Nuclear Physics B190[FS3] (1981) 61–92

COMPLETE S-MATRIX OF THE O(2N) GROSS-NEVEU MODEL

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Received 12 November 1980

We present the complete S-matrix of the O(2N) Gross-Neveu model including kinks, elementary fermions, and higher bound states. In addition to the S-matrix factorization, unitarity, and crossing conditions we make essential use of constraints which follow from the fact that particles in the spectrum are bound states of each other. A consistent solution can only be obtained if the kinks obey generalized statistics. Remarkably, some quantities related to this, such as "spins" and Klein factors, show Bott periodicity.

1. Introduction

In 1974 Gross and Neveu (GN) investigated a two-dimensional model of U(N) spinor fields which displays dynamical symmetry breaking and asymptotic freedom [1]. In terms of Majorana fields one obtains a lagrangian which actually has isotopic O(2N) symmetry [2]

$$\mathcal{L}_{\mathrm{GN}} = \sum_{a=1}^{2N} \bar{\psi}_a i\not{\partial}\psi_a + \tfrac{1}{2}g^2\left(\sum_{a=1}^{2N} \bar{\psi}_a\psi_a\right)^2 . \tag{1}$$

The spectrum of the O(2N) GN model has been analyzed with semiclassical methods [2]. It contains elementary fermions and many bound states of them. Moreover, there are also kinks [3], whose existence is related to the spontaneous chiral symmetry breaking.

Amazingly, the GN model also showed other unanticipated properties. It was found in $1/N$ expansion that there is no particle production [4]. This is related to the existence of infinitely many (non-local) conservation laws which imply soliton behaviour, an essential property for the determination of the scattering matrix [5].

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The S-matrix of the elementary fermions and their lowest bound states has been found by A. and Al. Zamolodchikov [4]. They used the procedure which is generally applicable to models with soliton behaviour. Factorization, unitarity, and crossing together with some minimality assumptions allow one to derive the S-matrix explicitly.

Witten proposed that the kinks are isospinors [6]. The factorization equations have been derived by Shankar and Witten for the kink-fermion and kink-kink scattering amplitudes [7]. But since the number of amplitudes grows with N and because of complicated unitarity and crossing relations they did not calculate the S-matrix.

In this paper we take advantage of additional constraints, which arise from the fact that particles in the spectrum are bound states of each other, to determine the complete S-matrix of the GN model. Some of our results have been reported previously [8].

We shall show in this paper that the elementary fermions and all their bound states may also be considered as kink-kink bound states. Their spectrum is depicted in fig. 1 which is taken from ref. [2]. We denote these bound states collectively by $b_n^{(r)}$. Here $n = 1,2,\dots,N-2$ is the "principle quantum number" which determines the mass:

$$m_n = 2m\sin\left(\frac{\pi n}{2\lambda}\right), \tag{2}$$

where

$$\lambda = N - 1, \tag{3}$$

and m is the mass of the kinks. Furthermore, $r = n, n-2, n-4, \dots, \geqslant 0$ is the rank, i.e. these bound states are antisymmetric tensors which transform according to the rth fundamental representation of the isospin group. The odd-rank tensors correspond to fermions while the even-rank tensors correspond to bosons. In particular, the lightest of these bound states are identical to the elementary fermions of the GN model.

Note that the mass formula (2) is identical to that in the sine-Gordon (SG) theory, alias the massive Thirring (MT) model [9]. In contrast to the SG theory, however,

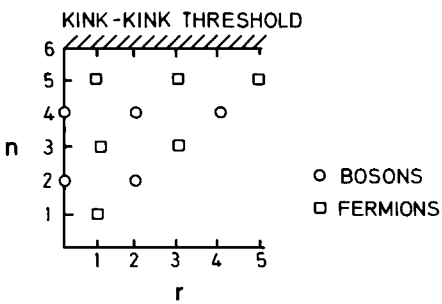


Fig. 1. Kink-kink bound states $b_n^{(r)}$ for $N = 7$.

where the parameter λ is a function of the coupling constant and varies continuously, in the GN model λ takes only integer values, cf. eq. (3). Despite this difference there exists a striking similarity between these models [2]. We shall see below that, apart from the statistics, the kinks of the GN model play the role of the solitons of the SG theory (or the elementary fermions in the MT model). Similarly, the bound states (both fermions and bosons) of the GN model correspond to the breathers of the SG theory. These connections will become evident from the fact that the S-matrix elements of the GN model can be conveniently written as a product of two terms. One factor (which contains the necessary bound states poles) is the corresponding SG S-matrix, while the other incorporates the dependence on the symmetry group.

This paper is organized as follows. In sect. 2 we recollect some well-known facts about the groups $O(2N)$ which determine the algebraic properties of the particles in the spectrum of the GN model. As a result we obtain algebraic rules, which particles may be considered as bound states of others. In sect. 3 we explain the concept of generalized statistics [10] and make a proposal for the "spins" of the kinks. At the cost of introducing certain Klein factors, the required ordinary statistics is obtained for the kink-kink bound states. The S-matrix of particles with generalized statistics contains, however, non-analytic phase factors. Following Köberle, Kurak and Swieca [11], we therefore introduce for all "physical" particles, corresponding auxiliary particles which have an S-matrix with the usual analytic behaviour. In sect. 4 we present our proposal for the kink-kink S-matrix and show its consistency with factorization, unitarity, crossing and bound-state relations and the assumed statistics. At this point it becomes clear that we had to pay special attention to the signs due to Klein factors since they crucially enter into the self-consistency conditions. Several technical details including the determination of the complete S-matrix from some of the consistency relations are relegated to the appendix. In sect. 5 we discuss some examples in more detail. The S-matrices of the GN models with O(4) and O(6) symmetry are shown to be related to the chiral $SU(N)$ theories and the O(8) model displays an extra symmetry (triality) between the kinks and elementary fermions. Finally, in sect. 6 we summarize our conclusions.

2. Group theoretical facts about $O(2N)$

In this section we recollect some well-known facts about the fundamental representations of the orthogonal groups $O(2N)$ for $N \geqslant 2$ [12]. We want to clarify some group theoretical aspects and fix the terminology before we plunge into the discussion of dynamical properties of the GN model*.

* This section may be omitted provided the reader is willing to accept statements (i)–(iii) at its very end.

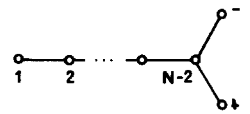


Fig. 2. Dynkin diagram for O(2N).

The group O(2N) has rank N, i.e., its Lie algebra has N simple roots. The Dynkin diagram is given in fig. 2. To every simple root there corresponds a fundamental representation. The group O(2N) has $N-2$ fundamental tensor representations ρ_r, i.e. representations by antisymmetric tensors of rank r ($r=1,2,\dots,N-2$). In addition it has two fundamental spinor representations ρ_+ and ρ_-. The representations corresponding to the endpoints of the branches of the Dynkin diagram, i.e. the vector representation ρ_1 and the fundamental spinor representations ρ_+ and ρ_-, are also called elementary representations. We shall see below that all other representations can be constructed from them. This fact will be very important for the determination of the S-matrix of the GN model. For convenience we denote the trivial (scalar) representation of O(2N) by ρ_0. It does not correspond to a simple root and is therefore not drawn in the Dynkin diagram.

Note the intimate connection between the spectrum of the GN model, fig. 1, and the Dynkin diagram fig. 2. In the GN model there exist exclusively such one-particle states which transform according to one of the fundamental (or the trivial) representations of O(2N).

2.1. SPINOR REPRESENTATIONS

In the following we want to discuss the elementary spinor representations in some detail. Although their properties can be derived from general considerations we prefer to think in terms of explicit matrix representations. This is not only for the sake of clarity, but these representations have a direct interpretation in the bosonized version of the GN model [7] as will become clear soon.

Explicit matrix representations of the spinor representations of O(2N) can be conveniently obtained from tensor products of the two-dimensional representation of SU(2). Let, for $n=1,2,\dots,N$,

$$\sigma_n^1,\ \sigma_n^2,\ \sigma_n^3$$

be SU(2) generators which act on different two-dimensional spaces

$$\sigma_n^a|\alpha_1\cdots\alpha_n\cdots\alpha_N\rangle=|\alpha_1\cdots\beta_n\cdots\alpha_N\rangle(\sigma^a)_{\beta_n\alpha_n},$$

where $(\sigma^a)_{\beta\alpha}$ are the Pauli matrices. The 2^N linearly independent states of the tensor product space may then be labelled by weight vectors

$$\alpha=\sum_{n=1}^{N}\alpha_n e_n$$

with components $\alpha_n = \pm\frac{1}{2}$ with respect to N orthogonal unit vectors e_n. The weight vectors α characterize the kinks of the GN model in the following sense [7]. In the bosonized version of the GN model the $2N$ Majorana fields are replaced by N Bose fields $\varphi_1,\dots,\varphi_N$. Then a static kink corresponds to a configuration with

$$\varphi_n(x=+\infty)-\varphi_n(x=-\infty)=\alpha_n\sqrt{\pi}, \tag{4}$$

for $n=1,2,\dots,N$.

We now consider the following $2N$ hermitian operators:

$$\gamma_{2n-1}=\prod_{k=1}^{n-1}\sigma_k^3\cdot\sigma_n^1, \tag{5a}$$

$$\gamma_{2n}=\prod_{k=1}^{n-1}\sigma_k^3\cdot\sigma_n^2, \tag{5b}$$

which are, respectively, real and symmetric or imaginary and antisymmetric. These operators form a Clifford algebra

$$\{\gamma_a,\gamma_b\}=2\delta_{ab}, \qquad (a,b=1,2,\dots,2N). \tag{6}$$

It is well known that these products of Pauli matrices (generalized γ-matrices) form an irreducible representation of the Clifford algebra. It is also well known that the commutators of Clifford operators are generators of orthogonal groups

$$M_{ab}=\tfrac{1}{2}(\gamma_a\gamma_b-\gamma_b\gamma_a). \tag{7}$$

This representation of O($2N$) in terms of γ-matrices, however, is reducible, which is seen as follows. There exists a hermitian operator

$$\Gamma\equiv(-i)^N\prod_{a=1}^{2N}\gamma_a=\prod_{n=1}^{N}\sigma_n^3 \tag{8}$$

which anticommutes with all generators of the Clifford algebra

$$\{\Gamma,\gamma_a\}=0, \qquad (a=1,2,\dots,2N). \tag{9}$$

It therefore commutes with all generators M_{ab} of O($2N$). Since

$$\Gamma^2=1, \tag{10}$$

its eigenvalues are ±1. In fact, in our representation Γ is already diagonal, cf. eq. (8). Let α^+ (α^-) be a weight vector with an even (odd) number of negative components, then

$$\Gamma|\alpha^\pm\rangle=\pm|\alpha^\pm\rangle, \tag{11}$$

and the combinations

$$\Gamma_{\pm} = \tfrac{1}{2}(1 \pm \Gamma) = (\Gamma_{\pm})^{\mathrm{T}} \tag{12}$$

are projection operators onto the subspaces of states with positive or negative (isotopic) chirality. Because of the existence of the operator Γ the group $O(2N)$ has two irreducible fundamental spinor representations, ρ_- which acts on the states $|\alpha^-\rangle$ with negative chirality and ρ_+ which acts on the states $|\alpha^+\rangle$ with positive chirality. The dimension of $\rho_\pm$ is 2^{N-1}. In the $O(2N)$ GN model the weight vectors α^+ (α^-) with an even (odd) number of negative components correspond to right-handed (left-handed) kinks.

Next we want to determine whether the spinor representations are real or complex. To this end we consider the unitary charge conjugation operator C with the property

$$C\gamma_a C^{-1} = \gamma_a^{\mathrm{T}} = \gamma_a^*, \tag{13}$$

which implies

$$CM_{ab}C^{-1} = M_{ab}^*. \tag{14}$$

In our representation C is given by

$$C = \prod_{n=1}^{N} (i\gamma_{2n}) = \prod_{k=1}^{N/2} \left(-i\sigma_{2k-1}^1 \sigma_{2k}^2\right) \tag{15a}$$

for even N and

$$C = \prod_{n=1}^{N} \gamma_{2n-1} = \sigma_1^1 \prod_{k=1}^{(N-1)/2} \left(-i\sigma_{2k}^2 \sigma_{2k+1}^1\right) \tag{15b}$$

for odd N. One easily proves that

$$C^{\mathrm{T}} = (-1)^{[N/2]} C \tag{16}$$

($[a]$ is the largest integer contained in a). Therefore C is antisymmetric if N is equal to 2 or 3 mod 4, and C is symmetric for N equal to 0 or 1 mod 4 (recall that we are considering only $N \geqslant 2$). From eqs. (15a, b) it is also evident that in our representation the operator C, being a tensor product of antidiagonal operators, is antidiagonal, too. It maps a state $|\alpha\rangle$ onto a state with the opposite weight vector $|-\alpha\rangle$. This means in particular that for even (odd) N the charge conjugation operator C does not change (changes) the chirality of a state

$$\Gamma C = (-1)^N C\Gamma. \tag{17}$$

TABLE 1
Periodicity of "spins", Klein factors, etc. in the O(2N) GN models for $N \geq 2$

N mod 8	$s_{++}=s_{--}$	s_{+-}	$s_{+\mathrm{F}}$	$s_{-\mathrm{F}}$	K'_+	K'_-	$K''_\pm$	$\varepsilon_{\mathrm{odd},N}$	$\varepsilon_{\mathrm{even},N}$	$\rho_\pm$
0,4	$0, \frac{1}{2}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	K_+	K_-	$K_\mp$	$-$	$+$	orthogonal
1,5	$\frac{1}{8}, \frac{5}{8}$	$\frac{3}{8}, \frac{7}{8}$	$\frac{1}{4}$	$\frac{3}{4}$	K	1	K	$-$	$-$	complex
2,6	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{2}, 0$	$\frac{3}{4}$	$\frac{3}{4}$	K_-	K_+	$K_\pm$	$+$	$-$	symplectic
3,7	$\frac{3}{8}, \frac{7}{8}$	$\frac{5}{8}, \frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{4}$	1	K	1	$+$	$+$	complex

Now eq. (14) states that our (reducible) representation of the group O(2N) is equivalent to its complex conjugate. If we project this equation with $\Gamma_\pm$ onto the irreducible subspaces, it implies in view of eq. (17) the following reality properties of the elementary spinor representations ρ_+ and ρ_-. Since for odd N the charge conjugation operator C changes the chirality of the states, ρ_+ is equivalent to ρ^*_-, $\rho_+ \sim \rho^*_-$ and vice versa. Therefore ρ_+ and ρ_- are complex for odd $N \geq 3$. On the other hand, for even N the operator C does not change the chirality, ρ_+ and ρ_- are real, $\rho_\pm \sim \rho^*_\pm$. More precisely, ρ_+ and ρ_- are real-orthogonal for $N = 0$ mod 4 (with C symmetric) but they are real-symplectic for $N = 2$ mod 4 (with C antisymmetric).

We shall see in sects. 3 and 4 that this pattern (Bott periodicity) is reflected in the S-matrices of the O(2N) GN models, cf. table 1.

In the context of the GN model, the above statements about the elementary spinor representations may be rephrased as follows. For odd N, the antiparticles of right-handed kinks are left-handed kinks and vice versa. Therefore for odd N it makes sense to call, e.g. the right-handed kinks simply kinks and the left-handed ones antikinks. On the other hand, for even N the antiparticles of right-handed kinks are again right-handed kinks and similarly for left-handed ones. Therefore in this case the chirality cannot distinguish kinks from antikinks; they are in the same representation.

2.2. CLEBSCH-GORDAN SERIES

We want to recall some facts about the Clebsch-Gordan series of the Kronecker product of fundamental representations of O(2N) [12]. We are interested to know which fundamental representations occur in the product of two fundamental representations. This knowledge will provide for the group theoretical basis of the various schemes in which the particles of the GN model may be considered as bound states of each other.

Obviously, the product of antisymmetric tensors of rank r and rank s contains an antisymmetric tensor of rank $r + s$,

$$\rho_r \otimes \rho_s \supset \rho_{r+s}. \tag{18}$$

In particular, the rank-r tensor representation is contained in the r-fold Kronecker product of the vector representation

$$\rho_r \subset (\rho_1)^r. \tag{19}$$

Next, the product of an elementary spinor representation and an even (odd)-rank tensor representation contains a spinor representation of the same (opposite) chirality,

$$\rho_\pm \otimes \rho_{2k} \supset \rho_\pm, \tag{20a}$$

$$\rho_\pm \otimes \rho_{2k+1} \supset \rho_\mp. \tag{20b}$$

In particular, a spinor representation of given chirality is contained in the product of the spinor representation of opposite chirality and a vector representation,

$$\rho_\pm \subset \rho_\mp \otimes \rho_1. \tag{21}$$

Thirdly, the product of two elementary spinor representations of the same (opposite) chirality contains every other fundamental tensor representation; more precisely,

$$\rho_\pm \otimes \rho_\pm \supset \rho_{N-2k}, \tag{22a}$$

$$\rho_\pm \otimes \rho_\mp \supset \rho_{N-1-2k}. \tag{22b}$$

This implies for even N that the even-rank tensor representations are contained in the products $\rho_\pm \otimes \rho_\pm$ but the odd-rank tensors are in the mixed product $\rho_+ \otimes \rho_-$, and vice versa for odd N. Now for even N the spinor representations are real, $\rho_\pm \sim \rho^*_\pm$, while for odd N they are complex, $\rho_\pm \sim \rho^*_\mp$. Therefore, the above statements can be reformulated in a more concise way which holds for any N;

$$\rho_{2k} \subset \rho_\pm \otimes \rho^*_\pm, \tag{23a}$$

$$\rho_{2k+1} \subset \rho_\pm \otimes \rho^*_\mp, \tag{23b}$$

in particular,

$$\rho_1 \subset \rho_\pm \otimes \rho^*_\mp. \tag{24}$$

We note in passing that eqs. (19) and (22) convincingly justify why ρ_1, ρ_+, and ρ_- are called elementary representations.

We already mentioned in subsect. 2.1 that in weight space the kinks of the O($2N$) GN model are represented by weight vectors with components $\pm\frac{1}{2}$. In the same sense, the elementary fermions correspond to the unit vectors $\pm e_n$. This means that in the bosonic representation the elementary fermions are in configurations whose asymptotic values differ by $\pm\sqrt{\pi}$ in exactly one component [7]. Obviously the weight vector of an elementary fermion may be obtained by the superposition of the weight vectors of two kinks which cancel in all but one component. Similarly the rank-r tensor states $b^{(r)}$ are represented by weight vectors with exactly r components equal to ± 1.

In sects. 3 and 4 we shall make particular use of eqs. (19), (24) and (21). They suggest the following properties of the O($2N$) GN model.

(i) The tensor states may be considered as bound states of elementary fermions.

(ii) The elementary fermions are kink-kink bound states. More precisely, for odd N they are bound states of two right-handed or two left-handed kinks, but for even N they are obtained from a right-handed and a left-handed kink.

(iii) A kink may be regarded as a bound state of a kink with opposite chirality and an elementary fermion.

Property (i) has already been used for the determination of the S-matrix of the elementary fermions and their lowest bound states [4], and (ii) has been discussed in ref. [7]. The third property is the essential new ingredient which enables us to calculate the kink-kink S-matrix [8].

3. Statistics

The kinks are the fundamental particles of the GN model in the same sense as the solitons are the fundamental particles of the SG theory. The elementary fermions in terms of which the GN model was originally formulated [1,2] are O($2N$) isovectors. They are kink-kink bound states quite analogously as the lowest SG breather (the elementary SG field) is a soliton-antisoliton bound state. It follows from property (ii) that for odd N the fermions are bound states of two right-handed or of two left-handed kinks. This implies that, at least for odd N, the kinks can be neither bosons nor fermions. They must obey generalized statistics [10].

3.1. GENERALIZED STATISTICS

In contrast to higher dimensions it is possible in two space-time dimensions that quantum fields have generalized bilinear commutation relations [10]

$$\phi_1(t,x)\phi_2(t,y)=\phi_2(t,y)\phi_1(t,x)\,\mathrm{e}^{2\pi i s_{12}\varepsilon(x-y)}, \tag{25}$$

where the generalized "spin" $s_{12}=s_{21}$ is a parameter (defined only modulo 1) which can be an arbitrary real number. Obviously, the usual commutation relations for bosons or fermions are obtained for $s=0$ and $s=\frac{1}{2}$ (mod 1), respectively, and the "relative spin" for particles of different types may be changed by units of $\frac{1}{2}$ by application of a Klein factor.

We propose the following values for the "spins" of the right-handed (+) and left-handed (−) kinks,

$$s_{++}=s_{--}=\tfrac{1}{8}N, \tag{26a}$$

$$s_{+-}=\tfrac{1}{8}N+\tfrac{1}{4}. \tag{26b}$$

This choice is motivated by four conditions:

(a) Kink-kink bound states corresponding to the fundamental representations ρ_r with odd (even) rank r are fermions (bosons), as required by the known particle spectrum [2].

(b) For $N=2,3,4$ the "spins" $s_{++}=s_{--}$ can be determined from the local isomorphisms $O(4)\simeq SU(2)\otimes SU(2)$, $O(6)\simeq SU(4)$, and the triality in O(8), cf. fig. 3. We show in sect. 5 that the isomorphisms correspond to model identities: the O(4) GN S-matrix is equal to a product of two chiral SU(2) S-matrices and the O(6) GN S-matrix is equal to the S-matrix of the chiral SU(4) model [11]. In the chiral SU(N) model, the "spin" of the fundamental particles is $s=\frac{1}{2}(1-1/N)$. Therefore, the "spins" are $\frac{1}{4}$ and $\frac{3}{8}$ for O(4) and O(6), respectively. On the other hand, for O(8) there exists a triality symmetry between the three elementary representations [12] and correspondingly between the fermions, the right-handed and the left-handed kinks [13]. This suggests for O(8), $s_{FF}=s_{++}=s_{--}=\frac{1}{2}$.

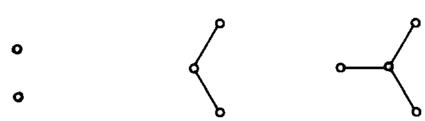
(c) We want a "smooth" N-behaviour of the "spin".

(d) The complete S-matrix of the O($2N$) GN model which we derive below is consistent with the assignments of eq. (26).

Since the "spins" are defined only modulo 1, eq. (26) implies a similar cyclic pattern as the Bott periodicity, cf. table 1. We shall see in sect. 4 that this periodicity shows up in the explicit S-matrix, too. The above choice of the "spins" turns out to be crucial for the self-consistency of our proposal.

Let us now give an algebraic formulation of the fact that elementary fermions are bound states of kinks. To this end we introduce fermion fields $\hat\psi_a$ $(a=1,2,\dots,2N)$ and "kink fields" $\hat\chi_\alpha$ $(\alpha=1,2,\dots,2^N)$. We do not pretend to have a well defined quantum field theoretic description of the kinks. This, however, is also not necessary for our purpose. We need the kink operators only "on the mass shell". We may write

$$\hat\psi_a(x)=\mathcal{N}[\hat\chi_\alpha\hat\chi_\beta](x)(\Gamma_+\gamma_a C)_{\alpha\beta}, \tag{27a}$$



a) b) c)

Fig. 3. Dynkin diagrams for (a) $O(4)\simeq SU(2)\otimes SU(2)$, (b) $O(6)\simeq SU(4)$, (c) O(8).

where $\mathfrak{N}$ is a suitable normal product prescription. The projection operator Γ_+ in the Clebsch-Gordan coefficient implies, as required by property (ii) stated at the end of sect. 2, that for odd N the elementary fermion is a bound state of two right-handed kinks whereas for even N it is a bound state of a right-handed and a left-handed kink. Alternatively, we may also express the fermions in terms of kinks with the opposite chiralities,

$$\hat{\psi}_a(x) = K^N \mathfrak{N}[\hat{\chi}_\alpha \hat{\chi}_\beta](x)(\Gamma_- \gamma_a C)_{\alpha\beta}. \tag{27b}$$

Here K is a Klein factor for kink fields,

$$K\hat{\chi}K = -\hat{\chi}, \qquad K^2 = K^+ K = 1, \tag{28}$$

which is necessary in order to have the same commutation relations of $\hat{\chi}$ with $\hat{\psi}$ as defined by eq. (27a) and by eq. (27b). For even N eqs. (27a) and (27b) are identical. One easily checks that with the choice (26) the $\hat{\psi}_a$ are indeed fermion fields which anticommute with each other, and the commutation relations between kinks and fermions are given by eq. (25) with

$$s_{+\mathrm{F}} = \frac{1}{4} + \frac{1}{2}\left[\frac{N}{2}\right],$$

$$s_{-\mathrm{F}} = \frac{1}{4} + \frac{1}{2}\left[\frac{N+1}{2}\right] \tag{29}$$

modulo 1. Note that $s_{-\mathrm{F}} = s_{+\mathrm{F}}$ for even N, but $s_{-\mathrm{F}} = s_{+\mathrm{F}} \pm \frac{1}{2}$ for odd N, cf. table 1.

In the SG theory, the soliton can be understood in the semiclassical limit as a coherent state of lowest breathers. In the quantum model this means that the soliton is a bound state of itself and a breather [8]. It is also known that in the semiclassical approximation a kink of the GN model may have a fermion "trapped" in it [2]. That is why we assume that a kink is a bound state of a kink and an elementary fermion, and in agreement with property (iii) there must be a relation of the form

$$\hat{\chi}^{\pm}_{\alpha}(x) = K'_{\pm} \mathfrak{N}[\hat{\psi}_a \hat{\chi}^{\mp}_{\beta}](x)(\gamma_a)_{\beta\alpha}, \tag{30}$$

where $\hat{\chi}^{\pm}_{\alpha} = (\Gamma_{\pm}\hat{\chi})_\alpha$. The Klein factors $K'_{\pm}$ can be determined from the requirement that both sides of eq. (30) have the same commutation relations with $\hat{\chi}^{\pm}_{\gamma}(y)$. One obtains, cf. table 1,

$$K'_+ = KK'_- = K_+^{[(N+2)/2]} K_-^{[(N+1)/2]}, \tag{31}$$

where

$$\hat{\chi}^{\pm}=K_{\mp}\hat{\chi}^{\pm}K_{\mp}=-K_{\pm}\hat{\chi}^{\pm}K_{\pm}, \tag{32}$$

$$K=K_{+}K_{-}. \tag{33}$$

So far we have obtained an admittedly formal version of the bound-state conditions which, however, respects the generalized commutation relations (25). In order to be able to derive the dynamical content of these equations (which is done in sect. 4) it is essential to disentangle the questions of particle statistics from the dynamical problems.

3.2. AUXILIARY PARTICLES

Following ref. [11] we consider the asymptotic creation operators $\hat{a}_A^{+\mathrm{in}}(\theta)$ and $\hat{a}_A^{+\mathrm{out}}(\theta)$ of the physical particles (the index A may be α for kinks and a for fermions, and θ is the usual rapidity variable: $p^1=m_A \operatorname{sh}\theta$). Then the commutation relations (25) imply

$$\hat{a}_A^{+\,\substack{\mathrm{in}\\ \mathrm{out}}}(\theta)\hat{a}_B^{+\,\substack{\mathrm{in}\\ \mathrm{out}}}(\theta')=\hat{a}_B^{+\,\substack{\mathrm{in}\\ \mathrm{out}}}(\theta')\hat{a}_A^{+\,\substack{\mathrm{in}\\ \mathrm{out}}}(\theta)\exp\{\mp 2\pi i s_{AB}\varepsilon(\theta-\theta')\}. \tag{34}$$

A solution of eq. (34) may be obtained in terms of auxiliary bosonic creation operators $a_A^{+\,\substack{\mathrm{in}\\ \mathrm{out}}}(\theta)$, namely

$$\hat{a}_A^{+\mathrm{in}}(\theta)=a_A^{+\mathrm{in}}(\theta)\exp\left\{2\pi i\int_{\theta}^{\infty}\mathrm{d}\theta' s_{AB}N_B^{\mathrm{in}}(\theta')\right\},$$

$$\hat{a}_A^{+\mathrm{out}}(\theta)=a_A^{+\mathrm{out}}(\theta)\exp\left\{2\pi i\int_{-\infty}^{\theta}\mathrm{d}\theta' s_{AB}N_B^{\mathrm{out}}(\theta')\right\}, \tag{35}$$

where $N_A=a_A^{+}a_A$.

We now assume, again following ref. [11], that the S-matrix of the auxiliary particles has the usual analytic behaviour. It follows from eqs. (35) that it coincides with that of the physical particles for a particular ordering of the creation and annihilation operators, namely

$$\langle 0|a_1^{\mathrm{out}}(\theta_1')\cdots a_n^{\mathrm{out}}(\theta_n')a_1^{+\mathrm{in}}(\theta_1)\cdots a_n^{+\mathrm{in}}(\theta_n)|0\rangle$$

$$=\langle 0|\hat{a}_1^{\mathrm{out}}(\theta_1')\cdots \hat{a}_n^{\mathrm{out}}(\theta_n')\hat{a}_1^{+\mathrm{in}}(\theta_1)\cdots \hat{a}_n^{+\mathrm{in}}(\theta_n)|0\rangle$$

$$\text{if } \theta_1'>\theta_2'>\cdots>\theta_n' \quad \text{and} \quad \theta_1>\theta_2>\cdots>\theta_n. \tag{36}$$

This implies for the two-particle S-matrix elements which are defined by

$$S|A(\theta_1)B(\theta_2)\rangle^{\mathrm{in}} = |C(\theta_1)D(\theta_2)\rangle^{\mathrm{in}}\ {}_{CD}S_{AB}(\theta_1-\theta_2),$$

$$S|\hat{A}(\theta_1)\hat{B}(\theta_2)\rangle^{\mathrm{in}} = |\hat{C}(\theta_1)\hat{D}(\theta_2)\rangle^{\mathrm{in}}\ {}_{CD}\hat{S}_{AB}(\theta_1-\theta_2), \tag{37}$$

the relation

$${}_{CD}S_{AB}(\theta) = {}_{CD}\hat{S}_{AB}(\theta)\exp\{2\pi i s_{CD}\varepsilon(\theta)\}. \tag{38}$$

Therefore, the physical S-matrix, $\hat{S}$, has a non-analytic dependence on the rapidity difference unless the "spin" s_{CD} is integer or half integer. In particular, we obtain for the scattering matrix of physical fermions ${}_{cd}\hat{S}_{ab}$ calculated in ref. [4] and the corresponding one for the auxiliary particles the relation

$${}_{cd}\hat{S}_{ab} = -\,{}_{cd}S_{ab}. \tag{39}$$

It is important to notice that the GN model in terms of the physical particles differs in some aspects from the model in terms of the auxiliary particles. Of course, in contrast to the physical particles, the auxiliary particles by definition have Bose statistics throughout. But there is also a higher degeneracy in the spectrum of the auxiliary particles than for the physical particles, cf. fig. 1. Let us give an example: for odd N, the bound state of two left-handed auxiliary kinks is not identical to that of two right-handed ones since their S-matrix elements for the scattering with other auxiliary particles differ in some cases by a sign. This is due to the Klein factor in eq. (27b) which may change the "spin" in eq. (38) by $\frac{1}{2}$. This extra degeneracy is discussed in more detail in appendix A.

4. Two-particle S-matrix

In this section we first present our proposal for the kink-kink S-matrix. The S-matrices for fermion-kink and for fermion-fermion scattering can be derived from it by the usual bound-state method [9, 14]. Then we show the consistency of these S-matrices with the factorization, unitarity, crossing and bound-state conditions. Finally, we also give the two-particle S-matrix for the scattering of higher bound states with elementary fermions.

4.1. KINKS AND ELEMENTARY FERMIONS

The two-particle S-matrix for auxiliary kinks has the following decomposition into invariant amplitudes [7]

$${}_{\gamma\delta}S_{\alpha\beta}(\theta) = 2^{-N}\sum_{r=0}^{2N}\frac{1}{r!}u_r(\theta)\sigma^{(r)}_{\gamma\beta}\sigma^{(r)}_{\delta\alpha}, \tag{40}$$

where

$$\sigma_{\alpha\beta}^{(r)} \equiv \sigma_{\alpha\beta}^{a_1\cdots a_r} = \frac{1}{r!} \sum_{\pi\in S_r} \operatorname{sign}(\pi)(\gamma^{a_{\pi_1}}\cdots\gamma^{a_{\pi_r}})_{\alpha\beta}$$

is an antisymmetric rank-r O($2N$) tensor, and by $\sigma_{\gamma\beta}^{(r)}\sigma_{\delta\alpha}^{(r)}$ we mean the contraction for fixed r, $\sigma_{\gamma\beta}^{a_1\cdots a_r}\sigma_{\delta\alpha}^{a_1\cdots a_r}$.

There exists a "conservation law for (isotopic) chirality" [6] which forbids the reflection of chirality, e.g.

$$(\Gamma_-)_{\gamma\gamma'\gamma'\delta} S_{\alpha'\beta}(\Gamma_+)_{\alpha'\beta} = 0 .$$

This implies the relations [7]

$$u_{N+r}(\theta) = (-1)^r u_{N-r}(\theta) . \tag{41}$$

Moreover, the amplitudes u_r with even (odd) index r contribute only to the scattering of kinks with the same (opposite) chirality.

We propose the invariant amplitudes to be given by

$$u_r(\theta) = -u^{\mathrm{SG}}(\theta)\exp\left\{\int_0^\infty \frac{dx}{x}\left[\frac{e^{-x(1+1/\lambda)}}{\operatorname{sh}(x/\lambda)}(1-e^{xr/\lambda}) + \frac{e^{-x(1+1/2\lambda)}\operatorname{sh}\frac{1}{2}x(1+1/\lambda)}{\operatorname{ch}\frac{1}{2}x\operatorname{sh}(x/\lambda)}\right]\operatorname{sh}\left(x\left(1-\frac{\theta}{i\pi}\right)\right)\right\} \tag{42a}$$

for $r \leqslant N$, and those for $N < r \leqslant 2N$ are obtained by eq. (41). For convenience we have in eq. (42a) split off the SG amplitude for the scattering of two solitons [9]:

$$u^{\mathrm{SG}}(\theta) = \exp\left\{\int_0^\infty \frac{dx}{x}\frac{\operatorname{sh}\left(\frac{1}{2}x(1-1/\lambda)\right)}{\operatorname{ch}\frac{1}{2}x\operatorname{sh}(x/2\lambda)}\operatorname{sh}\left(x\frac{\theta}{i\pi}\right)\right\}, \tag{42b}$$

for the special value of the coupling constant, $\lambda = N-1$. Note that the above exponential representations are valid only for values of θ such that the integrands do not increase exponentially for $x \to \infty$. In other regions of the complex θ-plane, the functions are defined by analytic continuation. The functions u^{SG} and u_r may be expressed in terms of (infinite) products of Γ-functions.

Since the elementary fermions may be considered as kink-kink bound states we can derive the S-matrices for fermion-kink and for fermion-fermion scattering from eqs. (40)–(42), cf. fig. 4. This is done explicitly in appendix B. We here only give the final results. The two-particle S-matrix for auxiliary fermions and kinks has the form

$${}_{b\beta}S_{a\alpha}(\theta) = (\Gamma^N)_{\beta\alpha}\delta^{ba}t_1(\theta) + (\Gamma^N\sigma^{ba})_{\beta\alpha}t_2(\theta), \tag{43}$$

Fig. 4. Fermion as kink-kink bound state in (a) fermion-kink scattering, (b) fermion-fermion scattering.

where the scattering amplitudes are given by

$$S_1 \equiv t_1 - t_2 = \frac{\theta - \frac{1}{2}i\pi(1+1/\lambda)}{\theta - \frac{1}{2}i\pi(1-1/\lambda)} S_2 \tag{44a}$$

and

$$\begin{aligned} S_2 &\equiv t_1 + (2N-1)t_2 \\ &= -\varepsilon_{1,N} S^{\mathrm{SG}}_{\mathrm{sol},b}(\theta) \exp\left\{\int_0^\infty \frac{\mathrm{d}x}{x} \frac{\mathrm{e}^{-x(1+1/2\lambda)}}{\mathrm{ch}\frac{1}{2}x} \mathrm{sh}\left(x\frac{\theta}{i\pi}\right)\right\}. \end{aligned} \tag{44b}$$

The SG amplitude for the scattering of a soliton and a lowest breather is [9]

$$S^{\mathrm{SG}}_{\mathrm{sol},b}(\theta) = \frac{\mathrm{sh}\theta + i\sin\left(\frac{1}{2}\pi(1+1/\lambda)\right)}{\mathrm{sh}\theta - i\sin\left(\frac{1}{2}\pi(1+1/\lambda)\right)}, \tag{44c}$$

and the sign $\varepsilon_{1,N}$ is a special case of the general expression, cf. table 1,

$$\varepsilon_{r,N} = (-1)^{(N/2+r)(N+1)}. \tag{44d}$$

Note that for odd N the S-matrices (43) for the interaction of auxiliary fermions with right-handed or left-handed auxiliary kinks differ by an overall sign. In the physical S-matrices, however, this sign difference is compensated, cf. eq. (38), since in this case the "spins" $s_{+\mathrm{F}}$ and $s_{-\mathrm{F}}$ differ by $\frac{1}{2}$, as was remarked after eq. (29).

The two-particle S-matrix for auxiliary fermions is

$${}_{cd}S_{ab}(\theta) = -\left\{\delta_{ab}\delta_{cd}\sigma_1(\theta) + \delta_{ac}\delta_{bd}\sigma_2(\theta) + \delta_{ad}\delta_{bc}\sigma_3(\theta)\right\}, \tag{45}$$

where the scattering amplitudes are given by

$$\begin{aligned} (S_0, S_+, S_-) &\equiv (2N\sigma_1 + \sigma_2 + \sigma_3, \sigma_2 + \sigma_3, \sigma_2 - \sigma_3) \\ &= \left(\frac{\theta - i\pi}{\theta + i\pi}, \frac{\lambda\theta - i\pi}{\lambda\theta + i\pi}, 1\right) S_-(\theta), \end{aligned} \tag{46a}$$

with

$$S_-(\theta)=S_{bb}^{\mathrm{SG}}(\theta)\exp\left\{-2\int_0^\infty\frac{\mathrm{d}x}{x}\,\frac{\mathrm{sh}(x/2\lambda)}{\mathrm{ch}\frac{1}{2}x}\,\mathrm{e}^{-x(1/2+1/2\lambda)}\mathrm{sh}\left(x\frac{\theta}{i\pi}\right)\right\}, \tag{46b}$$

and again we have extracted the SG amplitude for the scattering of two lowest breathers [9]:

$$S_{bb}^{\mathrm{SG}}(\theta)=\frac{\mathrm{sh}\,\theta+i\sin(\pi/\lambda)}{\mathrm{sh}\,\theta-i\sin(\pi/\lambda)}\,. \tag{46c}$$

Obviously, because of eq. (39), one obtains for the scattering of physical elementary fermions the S-matrix of ref. [4].

4.2. CONSISTENCY CONDITIONS

Now that we have listed the various two-particle amplitudes let us turn to the consistency relations which they have to fulfill. The factorization equations for kink-kink and kink-fermion [7] or fermion-fermion [4] scattering are easily checked:

$$u_{r+2}(\theta)=u_r(\theta)\frac{(1-r/\lambda)-(1-\theta/i\pi)}{(1-r/\lambda)+(1-\theta/i\pi)}, \tag{47a}$$

$$t_1(\theta)=2\lambda\left(\frac{\theta}{i\pi}-\tfrac{1}{2}\right)t_2(\theta), \tag{47b}$$

$$\sigma_2(\theta)=-\lambda\frac{\theta}{i\pi}\sigma_3(\theta). \tag{47c}$$

Similarly, unitarity for fermion-kink, fermion-fermion, and u-channel kink-kink scattering is obvious, since

$$|S_1(\theta)|=|S_2(\theta)|=|S_0(\theta)|$$

$$=|S_+(\theta)|=|S_-(\theta)|=|u_r(i\pi-\theta)|=1 \tag{48}$$

for real θ. The s-channel unitarity condition for kink-kink scattering is more cumbersome,

$$2^{-2N}\sum_{r,s}\binom{2N-t}{r}\binom{t}{s}u_{r+s}(-\theta)u_{t-s+r}(\theta)=\delta_{t,0},\qquad (t=0,1,\dots,2N). \tag{48a}$$

We have no general proof that eq. (42a) fulfills this relation, but we have verified it for special values of N and t, e.g. $N=2,3$ and $t=0,1,2N-1,2N$. In order to formulate the crossing relations, we have to be careful with the definition of antiparticles. First notice that with the shorthand notation

$$|\tilde{\alpha}\rangle \equiv |\beta\rangle C_{\alpha\beta},$$

where C is the charge conjugation matrix defined in eqs. (15a,b), the S-matrix fulfills the following relations:

$$_{\tilde{\alpha}\delta}S_{\tilde{\gamma}\beta}(i\pi-\theta) = {}_{\gamma\delta}S'_{\alpha\beta}(\theta) \equiv 2^{-N}\sum_r \frac{\varepsilon_{r,N}}{r!}u_r(\theta)\sigma^{(r)}_{\gamma\beta}\sigma^{(r)}_{\delta\alpha}, \tag{49a}$$

$$_{\tilde{\alpha}b}S_{\tilde{\beta}a}(i\pi-\theta) = -(-1)^N\ {}_{\beta b}S_{\alpha a}(\theta), \tag{49b}$$

$$_{\beta a}S_{\alpha b}(i\pi-\theta) = -\ {}_{\beta b}S_{\alpha a}(\theta), \tag{49c}$$

$$_{ad}S_{cb}(i\pi-\theta) = {}_{cd}S_{ab}(\theta), \tag{49d}$$

where $\varepsilon_{r,N}$ was defined previously (44d). Then eqs. (49) are the crossing relations provided we identify up to Klein factors the physical antiparticles $\bar{\hat{a}}$ and $\bar{\hat{\alpha}}$ with $\hat{a}$ and $\tilde{\alpha}$, respectively. More precisely, we must have

$$\hat{\psi}_{\bar{a}}(x) = K\hat{\psi}_a(x), \tag{50a}$$

$$\hat{\chi}^{\pm}_{\bar{\alpha}}(x) = K''_{\pm}\hat{\chi}^{\pm}_{\tilde{\alpha}}(x), \tag{50b}$$

where K is given by eqs. (32) and (33) and

$$K''_{\pm} = (K_{\pm})^{[(N+1)/2]}(K_{\mp})^{[(N+2)/2]}, \tag{51}$$

cf. table 1.

In the derivation of eq. (49a) we have used the identity

$$\sum_s f_{rs}u_s(\theta) = (-1)^{[r/2]}\sum_s \varepsilon_{s,N}f_{rs}u_s(i\pi-\theta), \tag{52}$$

where the coefficients f_{rs} arise in the Fierz transformation

$$\frac{1}{r!}\sigma^{(r)}_{\alpha\beta}\sigma^{(r)}_{\gamma\delta} = \sum_s f_{rs}\frac{1}{s!}\sigma^{(s)}_{\alpha\delta}\sigma^{(s)}_{\gamma\beta} \tag{53}$$

and are explicitly given by [15]

$$f_{rs} = (-1)^{[(r+s)/2]}2^{-N}\sum_t (-1)^t\binom{r}{t}\binom{2N-r}{s-t}. \tag{54}$$

We have no general proof of eq. (52), but it has been verified for $N=2,3$ with arbitrary θ and for general N with particular values of θ such as $\theta=0$, $i\pi$, $i\pi/\lambda$, etc.

Additional constraints on the two-particle S-matrix come from the bound-state relations. In the SG theory the soliton-antisoliton S-matrix has poles in the complex θ-plane which correspond to bound states, the breathers b_n, and there exist relations between various scattering amplitudes [9].

Similarly, the GN model has kink-kink bound states $b_n^{(r)}$ which transform according to the fundamental representations ρ_r of $O(2N)$ with the spectrum (2), cf. fig. 1. This implies that the amplitudes $u_r(\theta)$ have poles at

$$\theta_n = i\pi\frac{n}{\lambda}, \qquad \text{for } 0<n=r, r+2,\ldots<\lambda. \tag{55a}$$

In particular, the pole of $u_1(\theta)$ at $i\pi/\lambda$ corresponds to the elementary fermion as a kink-kink bound state as predicted by eq. (27). The condition (30) that a kink is a kink-fermion bound state requires a pole in the amplitude $S_2(\theta)$, cf. eq. (44), at

$$\theta=\tfrac{1}{2}i\pi(1+1/\lambda), \tag{55b}$$

and the poles in the amplitudes S_0 and S_-, cf. eq. (46), at

$$\theta=i\pi/\lambda \tag{55c}$$

describe the two-fermion bound states $b_2^{(0)}$ and $b_2^{(2)}$. The general bound-state scattering formula [14] (cf. appendix B) then gives relations between the scattering amplitudes. From the scattering of a fermion as a kink-kink bound state with a kink or a fermion, cf. fig. 4, we obtain, respectively, the conditions

$$t_1(\theta)=\varepsilon_{1,N}2^{-2N-1}\sum_r\binom{2N-1}{r}u_r(\theta_{13})u_{r+1}(i\pi-\theta_{23}) \tag{56a}$$

and

$$S_-(\theta)=4\lambda\left(1-\frac{\theta}{i\pi}\right)\left(1+\lambda\frac{\theta}{i\pi}\right)t_2(\theta_{13})t_2(\theta_{23}), \tag{56b}$$

with

$$\theta_{13}=\theta+\tfrac{1}{2}i\pi\left(1-\frac{1}{\lambda}\right),$$

$$\theta_{23}=\theta-\tfrac{1}{2}i\pi\left(1-\frac{1}{\lambda}\right). \tag{56c}$$

Fig. 5. Kink as kink-fermion bound state in (a) kink-kink scattering, (b) kink-fermion scattering.

Analogously, from the scattering of a kink as a kink-fermion bound state with a kink or a fermion, cf. fig. 5, we obtain, respectively,

$$u_r(\theta) = u_{r+1}(\theta'_{13})\varepsilon_{1,N} S_2(i\pi - \theta_{23}) \frac{2 - r/\lambda - \theta/i\pi}{2 - \theta/i\pi}$$

$$= u_{r+1}(\theta'_{23})\varepsilon_{1,N} S_2(i\pi - \theta_{13}) \frac{\theta/i\pi}{\theta/i\pi - r/\lambda} \tag{57a}$$

and

$$S_2(\theta) = S_2(\theta'_{13}) \cdot S_-(\theta_{23}) \frac{\theta/i\pi + \frac{1}{2} - 1/2\lambda}{\theta/i\pi + \frac{1}{2} + 1/2\lambda}, \tag{57b}$$

with

$$\theta'_{13} = \theta + i\frac{\pi}{\lambda},$$

$$\theta'_{23} = \theta - i\frac{\pi}{\lambda}. \tag{57c}$$

In the derivation of eqs. (56a) and (57a) the crossing relations (49a, b) and the factorization equations (47a, b) have been used. It can be verified that the relations (56) and (57) hold for the proposed S-matrices given by eqs. (40)–(46). In fact, eqs. (57) have been used to determine the amplitude S_2 from Zamolodchikovs' S_- and then u_r from S_2. These more lengthy calculations are relegated to appendix C.

4.3. HIGHER BOUND STATES

It is remarkable that in the GN model there exist exclusively such one-particle states which transform according to one of the fundamental (or the trivial) representations of O($2N$), cf. fig. 1. The higher bound states $b_n^{(r)}$ with rank $r < \lambda = N - 1$ are described by relations which generalize eqs. (27). For odd r, we obtain fermions

$$\hat{\psi}^{(r)}(x) = \mathfrak{N}[\hat{\chi}_\alpha \hat{\chi}_\beta](x)(\Gamma_+ \sigma^{(r)} C)_{\alpha\beta}$$

$$= K^N \mathfrak{N}[\hat{\chi}_\alpha \hat{\chi}_\beta](x)(\Gamma_- \sigma^{(r)} C)_{\alpha\beta}$$

$$= \mathfrak{N}[\hat{\bar{\psi}}_{a_1} \cdots \hat{\psi}_{a_r}](x), \tag{58a}$$

and for even r, bosons

$$\hat{\phi}^{(r)}(x) = K^{N+1}\mathcal{N}\left[\hat{\chi}_\alpha\hat{\chi}_\beta\right](x)\left(\Gamma_+\sigma^{(r)}C\right)_{\alpha\beta}$$
$$= K^{N+1}\mathcal{N}\left[\hat{\chi}_\alpha\hat{\chi}_\beta\right](x)\left(\Gamma_-\sigma^{(r)}C\right)_{\alpha\beta}$$
$$= \mathcal{N}\left[\hat{\bar{\psi}}_{a_1}\cdots\hat{\psi}_{a_r}\right](x). \tag{58b}$$

Then the S-matrix for the higher bound states may be determined. We restrict ourselves to the scattering of a higher bound state $b_n^{(r)}$ with an elementary fermion $f_a \equiv b_1^a$,

$$b_n^{(r)} + f_a \to b_n^{(s)} + f_b.$$

Upon application of the general bound-state formula (B1) of appendix B, from eqs. (58) and by properly taking into account the Klein factors we obtain the relation

$$ {}_{(s)b}S_{(r)a}(\theta) = (-1)^{(N+1)(r+1)}2^{-N}\left(\sigma^{(s)}C\right)^*_{\alpha'\beta'}\ {}_{\alpha' b}S_{\alpha n}\left(\theta + \tfrac{1}{2}i\pi\left(1-\frac{n}{\lambda}\right)\right)$$
$$\times\ {}_{\beta' n}S_{\beta a}\left(\theta - \tfrac{1}{2}i\pi\left(1-\frac{n}{\lambda}\right)\right)\left(\sigma^{(r)}C\right)_{\alpha\beta}\sqrt{\left|\frac{u_r}{u_s}\right|}\,\Bigg|_{\theta = i\pi n/\lambda},$$

and finally after some algebra

$$ {}_{(s)b}S_{(r)a}(\theta) = (-1)^{r+1}\Bigg\{\delta_{ab}\delta_{(s)(r)}\left[4\lambda^2\left(\frac{\theta}{i\pi}-\frac{n}{2\lambda}\right)\left(\frac{\theta}{i\pi}-1+\frac{n}{\lambda}\right)-2\lambda-1+2r\right]$$
$$+\left(\delta_{(bs)(ar)}-\delta_{ab}\delta_{(s)(r)}\right)\left[-4\lambda\left(1-\frac{\theta}{i\pi}\right)-2+2r\right]$$
$$+\left(\delta_{(as)(br)}-\delta_{ab}\delta_{(s)(r)}\right)\left[-4\lambda\frac{\theta}{i\pi}-2+2r\right]$$
$$-2\delta_{(bas)(r)}\left[(n-s)(2\lambda-s-n)\right]^{1/2}$$
$$-2\delta_{(s)}\delta_{(bar)}\left[(n-r)(2\lambda-r-n)\right]^{1/2}\Bigg\}$$
$$\times\frac{S^{\mathrm{SG}}_{b_n b_1}(\theta)}{4\lambda^2(\theta/i\pi+1-(n-1)/2\lambda)(\theta+(n+1)/2\lambda)}$$
$$\times\exp\left\{\int_0^\infty\frac{\mathrm{d}x}{x}\,\frac{\mathrm{e}^{-x(1/2-(n+1)/2\lambda)}+\mathrm{e}^{-x(3/2+(n-1)/2\lambda)}}{\mathrm{ch}\frac{1}{2}x}\,\mathrm{sh}\left(x\frac{\theta}{i\pi}\right)\right\}, \tag{59}$$

where

$$\delta_{(s)(r)} = 2^{-N}\operatorname{tr}(\sigma^{(s)}\sigma^{(r)}).$$

For $n=2$ and $r,s=0,2$, these S-matrices agree with those of ref. [4] which in view of the last equality in (58b) and eq. (55c) may also be calculated from the two-fermion S-matrix,

$$ {}_{(s)b}S_{(r)a}(\theta) = {}_{c'd'}\varphi^{*}_{(s)}\; {}_{c'b}S_{ca'}\left(\theta + \frac{i\pi}{2\lambda}\right)\; {}_{d'a'}S_{da}\left(\theta - \frac{i\pi}{2\lambda}\right)\; {}_{cd}\varphi_{(r)} \sqrt{\left|\frac{S_{(r)}}{S_{(s)}}\right|}\,\Bigg|_{\theta = i\pi/\lambda}, $$

where

$$ {}_{ab}\varphi_{(0)} = (2N)^{-1/2}\delta_{ab}, $$

$$ {}_{ab}\varphi_{cd} = 2^{-1/2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), $$

$$ S_{(0)} = S_0, \qquad S_{(2)} = S_- . $$

By means of eqs. (B.1) and (40) one can in principle also calculate the general kink-bound state S-matrix ${}_{(s)\beta}S_{(r)\alpha}$ and the general bound state-bound state S-matrix ${}_{(t)(u)}S_{(r)(s)}$, but the expressions get even more complicated than eq. (59).

5. Examples

In this section we discuss in more detail the S-matrices of the GN model for O(4), O(6), and O(8). We demonstrate that they are consistent with the local isomorphisms $O(4) \simeq SU(2) \otimes SU(2)$ and $O(6) \simeq SU(4)$, cf. figs. 3a, b, and with the triality for O(8) which is obvious from fig. 3c.

5.1. O(4)

Let us briefly recapitulate some properties of the chiral SU(N) model which is defined by the lagrangian [1]

$$ \mathcal{L} = \sum_{\alpha=1}^{N} \bar{\chi}_\alpha i\partial\!\!\!/ \chi_\alpha + \tfrac{1}{2}g^2\left[\left(\sum_\alpha \bar{\chi}_\alpha\chi_\alpha\right)^2 - \left(\sum_\alpha \bar{\chi}_\alpha\gamma^5\chi_\alpha\right)^2\right]. \tag{60} $$

The "spin" of the physical particles is [11]

$$ s = \tfrac{1}{2}\left(1 - \frac{1}{N}\right) \tag{61} $$

and the S-matrix of the auxiliary bosons is given by

$$ {}_{\gamma\delta}S^{\mathrm{SU}(N)}_{\alpha\beta}(\theta) = u_1^{\mathrm{SU}(N)}(\theta)\delta_{\gamma\alpha}\delta_{\delta\beta} + u_2^{\mathrm{SU}(N)}(\theta)\delta_{\gamma\beta}\delta_{\delta\alpha}, \tag{62a} $$

where

$$ u_1^{\mathrm{SU}(N)}(\theta) = -\frac{\Gamma(1-\theta/2\pi i)\Gamma(\theta/2\pi i - 1/N)}{\Gamma(1-\theta/2\pi i - 1/N)\Gamma(\theta/2\pi i)}, \tag{62b} $$

$$ u_2^{\mathrm{SU}(N)}(\theta) = -\frac{2\pi i}{N\theta} u_1^{\mathrm{SU}(N)}(\theta) \tag{62c} $$

(note that in ref. [11] the auxiliary particles are taken to be fermions; therefore their S-matrix differs from eq. (62) by a sign).

In the chiral SU(N) model there exist antisymmetric bound states $b^{(r)}$ of r fundamental particles for $r = 1,2,\ldots,N-1$. The states $b^{(r)}$ and $b^{(N-r)}$ are antiparticles of each other. In particular, in the chiral SU(2) model there exists only one doublet with [11]

$$ \hat{\chi}_{\bar{\alpha}} = \hat{\chi}^*_{\alpha} = K\varepsilon_{\alpha\beta}\hat{\chi}_{\beta}, \tag{63} $$

where K is a Klein factor

$$ K\hat{\chi}K = -\hat{\chi} $$

and by eq. (61) the "spin" equals

$$ s = \tfrac{1}{4}. \tag{64} $$

On the other hand, in the O(4) GN model there are doublets of right-handed and left-handed kinks but no bound states, cf. fig. 3a. For $N=2$ the particle-antiparticle relation given by eq. (50b) coincides with eq. (63). According to eq. (26a) the "spins" are

$$ s_{++} = s_{--} = \tfrac{1}{4}, \tag{65a} $$

which agrees with eq. (64). Really, this identify served to motivate formula (26a).

For the scattering of two auxiliary kinks of the same chirality one obtains from eqs. (40) and (41)

$$ \begin{aligned} {}_{\gamma\delta}S_{\alpha\beta} &= \tfrac{1}{4}\sum_{r\ \mathrm{even}} \frac{1}{r!} u_r \sigma^{(r)}_{\gamma\beta}\sigma^{(r)}_{\delta\alpha} \\ &= -u_2\delta_{\gamma\alpha}\delta_{\delta\beta} + \tfrac{1}{2}(u_0+u_2)\delta_{\gamma\beta}\delta_{\delta\alpha}, \end{aligned} \tag{66a} $$

and from eqs. (42a, b)

$$u_2 = \frac{\Gamma(1-\theta/2\pi i)\Gamma(\theta/2\pi i - \frac{1}{2})}{\Gamma(\frac{1}{2}-\theta/2\pi i)\Gamma(\theta/2\pi i)}, \tag{66b}$$

$$\tfrac{1}{2}(u_0(\theta)+u_2(\theta)) = \frac{i\pi}{\theta} u_2(\theta), \tag{66c}$$

which agrees with eqs. (62a, b, c) for $N=2$. Similarly, for the scattering of two kinks of opposite chirality one obtains, with

$$s_{+-} = \tfrac{1}{2}, \tag{65b}$$

the physical S-matrix

$$\begin{aligned} {}_{\gamma\delta}\hat{S}_{\alpha\beta} &= -\,{}_{\gamma\delta}S_{\alpha\beta} \\ &= -\tfrac{1}{4} \sum_{r\ \mathrm{odd}} \frac{1}{r!} u_r \sigma^{(r)}_{\gamma\beta} \sigma^{(r)}_{\delta\alpha} \\ &= \delta_{\gamma\alpha}\delta_{\delta\beta}, \end{aligned} \tag{67}$$

i.e. kinks of different chirality decouple.

Note that the amplitude $u_2(\theta)$ given by eq. (65b) also coincides with the SG soliton-antisoliton transmission amplitude $t^{\mathrm{SG}}(\theta) = u^{\mathrm{SG}}(i\pi-\theta)$, cf. eq. (42b) at the critical value $\lambda=0$ or $\beta^2=8\pi$ [9]. However, because of the anomalous statistics (65a, b) and, related to this, the anomalous crossing relation (49a), our proposed O(4) GN S-matrix cannot be consistently understood as a product of two SG S-matrices, as was argued in ref. [6].

5.2. O(6)

Similar to the previous example, we now show that the isomorphism O(6) $\simeq$ SU(4), cf. fig. 3b, leads to an identity between the O(6) GN model and the chiral SU(4) model. The four right-handed (left-handed) GN kinks correspond to the four fundamental SU(4) particles (antiparticles). The kinks have "spin" $s=\frac{3}{8}$. The S-matrix for the scattering of two kinks of the same chirality follows from eqs. (40) and (41):

$$\begin{aligned} {}_{\gamma\delta}S_{\alpha\beta}(\theta) &= \tfrac{1}{8} \sum_{r\ \mathrm{even}} \frac{1}{r!} u_r(\theta) \sigma^{(r)}_{\gamma\beta} \sigma^{(r)}_{\delta\alpha} \\ &= -u_2(\theta)\delta_{\gamma\alpha}\delta_{\delta\beta} + \tfrac{1}{4}(u_0(\theta)+u_2(\theta))\delta_{\gamma\beta}\delta_{\delta\alpha}, \end{aligned} \tag{68a}$$

where

$$u_2(\theta)=\frac{\Gamma(1-\theta/2\pi i)\Gamma\big(\theta/2\pi i-\frac{1}{4}\big)}{\Gamma\big(\frac{3}{4}-\theta/2\pi i\big)\Gamma(\theta/2\pi i)}, \tag{68b}$$

$$\tfrac{1}{4}(u_0(\theta)+u_2(\theta))=\frac{i\pi}{2\theta}u_2(\theta), \tag{68c}$$

which agrees with eqs. (62a, b, c) for $N=4$.

The other scattering amplitudes follow for both models analogously by using the crossing relation and the bound-state formula (B.1).

5.3. O(8)

We terminate our presentation of examples with the group O(8). This turns out to be a particularly beautiful case, a fact which is also suggested by the Dynkin diagram fig. 3c. There exists an extra symmetry (triality) between the three elementary representations [12]. In the GN model this implies a degeneracy between the two types of kinks and fermions not only for their masses [13]

$$m_1=2m\sin\tfrac{1}{6}\pi=m,$$

but also for their "spins"

$$s_{\mathrm{FF}}=s_{++}=s_{--}=\tfrac{1}{2}.$$

In the S-matrix this symmetry shows up in the identity of the unitary u-channel amplitudes for scattering of kinks with equal chirality and for fermion-fermion scattering. From eqs. (40)–(42) and (45), (46) we obtain

$$\begin{aligned} u_0(\theta)&=S_0(i\pi-\theta),\\ u_2(\theta)&=-S_-(i\pi-\theta),\\ u_4(\theta)&=S_+(i\pi-\theta). \end{aligned} \tag{69}$$

Correspondingly, for the scattering of kinks with different chiralities and for fermion-kink scattering we obtain from eqs. (40)–(44)

$$\begin{aligned} u_1(\theta)&=S_2(i\pi-\theta),\\ u_3(\theta)&=-S_1(i\pi-\theta). \end{aligned} \tag{70}$$

We finally remark that the O(8) GN model realizes a "perfect bootstrap" [13] in the sense that any of the elementary particles is a bound state of particles of the two other representations. And the boson (in the center of the Dynkin diagram) may be considered as a bound state of two particles of the same elementary representation.

6. Conclusions

Since we now have determined its total S-matrix, we have gained a rather complete knowledge of the $O(2N)$ Gross-Neveu model. Many properties of the particles in the spectrum are known: their masses, their behaviour under internal symmetry transformations, their statistics, and their scattering amplitudes. This brings the GN model almost to the same state as the sine-Gordon theory (alias massive Thirring model). Indeed, there is a close analogy between them. The GN kinks correspond to the SG solitons, the elementary GN fermions to the lowest breather (the SG field), and the higher bound states to the higher breathers. What makes the GN model a more interesting testing ground (from the point of view of elementary particle physics in four dimensions) are its additional properties such as dynamical symmetry breaking and asymptotic freedom.

What is still missing for the GN model is an analog to Coleman's correspondence between the SG theory and the MT model. This, of course, is related to the fact that, at this time, we do not have a satisfactory field theory of the kinks. We would like to remark that the perturbation expansion in the SG theory around $\beta = 0$ corresponds to the $1/N$ expansion in the GN model. Hence the kink S-matrix has no simple $N \to \infty$ limit, as the soliton S-matrix has a non-trivial $\beta \to 0$ limit which is the classical expression. On the other hand, in the MT model the perturbation expansion around $g = 0$ ($\lambda = 1$) corresponds to $N = 2$ in the GN model which is, of course, not a good expansion point. Also, the super-selection rules implied by topologically non-trivial configurations are quite different for theories with solitons or kinks.

It is interesting also to consider the $O(2N+1)$ models. These, however, are not related to Gross and Neveu's original $U(N)$ models. They also cannot be completely bosonized. The determination of their S-matrix is in progress.

We thank B. Schroer for discussions. One of us (H.J.T.) gratefully acknowledges the kind hospitality extended to him at the II. Institut für Theoretische Physik der Universität Hamburg where part of this work was performed. He also thanks P. Weisz for useful discussions and for a critical reading of the manuscript.

Appendix

(A) AUXILIARY FIELDS

In subsect. 3.2 we have introduced auxiliary particles (bosons) corresponding to the elementary physical particles. We have already pointed out that the auxiliary particle spectrum has a higher degeneracy. Indeed, we have to distinguish between different types of auxiliary elementary fermions and kinks, i.e. $\psi(x)$, $\dot{\psi}(x)$ and $\chi(x)$, $\dot{\chi}(x)$, $\ddot{\chi}(x)$. The bound-state relations for the auxiliary model which correspond to eqs. (27a, b) and (30) read in terms of auxiliary kink and fermion fields

$$\psi_a(x) = \mathfrak{N}[\chi_\alpha \chi_\beta](x)(\Gamma_+ \gamma_a C)_{\alpha\beta}, \tag{A.1a}$$

$$\dot{\psi}_a(x) = \mathfrak{N}[\chi_\alpha \chi_\beta](x)(\Gamma_- \gamma_a C)_{\alpha\beta}, \tag{A.1b}$$

$$\dot{\chi}^{\pm}(x) = \mathfrak{N}[\psi_a \chi_\beta^{\mp}](x)(\gamma_a)_{\beta\alpha}, \tag{A.2a}$$

$$\ddot{\chi}^{\pm}(x) = \mathfrak{N}[\dot{\psi}_a \chi^{\mp}](x)(\gamma_a)_{\beta\alpha}. \tag{A.2b}$$

The S-matrix elements for the scattering of particles and their dotted partners with another particle differ at most by a sign due to the Klein factors in eqs. (27b) and (30). This can be seen as follows for the scattering of bound states a and $\dot{a}$ with a kink α using eq. (38):

$$\begin{aligned} {}_{\dot{b}\beta}S_{\dot{a}\alpha}(\theta) &= \exp\{2\pi i s_{\dot{b}\beta}\varepsilon(\theta)\}\ {}_{\hat{b}\beta}\hat{S}_{\hat{a}\alpha}(\theta) \\ &= \exp\{2\pi i (s_{\dot{b}\beta} - s_{b\beta})\varepsilon(\theta)\}\ {}_{b\beta}S_{a\alpha} \\ &= (-1)^N\ {}_{b\beta}S_{a\alpha}, \end{aligned} \tag{A.3a}$$

where the particles $\hat{a}$, $\hat{b}$ are given by eq. (27b) without the Klein factor K^N. In the second equation of (A.3) the identity $K^+K = 1$ has been used in deriving the relation

$$\langle 0|\beta_{\text{out}}(\theta_2')\hat{b}_{\text{out}}(\theta_1')\hat{a}^+_{\text{in}}(\theta_1)\alpha^+_{\text{in}}(\theta_2)|0\rangle = \langle 0|\beta_{\text{out}}(\theta_2')\hat{\dot{b}}_{\text{out}}(\theta_1')\hat{\dot{a}}^+_{\text{in}}(\theta_1)\alpha^+_{\text{in}}(\theta_2)|0\rangle\,,$$

and for the third equation we inserted

$$s_{\dot{b}\beta} - s_{b\beta} = 0\ (\tfrac{1}{2}),$$

for N even (odd).

We also obtain

$$ {}_{b\beta}S_{a\dot{\alpha}} = -(-1)^N\ {}_{b\beta}S_{a\alpha}, \tag{A.3b}$$

and similarly for the scattering of auxiliary fermions

$$ {}_{cd}S_{ab} = {}_{\dot{c}d}S_{\dot{a}b} = {}_{\dot{c}\dot{d}}S_{\dot{a}\dot{b}}, \tag{A.4}$$

and for kink-kink scattering

$$ {}_{\gamma\delta}S_{\alpha\beta} = -(-1)^{[N/2]}\ {}_{\dot\gamma\delta}S_{\dot\varepsilon\kappa}(\Gamma)_{\varepsilon\alpha}(\Gamma^{1+N})_{\kappa\beta}, \tag{A.5a} $$

$$ = (-1)^{[N/2]}\ {}_{\ddot\gamma\delta}S_{\ddot\varepsilon\kappa}(\Gamma)_{\varepsilon\alpha}\left((-\Gamma)^{1+N}\right)_{\kappa\beta}. \tag{A.5b} $$

All these relations are consistent with the bound state S-matrices derived in appendix B and the S-matrix proposed by eqs. (40)–(42).

(B) BOUND-STATE RELATIONS

In this appendix we derive the consistency conditions (56) and (57) which follow from the assumption that the elementary fermion is a kink-kink bound state and the kink is a kink-fermion bound state as expressed by eqs. (A.1) and (A.2).

The general bound-state formula [14] reads

$$ \begin{aligned} &{}_{a'+b',c'}S_{a+b,c}(p_1+p_2,p_3) \\ &\quad = \operatorname*{Res}_{(p_1+p_2)^2=m^2_{a+b}}\ {}_{ab}\varphi^*_{a'+b'}\ {}_{a'b'}|S(p_1,p_2)|^{1/2}_{a''b''} \\ &\qquad \times {}_{a''c'}S_{a'''c''}(p_1,p_3)\ {}_{b''c''}S_{b'''c}(p_2,p_3)\ {}_{a'''b'''}|S(p_1,p_2)|^{-1/2}_{ab}\ {}_{ab}\varphi_{a+b}, \end{aligned} \tag{B.1} $$

if $a+b$ is a bound state of a and b with mass m_{a+b} described by a normal-product relation like

$$ \Phi_{a+b}(x) = \mathcal{N}[\Phi_a\Phi_b](x)\ {}_{ab}\varphi_{a+b}, \tag{B.2} $$

with

$$ \sum_{a,b} |{}_{ab}\varphi_{a+b}|^2 = 1\,. $$

Eqs. (A.1a, b) and (A.2a, b) are special cases of this general relation.

The relation (A.1a) which expresses the auxiliary fermion as a bound state of auxiliary kinks together with eq. (B.1) imply, in view of fig. 4a,

$$ {}_{b\beta}S_{a\alpha}(\theta) = 2^{1-N}(C^+\gamma_b)_{\delta'\gamma'}\ {}_{\gamma'\beta}S_{\gamma\varepsilon}(\theta_{13})\ {}_{\delta'\varepsilon}S_{\delta\alpha'}(\theta_{23})(\Gamma_+\gamma_a C)_{\gamma\delta}, \tag{B.3} $$

with θ_{13} and θ_{23} as defined by eq. (56c).

From the representation of the kink-kink S-matrix (40) together with (41), the crossing relation (49a), and the identity

$$2^{-N}\operatorname{tr}\left(\gamma^b\sigma^{(r)}\sigma^{(s)}\right)\left(\sigma^{(r)}\Gamma_+\gamma^a\sigma^{(s)}\right)_{\beta\alpha}$$

$$=\tfrac{1}{2}\left[1+(-1)^r\Gamma\right]_{\beta\gamma}(-1)^r r!s!$$

$$\times\left\{\left[\delta_{\gamma\alpha}\delta^{ba}\binom{2N-1}{r}+\sigma^{ba}_{\gamma\alpha}\left(2\binom{2N-2}{r-1}-\binom{2N-1}{r}\right)\right]\delta_{s,r+1}\right.$$

$$\left.-\left(r\to 2N-r, s\to 2N-s\right)\right\}, \tag{B.4}$$

we obtain

$$_{b\beta}S_{a\alpha}(\theta)=2^{1-2N}\sum_r(-1)^r\left[1+(-1)^r\Gamma\right]_{\beta\gamma}\varepsilon_{r+1,N}$$

$$\times\left[\delta_{\gamma\alpha}\delta^{ba}\binom{2N-1}{r}+\sigma^{ba}_{\gamma\alpha}\left(2\binom{2N-2}{r-1}-\binom{2N-1}{r}\right)\right]u_r(\theta_{13})u_{r+1}(i\pi-\theta_{23}) \tag{B.5}$$

which leads to eq. (56a).

Using the factorization equations (47a) and the representations (42) and (44) of the amplitudes u_r and t_i we find consistency of eq. (B.5) with the proposed S-matrix (40)–(44). If we perform the same calculation for the $\dot{\psi}$ as given by eq. (A.1b) we have to replace Γ and Γ_+ by $-\Gamma$ and Γ_-, respectively, in eqs. (B.3)–(B.5) and also in (43) which is consistent with eq. (A.3a).

Analogously to eq. (B.3) we obtain from fig. 4b for fermion-fermion scattering

$$_{cd}S_{ab}(\theta)=2^{1-N}(C^+\gamma_c)_{\beta'\alpha'}\;{}_{\alpha'd}S_{ab'}(\theta_{13})\;{}_{\beta'b'}S_{\beta b}(\theta_{23})(\Gamma_+\gamma_a C)_{\alpha\beta}$$

$$=\delta_{ab}\delta_{cd}\left[t_1t_2+t_2t_1+(2N-2)t_2t_2\right]$$

$$+\delta_{ac}\delta_{bd}\left[t_1t_1-(2N-3)t_2t_2\right]$$

$$+\delta_{ad}\delta_{bc}\left[-t_1t_2-t_2t_1+(2N-2)t_2t_2\right], \tag{B.6}$$

where the arguments θ_{13} and θ_{23} of the t-amplitudes have been suppressed. Using the factorization equation (47b), we obtain eq. (56b), and with the representations (44), (46) of the amplitudes t_i and σ_i we find consistency of eq. (B.6) with the proposed S-matrix (43)–(46). Again the same statement holds for $\dot{\psi}$, in agreement with eq. (A.4).

Let us now turn to eq. (A2a) which expresses the dotted auxiliary kink as a bound state of an auxiliary kink and an auxiliary elementary fermion. From fig. 5a we obtain [with θ'_{13} given by eq. (57c)]

$$
\begin{aligned}
{}_{\dot\gamma\delta}S_{\dot\alpha\beta}(\theta) &= \frac{1}{2N}(\gamma_b)_{\gamma\gamma'}\ {}_{\gamma'\delta}S_{\alpha'\beta'}(\theta'_{13})\ {}_{b\beta'}S_{a\beta}(\theta_{23})(\gamma_a)_{\alpha'\alpha} \\
&= \frac{1}{2N}2^{-N}\sum_r (\sigma^{(r)}\Gamma^N)_{\gamma\beta}\,\sigma^{(r)}_{\delta\alpha}(-1)^{r-1} \\
&\quad\times\left\{\frac{u_{r-1}}{(r-1)!}\left[t_1+(2N-2r+1)t_2\right]-\frac{u_{r+1}}{(r+1)!}\left[t_1-(2N-2r-1)t_2\right]\right\} \\
&= 2^{-N}\sum_r\frac{(-1)^{r-1}}{r!}u_{r+1}(\theta'_{13})S_2(i\pi-\theta_{23})\frac{2-r/\lambda-\theta/i\pi}{2-\theta/i\pi}(\sigma^{(r)}\Gamma^N)_{\gamma\beta}\sigma^{(r)}_{\delta\alpha},
\end{aligned}
\tag{B.7}
$$

where again the factorization equations (47a, b) and the crossing relation (49b) have been used. Since the amplitudes u_r for even (odd) r contribute only to the scattering of kinks with equal (opposite) chirality, eq. (B.7) together with (A.5a) imply the first part of eq. (57a). The second part of eq. (54a) follows from an interchange of the role of the particles with rapidity θ_1 and θ_2 in eq. (B.7). If we perform the same calculation for $\dot\chi$ as defined by eq. (A.2b) we obtain from eq. (A.3a) on the r.h.s. an extra factor $(-1)^N$ which is in agreement with eq. (A.5b).

Finally, from fig. 5b we obtain

$$
\begin{aligned}
{}_{\beta b}S_{\dot\alpha a}(\theta) &= \frac{1}{N}\gamma^d_{\beta\delta}\ {}_{\delta b}S_{\gamma e}(\theta'_{13})\ {}_{de}S_{ca}(\theta_{23})\gamma^c_{\gamma\alpha} \\
&= -(-\Gamma)^N_{\beta\alpha}\delta^{ba}\left[\frac{1}{2N}t_1(\sigma_1+2N\sigma_2+\sigma_3)+\frac{2N-1}{2N}t_2(\sigma_3-\sigma_1)\right] \\
&\quad-\left((-\Gamma)^N\sigma^{ba}\right)_{\beta\alpha}\left[\frac{1}{2N}t_1(\sigma_1-\sigma_3)+\frac{1}{2N}t_2((2N-4)\sigma_2\right. \\
&\qquad\left.-(2N-1)(\sigma_1+\sigma_3))\right],
\end{aligned}
\tag{B.8}
$$

which leads to eq. (57b). Again one can verify that these equations are consistent with the proposed S-matrix, and the same statement is true for analogous relations involving $\dot\chi$. We have thus demonstrated that the proposed S-matrix is consistent with all bound-state conditions concerning elementary fermions and kinks.

(C) DERIVATION OF THE S-MATRIX

In the previous appendix we have shown how eqs. (56) and (57) follow from the bound-state relations (A.1) and (A.2) and the factorization and crossing relations. Let us now demonstrate how for $N \geqslant 3$ one can derive the kink-fermion and kink-kink S-matrices from the known fermion-fermion S-matrix (for $N=2$ there are no fermions). This method is similar to that used in ref. [5].

We make the minimality assumption that the amplitudes $u_r(\theta)$ and $S_2(\theta)$ are analytic in the physical strip $0 \leqslant \operatorname{Im}\theta \leqslant \pi$ except for possible poles at

$$\theta = i\pi n/\lambda, \qquad (n = 1,2,\ldots,\lambda-1),$$

and

$$\theta = \tfrac{1}{2}i\pi(1 \pm 1/\lambda),$$

respectively, which may either directly correspond to bound states, cf. eqs. (55a,b), or are implied by crossing symmetry. We consider the functions

$$u_r'(\theta) = u_r(\theta)/u^{\mathrm{SG}}(\theta), \tag{C.1a}$$

$$t_i'(\theta) = t_i(\theta)/S^{\mathrm{SG}}_{\mathrm{sol},b}(\theta), \tag{C.1b}$$

$$\sigma_i'(\theta) = \sigma_i(\theta)/S^{\mathrm{SG}}_{bb}(\theta). \tag{C.1c}$$

They also satisfy eqs. (56) and (57) since the SG amplitudes fulfill the corresponding bound-state relations [9]. Moreover, the function $S_2'(\theta)$, which is defined from t_i' as in eq. (44b), is analytic in the strip $0 \leqslant \operatorname{Im}\theta \leqslant \frac{1}{4}\pi$ for $\lambda \geqslant 2$. The further minimality assumption that $S_2'(\theta)$ does not vanish in this region and the unitarity equation (48) imply that $\ln S_2'(\theta)$ is analytic for $|\operatorname{Im}\theta| \leqslant \frac{1}{4}\pi$ [note that a zero at $\theta = \frac{1}{2}i\pi(1-1/\lambda)$ which would be introduced into S_2' by $S^{\mathrm{SG}}_{\mathrm{sol},b}$ if S_2 did not have a pole there, is excluded by eq. (57b)]. Therefore, we may apply Cauchy's theorem and obtain the representation valid for $|\operatorname{Im}\theta| \leqslant \pi/2\lambda$:

$$\frac{S_2'(\theta)}{S_2'(0)} = \exp\left\{\frac{1}{2\pi i}\int_{\mathrm{C}} \frac{\mathrm{d}z}{\operatorname{sh}(z-\lambda\theta)}\,\frac{\operatorname{sh}\lambda\theta}{\operatorname{sh}z}\,\ln\frac{S_2'(z/\lambda)}{S_2'(0)}\right\}$$

$$= \exp\left\{\frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\operatorname{ch}(z-\lambda\theta)}\,\frac{\operatorname{sh}\lambda\theta}{\operatorname{ch}z}\,\ln\frac{S_2'(z/\lambda - i\pi/2\lambda)}{S_2'(z/\lambda + i\pi/2\lambda)}\right\}, \tag{C.2}$$

where the contour C encloses the strip $|\operatorname{Im} z| \leqslant \frac{1}{2}\pi$ counterclockwise. Using eq. (57b) we may now replace the ratio of S_2' functions at different arguments on the r.h.s. of eq. (C.2) by the known function S_-',

$$\frac{S_2'(\theta)}{S_2'(0)}=\exp\left\{\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\mathrm{ch}(z-\lambda\theta)}\,\frac{\mathrm{sh}\,\lambda\theta}{\mathrm{ch}\,z}(-2)\int_0^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\mathrm{e}^{-x(1+1/2\lambda)}}{\mathrm{ch}\frac{1}{2}x}\,\mathrm{sh}\frac{x}{2\lambda}\,\mathrm{ch}\left(x\frac{z}{i\pi\lambda}\right)\right\}. \tag{C.3}$$

After some algebra we obtain eq. (44b) up to an undetermined sign, since $|S_2'(0)|=1$ by unitarity.

Analogously we argue that the function $\ln u_1'(i\pi-\theta)$ is analytic for $|\mathrm{Im}\,\theta|\leqslant\pi/\lambda$, since again a possible zero of $u_1'(i\pi-\theta)$ at $\theta=i\pi/\lambda$ is ruled out by eq. (57a). Then Cauchy's theorem gives the representation

$$\frac{u_1'(i\pi-\theta)}{u_1'(i\pi)}=\exp\left\{\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\mathrm{ch}\left(z-\frac{1}{2}\lambda\theta\right)}\,\frac{\mathrm{sh}\lambda\theta}{\mathrm{ch}z}\ln\frac{S_2'\left(2z/\lambda-i\pi\left(\frac{1}{2}-1/2\lambda\right)\right)}{S_2'\left(2z/\lambda+i\pi\left(\frac{1}{2}-1/2\lambda\right)\right)}\right\}. \tag{C.4}$$

Since S_2' has already been determined, we obtain after some algebra eq. (42a) for $u_1(\theta)$, again up to an open sign. Next the function $u_0(\theta)$ is determined again from eq. (57a), and the remaining ones may be derived recursively from the factorization equation (47a).

So far we have determined the amplitudes u_r and t_i only up to a sign. It can be verified that the general "minimal" solution of eqs. (57a, b) is obtained from ours by the substitutions

$$u_r\to\varepsilon(\varepsilon')^r u_r,$$

$$t_i\to(\varepsilon')^{N+1}t_i,$$

with ε, $\varepsilon'=\pm1$. There exists only one alternative choice of phase factors which matches all other conditions, too. It is obtained by $\varepsilon=-\varepsilon'=1$, but it also requires the substitutions

$$s_{+-}\to s_{+-}-\tfrac{1}{2},$$

$$K'\to K^N K',$$

$$K''\to K^N K''.$$

This, however, means that there are no changes in the physical S-matrix, $\hat{S}$.

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