PLAQUETTE–PLAQUETTE CORRELATIONS IN THE SU(2) LATTICE GAUGE THEORY

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Monte Carlo measurements of plaquette–plaquette correlations in the 4-dimensional SU(2) lattice gauge theory are reported. In the continuum limit the glue ball mass (= inverse correlation length) is estimated to be $m_g = (3.7 \pm 1.2)\sqrt{K}$, where K is the string tension.

In this letter I consider the standard SU(2) lattice gauge theory [1] in four dimensions with lattice spacing a. The random variables $U(b) \in \mathrm{SU}(2)$ are attached to links b of the lattice and the expectation value of a gauge invariant observable $O\{U\}$ is given by:

$$\langle O\rangle = Z^{-1} \int \prod_b \mathrm{d}U(b)\, O\{U\} \times \exp\{-\tfrac{1}{2}\beta \sum_p \mathrm{Tr}[1-U(\dot{p})]\}. \qquad (1)$$

The summation is over all elementary plaquettes p of the lattice. $U(\dot{p}) = U(b_4)...U(b_1)$ is the parallel transporter around a plaquette p with boundary consisting of links $\boldsymbol{b}_1, ..., b_4$ and the normalization is $\langle 1\rangle = 1$.

In the low temperature (= weak coupling) limit $\beta = 4/g^2 \to \infty$ the theory is supposed to approach a phase transition and to define a quantum field theory in the continuum. If this is correct the correlation length ξ has for $\beta \to \infty$ to go to infinity as dictated by asymptotic freedom [2]:

$$\xi(\beta) = a \text{ const. } \beta^{-51/121} \exp(\tfrac{3}{11}\pi^2\beta)\,. \qquad (2)$$

According to renormalization group analysis any physical mass is in the low temperature limit related to the correlation length by

$$m = \text{const.}'\, \xi^{-1}\,. \qquad (3)$$

The constants are different for different masses.

Creutz [3,4] has carried out Monte Carlo measurements of the string tension K. In a region of β where $Ia, Ja \gtrsim \xi$ $(I, J \in \mathbf{N})$ the rectangular Wilson loop $W(I, J)$ of area $A = IJa^2$ and perimeter $P = 2(I+J)a$ is supposed to behave like

$$W(I,J) = \text{const.}'' \exp(-K\cdot A - m_\mathrm{p}\cdot P) + (\text{nonleading corrections})\,, \qquad (4)$$

and therefore the quantity [4]

$$\chi(I,J) = -\ln\left\{\frac{W(I,J)\cdot W(I-1,J-1)}{W(I,J-1)\cdot W(I-1,J)}\right\} \qquad (5)$$

measures directly the string tension K, provided that $(I-1)a \gtrsim \xi$ and $(J-1)a \gtrsim \xi$.

Computer measurements of $\chi(I, I)$ with I between 3 and 5 exhibit renormalization group behaviour of the string tension in a region starting at $\beta \approx 2.1$. Fig. 1 shows data of Creutz [4] and data which were obtained in collaboration with Stehr [5] [‡1]. $\chi(3,3)$ fits the asymptotic weak coupling behaviour of the string tension up to $\beta \approx 2.5$. This gives $2a$ at $\beta = 2.5$ as an upper bound for the correlation length. Consistency with the observed asymptotic behaviour of the string tension requires that ξ behaves according to (2) down to $\beta \approx 2.1$. This extrapolation of the upper limit gives $\xi < \frac{3}{4}a$ at $\beta = 2.1$ (cf. fig. 2). It is indeed very remarkable that the calculation of Creutz requires that the asymptotic weak coupling behaviour, which is expected for a large correlation length, starts at a value of β, where $\xi(\beta)$ is still much less than one lattice spacing. Only this property enables (in this approach) the calculation

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[‡1] In ref. [5] correlations between Wilson loops and plaquettes are considered.

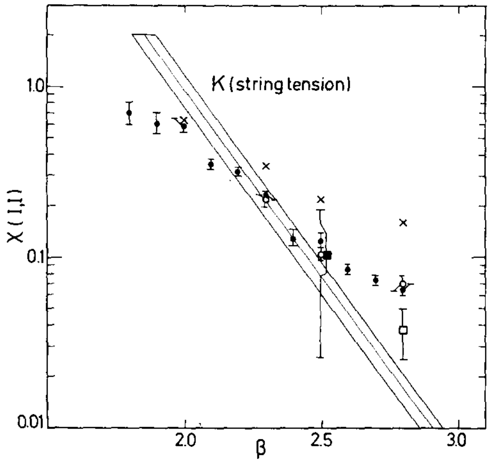

Fig. 1. Data for $\chi(I,I)$ in units of $[a^{-2}]$. From Creutz [4]: × ($I = 2$), ○ ($I = 3$), □ ($I = 4$) and from ref. [5]: ● ($I = 3$), ■ ($I = 5$).

of relations between physical quantities of the continuum theory on small lattices of size 10^4 and less.

To the knowledge of the author there exists no theoretical reasons to expect the beginning of the asymptotic weak coupling behaviour at such a small correlation length. All our further considerations are based on the assumption that the points for $\chi(I,I)$ [essentially $\chi(3,3)$] in fig. 1 show the expected asymptotic behaviour not only by some strange accident. It remains to find out in the future whether this rapid crossover from strong to weak coupling behaviour is an artifact of the four dimensional SU(2)

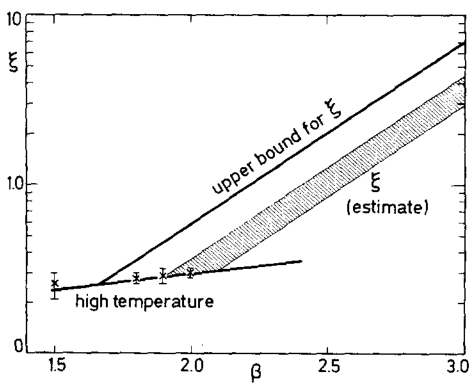

Fig. 2. Correlation length: Lowest order high temperature expansion and Monte Carlo points for $-1/\ln q(\beta)$ up to $\beta = 2.0$; upper bound for large β and estimate relying on the present Monte Carlo data for large β.

gauge theory or a common feature of a wider class of theories. From this point of view conclusions for the string tension of SU(3) [4,6], which do not explicitly exhibit the asymptotic behaviour of $\chi(I,I)$, but rely essentially on assuming analogy to SU(2), are not quite convincing.

It would of course be desirable to carry out direct Monte Carlo measurements of the SU(2) correlation length. For that reason I consider in the following correlations of parallel plaquettes, which are separated (in one of the directions orthogonal to the planes spanned by the plaquettes) by a distance $x = na$ ($n = 0, 1, 2, ...$). Let $p(x) = \frac{1}{2} \operatorname{Tr} U(\dot{p})$ for the plaquette p located at x. The correlation function

$$\rho(n) := \langle p(x)p(0)\rangle - \langle p(0)\rangle^2 \tag{6}$$

falls off exponentially like $\exp(-m_g x)$ for $x = na \to \infty$. Here $m_g = 1/\xi$ is the glue ball mass. In lowest order high temperature (= strong coupling) expansion one obtains easily

$$m_g = -4 \ln u, \quad u = I_2(\beta)/I_1(\beta) = \tfrac{1}{4}\beta + O(\beta^3), \tag{7}$$

and I_υ denotes a Bessel function (cf. ref. [7, 8.445]). The extrapolation of the upper bound hits the lowest order high temperature expansion at $\beta \approx 1.7$ (cf. fig. 2). Together with the observed asymptotic weak coupling behaviour of the string tension for $\beta \gtrsim 2.1$ this suggests a crossover from strong to weak coupling behaviour in the region $1.7 \leqslant \beta \leqslant 2.1$. It is amusing to note that this is also the region where a "roughening temperature" is conjectured in recent publications [8]. Whether there is a natural connection between the roughening temperature and the (fast) crossover is, however, unclear. In this context it is also notable that the convergence between random and ordered start at $\beta = 1.9$ was extremely bad.

Using the heat bath method [‡2] I have carried out Monte Carlo measurements of the correlation function $\rho(n)$ up to $n = 3$ on 6^4 and 9^4 lattices. For $\rho(3)$ the obtained values are already much smaller than the error bars and inconclusive. It is therefore not possible with my calculations to obtain the correlation length directly by means of the asymptotic exponential behaviour. In the region $\xi < 1a$ the values

[‡2] For the SU(2) lattice gauge theory a detailed description of the heat bath method is given in ref. [3].

of x for $n = 1, 2$ are, however, already large compared with the correlation length and expected to be sensitive with respect to the exponential fall-off. The difficulty is that for such small n the exponential law may be modified by some unknown β-dependent power law. I.e., for $n = 1, 2$ we can only expect a fit of the data by means of

$$\rho(n) = \text{const.}\,(\beta)(\mu x)^{-\alpha(\beta)}\exp(-m_g x)\,. \qquad (8)$$

Here μ is a new mass, which drops out in the important ratio $q(\beta) := \rho(2)/\rho(1) = 2^{-\alpha}\exp(1/\xi)$. For $\partial\xi/\partial\beta > 0$ and $\partial\alpha/\partial\beta = 0$ the ratio $q(\beta)$ is a monotonically increasing function of β. If also $\partial\alpha/\partial\beta > 0$ (as seems to be the case in the crossover region) the behaviour of $q(\beta)$ is in general unpredictable. If the numerical value of ξ is, however, small compared to $1a$ and $\xi(\beta)$ is strongly increasing, then the increase of $\exp(-1/\xi)$ would be rather drastic (in a short range of β) and should there dominate the β-dependence of α.

The data for $\rho(1)$ and $\rho(2)$ are summarized in fig. 3. They rely on altogether 16 045 sweeps through a 6^4 and 8482 sweeps through a 9^4 lattice. The error bars are obtained from the standard deviation with respect to values, which are obtained from reasonable large subsets of measurements. A more detailed analysis and data on additional quantities will be published elsewhere. The quantity $\rho(2)$ is already numerically very small and just at the border line of what can be computed within the abilities of the present "standard" computer program.

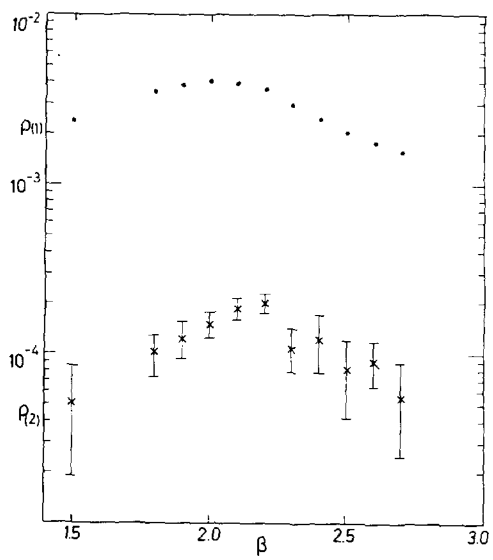

Fig. 3. Data for $\rho(n)$ (n = 1, 2). The error bars for $\rho(1)$ are negligible on the used scale).

For $\beta \leqslant 2.0$ both $\rho(1)$ and $\rho(2)$ increase strongly as functions of β, such that $1/[-\ln q(\beta)]$ is within the error bars consistent with the result expected for ξ by the lowest order high temperature expansion (cf. fig. 2). Around $\beta \approx 2.0$ $\rho(1)$ reaches a maximum and is then decreasing, whereas $\rho(2)$ is still increasing up to $2.1 < \beta \lesssim 2.2$. On a linear scale this leads to a drastically stronger increase of the quotient $q(\beta)$ for $2.0 \leqslant \beta \leqslant 2.2$ than for $\beta \leqslant 2.0$. In view of the observed weak coupling behaviour of the string tension for $\beta \geqslant 2.1$ it seems to be natural to interpret the data between $\beta = 2.0$ and $\beta = 2.2$ as a signal for the onset of the asymptotic exponential behaviour (2) of the correlation length in this region. Simultaneously α increases from $\alpha \approx 0$ at $\beta = 1.8$ to $\alpha \approx 1.1$ at $\beta = 2.2$. This takes into account that some weak crossover effect for $1.8 \leqslant \beta \leqslant 2.0$ may be completely obscured by an increase of $\alpha(\beta)$, Our interpretation is further supported by the fact, that $2.0 \leqslant \beta \leqslant 2.2$ is the only part of the measured region $1.5 \leqslant \beta \leqslant 2.7$ where an exceptional increase of $q(\beta)$ is observed. For $\beta \geqslant 2.2$ the exponential increase of the correlation length effects the quotient $q(\beta)$ less and less drastic and the power law becomes dominant. The data indicate a further increase of α to $\alpha(2.7) \approx 3.8$. They are not accurate enough to admit a more precise determination of the dependence of α on β.

Measurements of the described type allow a rather accurate determination of the maxima which $\rho(1)$ and $\rho(2)$ take as functions of β. The functions $\rho(n)$ ($n = 1, 2, ...$) become zero for $\beta \to 0$ as well as for $\beta \to \infty$. It is plausible that $\rho(n)$ as function of β is monotonically increasing up to $\beta = \beta_n$, where the correlation length ξ reaches a certain value $\xi_n = \xi(\beta_n)$ such that $\xi_{n_1} > \xi_{n_2}$ for $n_1 > n_2$, and is then monotonically decreasing. A theoretical estimate of ξ_n is desirable, since it would allow an independent check of our interpretation of the data. The maximum of $\rho(0)$ is 0.25 at $\beta = 0$. Therefore the maxima of $\rho(1)$ and $\rho(2)$ are very near to each other compared with the maxima of $\rho(0)$ and $\rho(1)$. This suggests a rapid increase of the correlation length in the region $\beta_1 \leqslant \beta \leqslant \beta_2$ in support of our previous considerations.

Relying on the given interpretation our data relate the glue ball mass $m_g = 1/\xi$ to the string tension, which was first measured by Creutz [3,4]. For large β the correlation length ξ is estimated to be in or near the dashed region of fig. 2. Here we have taken into account an onset of the asymptotic weak coupling behaviour of the correlation for $2.0 < \beta < 2.2$ and that some unknown crossover effect for $1.8 < \beta \leqslant 2.0$ would additionally enhance the value of ξ. Comparison of the values of m_g from fig. 2 and of $\sqrt{K}$ from fig. 1 (e.g. at $\beta = 2.5$) leads to

$$m_g = (3.7 + 1.2)\sqrt{K} . \tag{9}$$

In comparison the upper bound of the correlation length gives $m_g > 1.8\sqrt{K}$ (taking the mean value for $\sqrt{K}$).

The string tension sets the scale of the lattice cutoff Λ^L (cf. e.g. ref. [3]). Using the relation of Hasenfratz and Hasenfratz [9] between the conventional scale Λ^M in momentum space and Λ^L all quantities can of course easily be expressed in terms of Λ^M. For SU(3) this would be of phenomenological interest. Creutz [4] (also [6]) determines the string tension for the SU(3) theory by using values for $\chi(2, 2)$ and one value for $\chi(3, 3)$. If we take this value for $\chi(3, 3)$ seriously we obtain $m_g > 0.5$ $[a^{-1}]$ for $\sqrt{K} \approx 0.25$ $[a^{-1}]$ and therefore the lower bound $m_g > 2\sqrt{K}$. With the value (used by Creutz [4]) $\sqrt{K} \approx 400$ MeV this gives $m_g \gtrsim 800$ MeV.

In summary our data for SU(2) strongly indicate relation (9). Several assumptions concerning the interpretation are, however, involved. A measurement of $\rho(3)$ and $\rho(4)$ would presumably remove these ambiguities. This would require computer programs running faster by a factor 10^2 [for $\rho(3)$] or even 10^4 [for $\rho(4)$]. A factor 100 seems to be within the possibilities of computer engineering nowadays. This would also allow reliable data on SU(3) to be obtained, which decide the question, whether an onset of the asymptotic low temperature behaviour at a correlation length $\xi < 1a$ is also true in that physically interesting case.

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Note added. After submission of this paper I have learned that similar calculations were done by G. Bhanot and C. Rebbi.

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