ON THE CONNECTION BETWEEN THE Λ-PARAMETERS OF EUCLIDEAN LATTICE AND CONTINUUM QCD

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The origin of the discrepancy between recent calculations of the connection between the Λ-parameters of lattice and continuum pure gauge theory is explained. The calculation is extended to include fermions.

The continuum limit of SU(N) euclidean lattice gauge theory is expected to be obtained by taking the limit lattice spacing $a \to 0$, and the bare coupling $g_0 \to 0$ such that the renormalisation group invariant mass parameter

$$\Lambda_L \equiv a^{-1} \exp(-1/2\beta_0 g_0^2)\,(\beta_0 g_0^2)^{-\beta_1/2\beta_0^2}\,[1 + O(g_0^2)] \,, \tag{1}$$

tends to a finite non-zero limit. Here β_0, β_1 are the universal first two coefficients of the Callan–Symanzik β-function [1]

$$\beta_0 = (4\pi)^{-2}[\tfrac{11}{3}N - \tfrac{4}{3}T(R)n_f] \,, \quad \beta_1 = (4\pi)^{-4}[\tfrac{34}{3}N^2 - \tfrac{20}{3}NT(R)n_f - 4C_2(R)T(R)n_f] \,. \tag{2}$$

The actual value of Λ_L is arbitrary: it just fixes the mass scale of the continuum theory. Physical masses are expressed as (in principle calculable) constants times Λ_L. For example, from Monte Carlo studies Creutz [2] estimated for the string tension for the case $N = 2$ in the absence of fermions ($n_f = 0$)

$$\alpha^{1/2} \approx 70\,\Lambda_L \,. \tag{3}$$

To give such results phenomenological significance it is necessary to connect Λ_L to corresponding Λ parameters defined in the continuum theory:

$$\Lambda \equiv M \exp[-1/2\beta_0 g^2(M)]\,(\beta_0 g^2(M))^{-\beta_1/2\beta_0^2}[1 + O(g^2(M))] \,. \tag{4}$$

Recently, Hasenfratz and Hasenfratz [3] and Dashen and Gross [4] have performed the calculation (for the case $n_f = 0$) and they both find Λ_{MOM}/Λ_L is a large number (≈ 60 for $N = 2$) so that one obtains from (3) the phenomenologically satisfying result

$$\alpha^{1/2} \approx \Lambda_{MOM} \,. \tag{5}$$

The precise values of Λ_{MOM}/Λ_L obtained by the two groups, however, differ by 5%. Although such a small difference is phenomenologically unimportant, especially since the effect of massless fermions were not taken into account, the origin of the discrepancy should be theoretically understood. It is one of the purposes of this note to clarify this point. The other is to extend the result to include fermions ($n_f \neq 0$). These results will become relevant when Monte Carlo or equivalent methods are extended to include fermions [5] or when strong coupling estimates are made [6] analogous to those already performed for the case $n_f = 0$ [7]. The results are presented in table 1 and as one sees the n_f dependence is not too dramatic for small n_f.

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Table 1
Ratios of various Λ parameters.

	$N=2$					$N=3$				
	$n_f=0$	1	2	3	4	$n_f=0$	1	2	3	4
$\frac{\Lambda_{\mathrm{MIN}}}{\Lambda_{\mathrm{L}}}$	7.46	9.27	12.09	16.85	25.81	10.85	12.78	15.41	19.09	24.49
$\left(\frac{\Lambda_{\mathrm{MOM}}}{\Lambda_{\mathrm{L}}}\right)_{\alpha=1}$	57.40	61.39	66.64	73.85	84.26	83.42	89.24	96.36	105.24	116.57
$\left(\frac{\Lambda_{\widetilde{\mathrm{MOM}}}}{\Lambda_{\mathrm{MIN}}}\right)_{\alpha=1}$	6.40	6.42	6.46	6.51	6.56	6.40	6.41	6.44	6.46	6.49

Just as the continuum Λ depends on the renormalisation scheme employed, the lattice Λ_{L} depends on the choice of lattice action. The calculations reported here refer to the standard Wilson action [9].

$$S = -\sum_x \bar{\psi}_x \psi_x + K \sum_{x,\mu} [\bar{\psi}_x (1-\gamma_\mu) U_{x,\mu} \psi_{x+\hat{\mu}} + \bar{\psi}_{x+\hat{\mu}} (1+\gamma_\mu) U^\dagger_{x,\mu} \psi_x]$$

$$+ g_0^{-2} \sum_{x,\mu\neq\nu} \mathrm{tr}(U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu} - 1) . \tag{6}$$

Now it follows from eqs. (1), (4) that

$$\ln(\Lambda/\Lambda_{\mathrm{L}}) = \ln aM - (1/2\beta_0)(g^{-2}(M) - g_0^{-2}) + \mathrm{O}(g^2) \tag{7}$$

and thus as it has been stressed by Celmaster and Gonsalves [8] and Parisi [10] to extract the relation between various Λ parameters it is sufficient to obtain the relationship between g_0 and $g(M)$ correct to first non-trivial order which can be done by suitable one-loop calculations. I will first summarize the (different) calculations of the two groups [3,4] referred to above and explain the origin of the discrepancy between their results and then list the results for the case $n_f \neq 0$.

Hasenfratz and Hasenfratz [3] evaluate two- and three-point functions in weak coupling lattice gauge theory. Specifically for computational ease they calculate the three-point function $\Gamma^{abc}_{\mu\nu\omega}(p,q,r)$ at the asymmetric point $r=0, p^2=q^2=M^2$ in the Feynman gauge. Requiring $\Gamma^{abc}_{\mu\nu\omega}(p,-p,0)|_{p^2=M^2}$ to have no correction proportional to the bare vertex defines the $\widetilde{\mathrm{MOM}}$ renormalisation scheme. This calculation, which is in principle straightforward but technically involved, yields a value for $\Lambda_{\mathrm{L}}/\Lambda_{\widetilde{\mathrm{MOM}}}$. Finally to obtain the relation between Λ_{L} and more conventionally employed Λ parameters, for example Λ_{MIN} defined by minimal subtraction using dimensional regularisation, a continuum calculation of $\Lambda_{\widetilde{\mathrm{MOM}}}/\Lambda_{\mathrm{MIN}}$ is performed. Dashen and Gross [4] use a background field method generalised to lattice gauge theories. The ratio of the lattice and continuum partition functions Z_a, Z in a weak background field $A^{(0)}_\mu$ for weak coupling is calculated and one finds (for $a \to 0$):

$$\ln(Z_a/Z) = [-\tfrac{1}{4}(g_0^{-2} - g^{-2}(M)) - \tfrac{1}{2}\beta_0 \ln aMC] \int \mathrm{d}^4x\, F^{(0)}_{\mu\nu} F^{(0)}_{\mu\nu} . \tag{8}$$

Requiring $Z_a = Z$ yields, using eq. (7):

$$\Lambda_{\mathrm{L}}/\Lambda = C . \tag{9}$$

The method is rather elegant and can be done relatively easily by hand. Care has to be taken, however, to correctly introduce the infrared and ultraviolet regularisations. Using Pauli–Villars regularisation Dashen and Gross end up with a value for Λ_L/Λ_{PV}. Finally, to compare with the result of Hasenfratz and Hasenfratz they use a value of $\Lambda_{PV}/\Lambda_{MIN}$ implicitly calculated by 't Hooft [11]. Repeating the calculation of Dashen and Gross with the small modification of using directly dimensional regularisation in the continuum part of the calculation I find complete agreement with Hasenfratz and Hasenfratz. The error mus thus arise in the transformation from Pauli–Villars to dimensional schemes. Indeed one small error is manifest in section 13 of 't Hooft's important paper [11], in which he considers this relationship.

't Hooft's reasoning is based on merely comparing the integrals occurring in the Pauli–Villars and dimensional schemes. In particular in the contribution which arises from the ghosts and the part of the gaussian vector field fluctuations which "behave like scalars", 't Hooft [11] argues one should replace ($\nu = 4 - \epsilon$, and M_0 the Pauli–Villars regulator mass, $J \equiv \frac{1}{2}(\ln 4\pi - \gamma) = 0.9769042$)

$$\ln M_0 \to 1/\epsilon + J - \tfrac{5}{12} \,. \tag{10}$$

However, in ν dimensions there are ν equivalent scalar fields (and two ghosts) and hence the coefficient multiplying the integrals is implicitly ϵ-dependent. This must be taken into account since the integrals are singular as $\epsilon \to 0$. Thus in the contribution referred to above one should replace

$$\ln M_0 \to (1 - \tfrac{1}{2}\epsilon)(1/\epsilon + J - \tfrac{5}{12}) \sim 1/\epsilon + J - \tfrac{11}{12} \,. \tag{11}$$

The other (larger) term arising from the vector field fluctuation has no such ϵ-dependent factor.

Taking this into account one would extract from 't Hooft's calculation,

$$(\Lambda_{PV}/\Lambda_{MIN})|_{n_f=0} = \exp(J + \tfrac{1}{12}) \,, \tag{12}$$

and inserting this in Dashen and Gross's computation the discrepancy with Hasenfratz and Hasenfratz is resolved [‡1]

Results: Using the background field method the contribution Z_a^f from the fermions to the lattice partition function in the weak field, weak coupling approximation is given by taking the limit:

$$8K \to 1 - \tfrac{1}{4} m_f a$$

of the expression

$$\ln Z_a^f = -\frac{1}{2}\left(\int d^4x\, F_{\mu\nu}^{(0)} F_{\mu\nu}^{(0)}\right) T(R) n_f \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \{1/G(p)^2 - \tfrac{1}{2}[\partial_1^p \partial_2^p \ln G(p)]^2\} \,, \tag{13}$$

where $G(p)$ is given by

$$G(p) = \sum_{\mu=1}^{4} \sin^2 p_\mu + (1/4K^2)\left(1 - 2K \sum_{\mu=1}^{4} \cos p_\mu\right)^2 \,. \tag{14}$$

The infrared singularity as $m_f \to 0$ is cancelled by the continuum contribution. Adding the fermion and vector (and ghost) contributions one obtains the result for the ratio of lattice and continuum Λ-parameters;

$$\Lambda_L/\Lambda_{MIN} = \exp\{J + \beta_0^{-1}\,[1/16N - NP + T(R) n_f P_3]\} \,, \tag{15}$$

where

$$P = \tfrac{1}{48}(\tfrac{10}{3}P_1 + 88P_2 + \tfrac{3}{2} - 1/2\pi^2) \,. \tag{16}$$

‡1 Unfortunately there still remains the possibility that eq. (12) is incorrect due to a mistake in the Pauli–Villars calculation itself. (The same mistake would have been made by both 't Hooft and Dashen and Gross and would be cancelled in the finally quoted Λ_{MIN}/Λ_L.) Such a possibility has been recently emphasized to me by P. Hasenfratz who also referred me to Shore [12] who has previously stressed this point.

and where P_1, P_2, P_3 are the following integrals:

$$P_1 \equiv \int_0^\infty \mathrm{d}\beta\, \mathrm{e}^{-8\beta} I_0^4(2\beta)\,, \tag{17}$$

$$P_2 \equiv \int_0^\infty \mathrm{d}\beta \{\beta \mathrm{e}^{-8\beta} I_0^4(2\beta) - (1-\mathrm{e}^{-\beta})/16\pi^2\beta\}\,, \tag{18}$$

$$P_3 \equiv \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left\{ \{1/G(p)^2 - \tfrac{1}{2}[\partial_1^p \partial_2^p \ln G(p)]^2\}_{K=\frac{1}{8}} \sum_{\mu=1}^{4} \theta(\pi - |p_\mu|) - \tfrac{2}{3}[1/(p^2)^2 - 1/(p^2+1)^2] \right\}. \tag{19}$$

The integrals P_1 and P_2 have been numerically evaluated by Dashen and Gross [4] to high accuracy:

$$P_1 = 0.1549334 \quad P_2 = 0.0240132\,, \tag{20}$$

giving

$$P = 0.0849780 \tag{21}$$

These integrals have been checked by Stehr (to order 10^{-5}). Stehr has also estimated P_3 using a straightforward Monte Carlo program:

$$P_3 = 0.006887 \pm (0.000009 \text{ statistical error})\,. \tag{22}$$

Using the same program for the evaluation of P_2 in its equivalent form as a 4-dimensional integral Stehr found 0.0246. We thus cannot rule out a similar systematic error of ≈2% in the above result eq. (22). The values of $\Lambda_{\mathrm{MIN}}/\Lambda_{\mathrm{L}}$ following from eqs. (15), (21), (22) are given in table 1 for all the fermions in the fundamental representation i.e. $T(R) = 1/2$, for $n_\mathrm{f} = 0, 1, 2, 3, 4$ and for $N = 2, 3$. These can be converted to values for $\Lambda_\mathrm{L}/\Lambda_{\mathrm{MOM}}$ using the results of Celmaster and Gonsalves [8]. In table 1 the results for $(\Lambda_{\mathrm{MOM}}/\Lambda_\mathrm{L})_{\alpha=1}$(Feynman gauge) are also included

$$(\Lambda_{\mathrm{MOM}}/\Lambda_\mathrm{L})_{\alpha=1} = \exp(\beta_0^{-1}\{N[(1/96\pi^2)(23+\tfrac{1}{6}I) + P - 1/16N^2] - T(R)n_\mathrm{f}[(1/12\pi^2)(1+\tfrac{2}{3}I) + P_3]\})\,, \tag{23}$$

where

$$I \equiv -2\int_0^1 \frac{\ln x}{x^2 - x + 1} = 2.3439072\,\ldots\,. \tag{24}$$

In addition in table 1 appear the values for $(\Lambda_{\widehat{\mathrm{MOM}}}/\Lambda_{\mathrm{MIN}})_\alpha{=}1$, which may be of some use in future calculations. Defining the functions F_i, $i = 0, 1, 2$ in the continuum theory by

$$\Gamma^{abc}_{\mu\nu\omega}(p, -p, 0)|_{p^2=M^2} = g f_{abc}[(2\delta_{\mu\nu}p_\omega - \delta_{\mu\omega}p_\nu - \delta_{\nu\omega}p_\mu)(F_0(M^2) + Z_1\mu^\epsilon) + 2\delta_{\mu\nu}p_\omega F_1(M^2) + 2p_\mu p_\nu P_\omega F_2(M^2)]\,, \tag{25}$$

I find in the Feynman gauge using dimensional regularisation (and the trace of the unit matrix in ν dimensions to be [4]:

$$F_0(M^2)|_{\alpha=1} = (g^2/16\pi^2)\{(-\tfrac{4}{3}N + \tfrac{8}{3}T(R)n_\mathrm{f})[1/\epsilon + J - \ln(M/\mu)] - \tfrac{35}{18}N + \tfrac{20}{9}T(R)n_\mathrm{f}\}\,, \tag{26}$$

$$F_1(M^2)|_{\alpha=1} = (g^2/16\pi^2)[\tfrac{2}{3}N - \tfrac{4}{3}T(R)n_\mathrm{f}]\,, \tag{27}$$

The results for $n_f = 0$ agree with those of Hasenfratz and Hasenfratz [3]. Using the well-known result [1] for the wave function renormalisation constant Z_3 one then obtains

$$(\Lambda_{\widetilde{MOM}}/\Lambda_{MIN})_{\alpha=1} = \exp[J + (29N - 10T(R)n_f)/(33N - 12T(R)n_f)] \ . \tag{28}$$

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