. I remark that the π^2 terms are cancelled in eq. (10). Therefore it is not obvious that the π^2 terms in formula (3.26) of ref. [1] have to be interpreted as large soft gluon contributions.

In ref. [1] it has been observed that in the large jet mass limit eq. (5) contains a single logarithmic term of the form

$$(\alpha_s/2\pi)(\tfrac{2}{3}T_R - \tfrac{11}{6}N_C)\ln(y_{13}y_{23}). \tag{11}$$

This term can be exactly cancelled if the coupling constant is evaluated not at Q^2 but at $s_{13} \cdot s_{23}/Q^2$ [‡5]

[‡5] Indeed $\alpha_s(\kappa Q^2)/2\pi = 1/\{b_0 \ln(Q^2/\Lambda^2)[1 + (\ln\kappa/\ln Q^2/\Lambda^2)]\} \approx (\alpha_s(Q^2)/2\pi)(1 - (\alpha_s/2\pi)b_0 \ln\kappa)$, where $b_0 = 11 N_C/6 - 2 T_R/3$.

which provides the natural scale of the momentum of the quark gluon vertex function in the three-jet configurations. In other words, if the three-point vertices are renormalized at $(y_{13}y_{23}Q^2)$, the loop corrections also give single logs and cancel the contribution (11) obtained by appropriate integration over the pole terms. This cancellation mechanism will appear also in higher order corrections. Therefore better convergence is expected if the three-jet cross section is expanded in powers of

$$\alpha_s^{(y)} = 2\pi/b_0/\ln[y_{13}y_{23}Q^2/\Lambda^2]\ . \tag{12}$$

In order to study the behaviour of the $O(\alpha_s^2)$ corrections we have calculated the thrust distribution with various thrust cut-off (T_c) and jet mass (m_J) values. The total $O(\alpha_s^2)$ corrections are evaluated according to the formulae of ref. [1]. The total thrust distribution is splitted into the sum of three-jet and four-jet like contributions

$$\mathrm{d}\sigma^{3\mathrm{jet}} = \mathrm{d}\sigma^{(s)} + \mathrm{d}\sigma^{(3)} + \mathrm{d}\sigma^{(4)}_{(3)}\ ,$$
$$\mathrm{d}\sigma^{4\mathrm{jet}} = \mathrm{d}\sigma^{(4)}_{(4)}\ , \tag{13}$$

where $\mathrm{d}\sigma^{(s)} + \mathrm{d}\sigma^{(3)}$ is defined by formulae (5)–(7). The $\mathrm{d}\sigma^{(4)}$ cross section is calculated with the four-jet matrix elements squared minus the pole terms of ref. [1], but integrated over the y_{ij} singularity only up to the cut-off values x_1, x_2, x_3 [see eq. (8)]. Furthermore the $\mathrm{d}\sigma^{(4)}$ cross section is splitted into a three-jet like and a four-jet like contribution. The $\mathrm{d}\sigma^{(4)}$ contribution is three-jet like $[\mathrm{d}\sigma^{(4)} = \mathrm{d}\sigma^{(4)}_{(3)}]$ if any of the invariants s_{ij} is smaller than m_J^2 otherwise it is four-jet like $[\mathrm{d}\sigma^{(4)} = \mathrm{d}\sigma^{(4)}_{(4)}]$.

Obviously the $\mathrm{d}\sigma^{3\mathrm{jet}}$ contribution must be fragmented as three-jet and the $\mathrm{d}\sigma^{4\mathrm{jet}}$ contribution as four-jet processes. The cross sections $\mathrm{d}\sigma^{3\mathrm{jet}}$ and $\mathrm{d}\sigma^{4\mathrm{jet}}$ depend sensitively on the jet mass value m_J.

The Born term $[O(\alpha_s)$ contribution] is calculated with three different definitions of α_s. First we use α_s as defined in eq. (9) $[\mathrm{d}\sigma_B^{(s)}]$, next we drop the term $(b_1/b_0)\ln\ln(Q^2/\Lambda^2)$ in the denominator in eq. (9) and change the normalization of Q^2 to $(1-T)Q^2$.

$$\alpha_s^T/2\pi = [b_0 \ln[(1-T)Q^2/\Lambda^2]]^{-1}\ , \tag{14}$$

where T denotes thrust. The corresponding cross section is $\mathrm{d}\sigma_B^T$. Finally, we use $\alpha_s^{(y)}$ as given by eq. (12) and the Born cross section is denoted by $\mathrm{d}\sigma_B^{(y)}$. Numerical values of integrated cross sections with $T_c = 0.75, 0.85, 0.95$ and $m_J = 3$ or 5 GeV are given in table 1 [‡6].

I confirm the result of ref. [1] that the corrections are large and approximately independent of T_c in comparison with $\sigma_B^{(s)}$. However, I also find that the change in the normalization of α_s is very important. The size of the corrections is 10%–45% and decreases with increase of T_c if we make a comparison with $\mathrm{d}\sigma_B^{(y)}$. Furthermore with $m_J = 5$ GeV they can be decomposed into a small negative correction = −(3%–5%) to $\mathrm{d}\sigma^{3\mathrm{jet}}$ and somewhat larger corrections ≈15%–50% to $\mathrm{d}\sigma^{4\mathrm{jet}}$ with increasing ratio of $\mathrm{d}\sigma^{4\mathrm{jet}}/\mathrm{d}\sigma_B^{(y)}$ as T_c is decreased.

‡6 The cross sections are calculated for four quarks (u, d, s, c) and a value of $\Lambda = 250$ MeV is used. The numbers have integrating errors of ≈2%.

Table 1
Total cross section values in pb with various thrust cut-off T_c. Three different $O(\alpha_s^{(i)})$ cross section values are given ($\sigma_B^{(s)}, \sigma_B^{(T)}, \sigma_B^{(y)}$). $\sigma^{(\mathrm{all})}$ denotes the total cross section up to order α_s^2 with $T < T_c$, $\Lambda^{\mathrm{MS}} = 250$ MeV and $n_f = 4$. Its three and four jet like parts (σ^{3j} and σ^{4j}) are calculated with $m_J = 3$ GeV and 5 GeV.

m_J (GeV)	T_c	$\sigma_B^{(s)}$	$\sigma_B^{(T)}$	$\sigma_B^{(y)}$	σ^{3j}	σ^{4j}	$\sigma^{(\mathrm{all})}$	$R_s = \frac{\sigma^{(\mathrm{all})}}{\sigma_B^{(s)}}$	$R_y = \frac{\sigma^{(\mathrm{all})}}{\sigma_B^{(y)}}$	$R_y^{(3)} = \frac{\sigma^{3j}}{\sigma_B^{(y)}}$	$\frac{\sigma^{4j}}{\sigma^{3j}}$
3	0.75	1.81	2.34	2.64	1.30	2.62	3.92	2.16	1.48	0.49	2.01
3	0.85	11.9	16.0	18.3	10.5	13.2	23.7	1.99	1.30	0.57	1.26
3	0.95	57.7	85.2	105	77.0	40.0	117.0	2.02	1.11	0.77	0.52
5	0.75	1.81	2.34	2.64	2.50	1.35	3.85			0.95	0.54
5	0.85	11.9	16.0	18.3	17.6	5.68	23.3			0.96	0.32
5	0.95	57.7	85.2	105	103	13	116			1.10	0.13

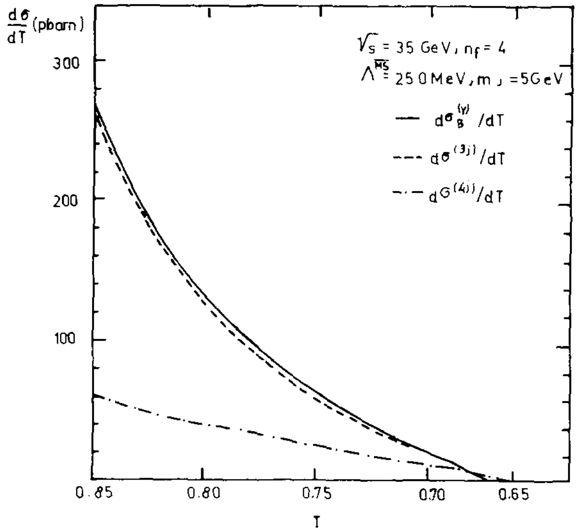

Fig. 1. Thrust distributions $d\sigma^{(3j)}/dT$ (dashed line), $d\sigma^{(4j)}/dT$ (dotted line) and $d\sigma_B^{(y)}/dT$ (solid line) for $T < 0.85$. ($\Lambda^{MS} = 250$ MeV, $m_J = 5$ GeV, $n_f = 4$). The cross section values are given in pb.

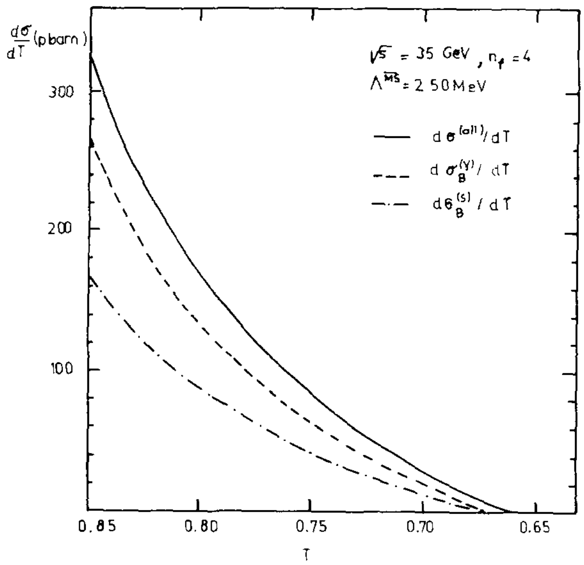

Fig. 2. Thrust distribution $d\sigma^{(all)}/dT$ (solid line), $d\sigma^{(y)}/dT$ (dashed line) and $d\sigma_B^{(s)}/dT$ (dotted line). The parameter values and normalization are the same as in fig. 1.

The thrust distributions $d\sigma^{(3j)}/dT$, $d\sigma^{(4j)}/dT$ and $d\sigma_B^{(y)}$ are plotted in fig. 1, for $m_J = 5$ GeV, while in fig. 2 we plotted the total thrust distribution in comparison with $d\sigma_B^{(y)}/dT$ and $d\sigma_B^{(s)}/dT$.

We remark that since we could extend the numerical evaluation up to $T_c = 0.95$, with the known value of the total cross section [8] in the $\overline{MS}$ scheme, we can calculate the normalization of the squared two-jet contribution

$$\sigma_{2jet}(T_c) = \sigma_{total} - \sigma_{3jet}(T_c) - \sigma_{4jet}(T_c) .$$

With $m_J = 5$ GeV, $T_c = 0.95$ and $\Lambda = 250$ MeV for four quarks we obtain

$$\sigma_{2j} : \sigma_{3j} : \sigma_{4j} = 123 \text{ pb} : 102 \text{ pb} : 12 \text{ pb}$$

$$= 0.52 : 0.43 : 0.05 .$$

This ratio can be changed with decreasing T_c [9].

In conclusion, the perturbative expansion of the three-jet cross section in powers of $\alpha_s(Q^2)$ in the $\overline{MS}$ scheme may contain large logarithmic terms of $\ln^k(y_{12}y_{23})$ which can be absorbed into the change of scale of the four-momentum squared which determines the strength of the running coupling constant $[\alpha_s(Q^2) \to \alpha_s(y_{12}y_{23}Q^2)]$. A systematic study of these logarithmic terms is required in order to clarify whether the perturbative expansion of multijet production in e^+e^- annihilation is convergent or not.

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