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SCHRÖDINGER REPRESENTATION AND CASIMIR EFFECT IN RENORMALIZABLE QUANTUM FIELD THEORY

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We show that the Schrödinger representation exists in renormalizable quantum field theory to all orders in the perturbation expansion. In this sense, completeness of the Schrödinger states also holds. However, the field operator that is being diagonalized on a smooth three-dimensional hypersurface differs from the usual renormalized one by a factor that diverges logarithmically if the distance from the hypersurface goes to zero. This requires a limit procedure to be employed if expectation values of the renormalized field operator are to be computed in this representation. The Schrödinger functional differential operator involves point splitting Δ and has coefficients depending logarithmically on Δ, and also some by factors Δ^{-1}, Δ^{-2}, Δ^{-4}. Details are given for the massless ϕ_4^4 theory, but the extension to other models, in particular with spin-$\frac{1}{2}$ fermions, is outlined. The Casimir potential for disjoint surfaces is shown to be finite to all orders in the perturbation expansion, and computed for a pair of parallel plates to first order in massless ϕ_ν^4.

1. Introduction

Soon after the invention of quantum field theory, its Schrödinger representation was also known, and it has been mentioned since in some text books [1]. Calculations, however, were first done in the interaction representation, which is formally related to the Schrödinger representation by a change of basis. Later, covariant four-dimensional formalisms (S-matrix calculus, Green functions, and in particular functional integrals) were used almost exclusively. Even more than the interaction representation, which preserves a certain conceptual role in scattering theory, the Schrödinger representation fell into disrepute, the more so since it seems to have been considered to be non-renormalizable, as the interaction representation indeed is [2, 3].

More recently, however, the search for non-perturbative methods in strong-interaction theory led to the discussion of highly non-pointlike objects (dual strings, Wilson loops). The dynamics of such objects was formulated in terms of QFTs with boundaries [4, 5], an attractive setting due to the high flexibility and perfection of the QFT formalism. Prescribing the value of the quantum field on a boundary, however, means using the Schrödinger representation, slightly extended from flat to curved boundaries. This led us to consider renormalizable QFTs with boundaries, since non-renormalizable theories already pose unsolved problems in infinite space-time.

Superrenormalizable theories, on the other hand, offer no particular difficulty in this respect and, above all, do not appear* to describe interesting physics here.

We show here that in renormalizable models, the Schrödinger wave functional exists to all orders in perturbation theory, and give what we believe to be strong arguments that the Schrödinger functional differential operator that appears in the Schrödinger equation does so as well. For simplicity, we treat only the ϕ_4^4 theory in detail, and mostly choose it massless since masses are inessential here, but describe the principles of extension to other models. Dimensional regularization is used and mostly the euclidean frame, since the transition to the Minkowski frame is obvious. Renormalization group equations are given at every stage.

In sect. 2, we discuss how boundary conditions (in particular, homogeneous and inhomogeneous Dirichlet ones) are implemented by surface interaction. That the latter can be introduced in the renormalized infinite-space theory and hereby only causes divergences absorbable in logarithmic factors is crucial for the renormalizability, as we show in sect. 3. An immediate consequence is the finiteness of the Casimir effect (for a pair of parallel plates, for instance) to all orders, and we compute it to first order in massless ϕ_ν^4 theory in sect. 4. In sect. 5 we show that if the renormalized field operator, or its derivative in the normal direction, approaches the boundary plane, a factor, different in the two cases, with logarithmic dependence on the distance from the boundary must be applied to keep matrix elements finite. In sect. 6 we construct the Schrödinger functional differential operator, which requires point splitting, as it does in the free field theory. Completeness and unitarity are discussed in sect.7. That the field operator that is being diagonalized is not the renormalized one forces us to use a limit procedure if expectations of the field operator are to be computed. We discuss the extension of our method to other models in sect. 8, and give some details for the spin-$\frac{1}{2}$ Majorana field in an appendix. We also point out the divergences that arise if the transition to the interaction representation is attempted. We note that, unfortunately, the present methods are not applicable to the string lagrangians [4] as long as these are not shown to be renormalizable in infinite space. Conclusions are stated in sect. 9. Some technical material is relegated to the appendices.

2. Boundary conditions by surface interactions

Consider the theory of a free one-component scalar field with (euclidean) action density in ν dimensions:

$$L_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 \,. \tag{2.1}$$

Let Γ be a simply connected (not necessarily bounded) region of space with $\nu-1$ dimensional smooth boundary $\partial\Gamma$, described by $f(x)=0$, $x\in R^\nu$, with $f(x)>0$ in Γ

* Recently, however, Migdal [5] has proposed a free-fermion theory with boundaries for model use in QCD.

and $f(x)<0$ in the complementary region Γ'. Consider now the augmented action density

$$L_\Gamma = L_0 - \delta(f(x))\phi(x-0\partial f)\partial_\mu\phi(x)\partial_\mu f(x)\,, \tag{2.2}$$

where $-0\partial f$ is an infinitesimal vector pointing from $x\in\partial\Gamma$ outwards. We show in appendix A that the functional integral with source is

$$\int \mathscr{D}\phi\,\exp\left[\int L_\Gamma + \int J\phi\right]$$
$$= \mathrm{const}\,(\Gamma)\,\exp\left[\tfrac{1}{2}\int_\Gamma\int_\Gamma J(x)G_\mathrm{D}^\Gamma(x,x')J(x') + \tfrac{1}{2}\int_{\Gamma'}\int_{\Gamma'} J(x)G_\mathrm{N}^{\Gamma'}(x,x')J(x')\right]. \tag{2.3}$$

Here G_D^Γ is the Dirichlet Green function in Γ,

$$(m^2-\partial_x^2)G_\mathrm{D}^\Gamma(x,x') = \delta(x-x')\,,\qquad x,x'\in\Gamma\,, \tag{2.4a}$$

$$G_\mathrm{D}^\Gamma(x,x') = 0\,,\qquad x\in\partial\Gamma\,,\qquad x'\in\Gamma\,, \tag{2.4b}$$

and $G_\mathrm{N}(x,x')$ the Neumann Green function* in Γ',

$$(m^2-\partial_x^2)G_\mathrm{N}^{\Gamma'}(x,x') = \delta(x-x')\,,\qquad x,x'\in\Gamma'\,, \tag{2.5a}$$

$$\partial_n G_\mathrm{N}^{\Gamma'}(x,x') = 0\,,\qquad x\in\partial\Gamma\,,\qquad x'\in\Gamma'\,, \tag{2.5b}$$

with ∂_n the (to Γ', interior) normal derivative at x.

We can say, for short, that in (2.3)

$$\phi(x)\to 0\,,\qquad x\to\partial\Gamma \text{ from } \Gamma\,, \tag{2.6a}$$

$$\partial_n\phi(x)\to 0\,,\qquad x\to\partial\Gamma \text{ from } \Gamma'\,, \tag{2.6b}$$

meant in the sense of arguments of correlation functions. Note that in (2.3) there are no correlations between points in Γ and in Γ'; that is, Γ and Γ' have been decoupled from each other by the surface interaction.

We recall the familiar relations

$$G_\mathrm{D}^\Gamma(x,x') = G_\mathrm{D}^\Gamma(x',x)\,,\qquad G_\mathrm{N}^{\Gamma'}(x,x') = G_\mathrm{N}^{\Gamma'}(x',x)\,, \tag{2.7a}$$

$$-\lim_{\substack{x\to\partial\Gamma \text{ from } \Gamma\\ x'\in\partial\Gamma}} G_\mathrm{D}^\Gamma(x,x')\overleftarrow{\partial}'_n = \delta(x,x')\,, \tag{2.7b}$$

$$-\lim_{\substack{x\to\partial\Gamma \text{ from } \Gamma'\\ x'\in\partial\Gamma}} \partial_n G_\mathrm{N}^{\Gamma'}(x,x') = \delta(x,x')\,, \tag{2.7c}$$

* If Γ' is infinite and $m^2=0$, G_N vanishes at infinity strongly enough to render this function unique. (See also appendix A.)

where $\delta(x, x')$ is the δ-function on $\partial\Gamma$. The function

$$\lim_{\substack{x\to\partial\Gamma \text{ from } \Gamma\\ x'\in\partial\Gamma}} \partial_n G_{\mathrm{D}}^{\Gamma}(x, x')\overleftarrow{\partial}'_n = \lim_{\substack{x'\to\partial\Gamma \text{ from } \Gamma\\ x\in\partial\Gamma}} \partial_n G_{\mathrm{D}}^{\Gamma}(x, x')\overleftarrow{\partial}'_n \equiv \partial_n G_{\mathrm{D}}^{\Gamma}\overleftarrow{\partial}'_n \tag{2.8}$$

is a negative definite kernel on $\partial\Gamma$ which will later appear often.

Inhomogeneous Dirichlet boundary conditions are implemented by replacing L_Γ of (2.2) by

$$L_{\Gamma A} = L_\Gamma + \delta(f(x))A(x)\partial_\mu\phi(x)\partial_\mu f(x). \tag{2.9}$$

In simplified notation,

$$L_{\Gamma A} = -\tfrac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 + \delta(\sigma)\phi\,\partial_n\phi - \delta(\sigma)A\,\partial_n\phi \tag{2.10}$$

where the operator ordering in the third term, specified in (1.2), must be kept in mind. From (2.3) it follows by substitution for J that

$$\int \mathscr{D}\phi \exp\left[\int L_{\Gamma A} + \int J\phi\right]$$

$$= \mathrm{const}\,(\Gamma)\exp\left[\tfrac{1}{2}\int_\Gamma\int_\Gamma JG_{\mathrm{D}}^{\Gamma}J + \tfrac{1}{2}\int_{\Gamma'}\int_{\Gamma'} JG_{\mathrm{N}}^{\Gamma'}J\right.$$

$$\left. + \tfrac{1}{2}\int_{\partial\Gamma}\int_{\partial\Gamma} A\partial_n G_{\mathrm{D}}^{\Gamma}\overleftarrow{\partial}'_n A - \int_{\partial\Gamma}\int_\Gamma A\partial_n G_{\mathrm{D}}^{\Gamma} J\right], \tag{2.11}$$

where, however, we have discarded a singular linearly divergent term proportional to $\int_{\partial\Gamma} A^2$ in the exponent which, to allow explicit subtraction in (2.10), requires smearing the last $\partial_n\phi$ in (2.10) with an L^2 function within a layer adjoining $\partial\Gamma$ inside Γ and letting that layer shrink to zero width. (We give this formulae in subsect. 5.2.) From (2.11) and (2.7b) it now follows that (2.6a) is replaced by

$$\phi(x) \to A(x\in\partial\Gamma), \qquad x\to\partial\Gamma \text{ from } \Gamma, \tag{2.12}$$

while (2.6b) remains unchanged.

If $\partial\Gamma$ is a plane, there is no difficulty in taking space parallel to the plane $3-\varepsilon$ dimensional ($\varepsilon > 0$), such that $\nu = 4-\varepsilon$. We give formulae, to be used later, for this case. In (2.2), we set $f(x) = y$, with* $\boldsymbol{x}$ the $3-\varepsilon$ dimensional coordinate along the plane and the y-axis pointing into Γ orthogonal to the plane, such that $y, y' > 0$ is the Dirichlet and $y, y' < 0$ the Neumann region. To simplify, we set $m = 0$ and then have

$$G_{\mathrm{N}}^{\mathrm{D}}(x, x') = \tfrac{1}{4}\pi^{-2+\varepsilon/2}\Gamma(1-\tfrac{1}{2}\varepsilon)$$

$$\times\{[(\boldsymbol{x}-\boldsymbol{x}')^2 + (y-y')^2]^{-1+\varepsilon/2} \mp [(\boldsymbol{x}-\boldsymbol{x}')^2 + (y+y')^2]^{-1+\varepsilon/2}\}, \tag{2.13}$$

with Fourier transforms

$$\int \mathrm{d}\boldsymbol{x}\, G_{\mathrm{N}}^{\mathrm{D}}(\boldsymbol{x}y, \boldsymbol{x}'y')\,\mathrm{e}^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}$$

$$= (2k)^{-1}[\mathrm{e}^{-k|y-y'|} \mp \mathrm{e}^{-k|y+y'|}] \equiv \tilde{G}_{\mathrm{N}}^{\mathrm{D}}(k, yy'), \tag{2.14}$$

* All bold face arguments are $3-\varepsilon$ dimensional.

where $k=|\boldsymbol{k}|$ (or $=(\boldsymbol{k}^2+m^2)^{1/2}$ if $m>0$). One notes the singularity in

$$\partial_y \partial_{y'} \tilde{G}_{\mathrm{D}}(k, yy') = \delta(y-y') - k^2 \tilde{G}_{\mathrm{N}}(k, (-y)(-y')) . \tag{2.15}$$

Eq. (2.8) becomes

$$\mathrm{FT}\, \partial_n G_{\mathrm{D}} \overleftarrow{\partial}'_n = -k . \tag{2.16}$$

For one argument on the boundary with inner normal derivative,

$$\partial_{y'} G_{\mathrm{D}}(\boldsymbol{x}y, \boldsymbol{x}'y')|_{y'=0} = \pi^{-2+\varepsilon/2} \Gamma(2-\tfrac{1}{2}\varepsilon) y [(\boldsymbol{x}-\boldsymbol{x}')^2 + y^2]^{-2+\varepsilon/2} , \tag{2.17}$$

with Fourier transform e^{-ky}.

The Fourier transform with respect to all variables is (again, for $m \geqslant 0$)

$$\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{iqy} \int_{-\infty}^{\infty} \mathrm{d}y'\, \mathrm{e}^{iq'y'} [\theta(y)\theta(y')\tilde{G}_{\mathrm{D}}(k, yy') + \theta(-y)\theta(-y')\tilde{G}_{\mathrm{N}}(k, yy')]$$
$$= 2\pi\delta(q+q')(k^2+q^2)^{-1} - k^{-1}(k^2+q^2)^{-1}(k^2+q'^2)^{-1}[k^2+ik(q+q')+qq']. \tag{2.18}$$

Here the first term is the free-space function, and the second term, being generated by the surface interaction in (2.2), is a sum of factorizing parts. Eq. (2.18) is less singular than are the Fourier transforms of the Dirichlet or Neumann function.

3. Renormalization

3.1. REGULARIZATION AND DIVERGENCES

To be able to add the interaction term to (2.10), we must choose a regularization. While the renormalized theory is fixed by the renormalization conditions for the superficially divergent (s.d.) vertex functions (the amputated one-particle irreducible (1PI) parts of the connected Green functions) alone, the precise form of the counter terms depends on the regularization employed. For most of our considerations, a plane $\partial\Gamma$ is sufficient, and then, for calculations, dimensional regularization is the most convenient: $3-\varepsilon$ space dimensions ($\operatorname{Re}\varepsilon>0$), one (euclidean or minkowskian) time dimension. This is as effective as introducing a lattice in space while keeping time continuous, but by itself does not break Lorentz invariance such that renormalization of the speed of light is not needed. Leaving the time direction unregularized means, however, that the ordering prescriptions in (2.2) and (2.9) must be kept in mind and that the form of the counter terms to these "bare" terms is, in general, not the one expected on the basis of naive canonical reasoning. (Pauli–Villars regularization is stronger and more generally applicable, but more cumbersome to calculate with.)

Therefore, with plane $\partial\Gamma$, we add to (2.10) the interaction term

$$L_{\mathrm{int}} = -\tfrac{1}{24} g \mu^{\varepsilon} \phi^4 \tag{3.1}$$

and counter terms to cancel the singularities proportional to ε^{-n} developing in Green functions when $\varepsilon \searrow 0$. Such singularities stem from coalescing vertices in (here, euclidean) s.d. 1PI Feynman graphs. Counter terms need be associated only with final subtractions, since divergences due to the coalescence of a subset of the vertices are compensated by the counter terms for final subtractions of the corresponding subgraph [3, 6]. In the final subtraction, all vertices of the graph coalesce.

It is now convenient to separate the Dirichlet and Neumann propagator (2.13) or (2.14) or (2.18) into the free-space part and the remainder, which involves the surface vertex in (2.2) at least once and which we call the surface propagator. [This separation is also manifest for non-planar $\partial\Gamma$, cf. (A.11), (A.14).] If a graph has at least one surface propagator, it can coalesce only on $\partial\Gamma$. Thus, the 1PI graphs requiring final subtraction are either free-space ones, if they involve free-space propagators only, or surface ones. The free-space ones require in L the usual covariant counter terms,

$$\Delta L = -(Z_3-1)\tfrac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - (Z_1-1)\tfrac{1}{24}g\mu^{\varepsilon}\phi^4 - (Z_2-1)\tfrac{1}{2}m^2\phi^2\,, \tag{3.2}$$

since the s.d. of these graphs is the usual one:

$$\mathbb{D}_\infty = 4 - E\,, \tag{3.3}$$

where E is the (even) number of external lines.

3.2. SURFACE GRAPHS AND SURFACE COUNTER TERMS

It here suffices to consider $\partial\Gamma$ as effectively flat. The singularity of a surface propagator, with both endpoints close to $\partial\Gamma$ and also to each other, is the same as for the free-space propagator, see (2.13). Therefore, if one vertex in a coalescing group of vertices is fixed near $\partial\Gamma$, the s.d. therefrom is the same as for the corresponding free-space graph. There is in addition, however, the integration of the fixed vertex over a small distance across $\partial\Gamma$. This reduces the s.d. relative to the free-space graph by one, such that

$$\mathbb{D}_{\partial\Gamma} = 3 - E\,. \tag{3.4}$$

Thus, for $E=0$, a "vacuum" graph, the divergence is cubic (we return to this in sect. 4), and for $E=2$, linear. The latter requires one to add to the action density

$$\Delta L_{\partial\Gamma} = (Z_4-1)\delta(\sigma)\phi\,\partial_n\phi + (c_1\Lambda + c_2 R^{-1}\ln\Lambda)\delta(\sigma)\phi^2\,. \tag{3.5}$$

In dimensional regularization, Z_4-1 can be chosen as a power series in ε^{-1}. The linear divergence also requires the term proportional to the cutoff (up to logarithms), absent if dimensional regularization were sufficient, and the term proportional to a typical curvature R^{-1} of $\partial\Gamma$, with logarithmically divergent coefficient (here indicated symbolically), vanishing for flat $\partial\Gamma$.

Lastly we consider the effect of the A-vertex in (2.10). It binds one external leg of the graph with a normal derivative to the surface. The s.d. of a surface graph is then the same as if that line were amputated, since a gain of three powers $(2+1)$ from the line and derivative is compensated by the same loss due to the three-dimensional integration over the surface, supposing that the endpoint on the surface is smeared with a smooth function A. (It actually suffices that the endpoint does not coincide with some other endpoint of the graph on $\partial\Gamma$.) Thus, altogether

$$\mathbb{D}_{\partial\Gamma A} = 3 - E - E_A , \tag{3.6}$$

and $\mathbb{D} = 1$ for $E = E_A = 1$ and for $E = 0$, $E_A = 2$. This demands in addition to (3.5) the counter terms

$$\begin{aligned} \Delta L_{\partial\Gamma A} = &-(Z_5 - 1)\delta(\sigma)A\partial_n\phi + (c_3\Lambda + c_4 R^{-1}\ln\Lambda)\delta(\sigma)A\phi \\ &+ (c_5\Lambda + c_6 R^{-1}\ln\Lambda)\delta(\sigma)A^2 . \end{aligned} \tag{3.7}$$

Again, $Z_5 - 1$ can be chosen in dimensional regularization to be a power series in ε^{-1}, and what was said after (3.5) applies to the c-terms. Collecting the action and also adding, for completeness, the source term for ϕ^2 we have

$$\begin{aligned} L = &-\tfrac{1}{2}Z_3\partial_\mu\phi\partial_\mu\phi - \tfrac{1}{24}Z_1 g\mu^\varepsilon\phi^4 - \tfrac{1}{2}Z_2 m^2\phi^2 + J\phi + \tfrac{1}{2}KZ_2\phi^2 \\ &+ Z_4\delta(\sigma)\phi\partial_n\phi - Z_5\delta(\sigma)A\partial_n\phi + Z_6\mu^{-\varepsilon}m^2K - Z_6\tfrac{1}{2}\mu^{-\varepsilon}K^2 \\ &+ Z_7\delta(\sigma)\mu^{-\varepsilon}\partial_n K + c\text{-terms} , \end{aligned} \tag{3.8}$$

where the c-terms now also encompass

$$(c_7\Lambda + c_8 R^{-1}\ln\Lambda)\delta(\sigma)\mu^{-\varepsilon}K .$$

The perturbation expansion with (3.8) (setting $K \equiv 0$) is discussed in appendix B, and here we only summarize the results.

The calculations on the Dirichlet and on the Neumann side can be done separately, due to the decoupling of the two regions mentioned after (2.3). The (unamputated!) Green functions on the Dirichlet side obey the Dirichlet condition where $A = 0$, and are, as a consequence of this, independent of the renormalization condition that fixes the choice of Z_4 which, in dimensional renormalization, is

$$Z_4 = 1 + \tfrac{1}{16}\pi^{-2}\varepsilon^{-1}g + \mathrm{O}(g^2) . \tag{3.9}$$

(That it differs from the usual

$$Z_3 = 1 - (3\cdot 2^{10}\pi^4\varepsilon)^{-1}g^2 + \mathrm{O}(g^3) \tag{3.10}$$

is due to the incompleteness of the regularization as emphasized in subsect. 3.1.) Also the c-terms in (3.5) are, for (unamputated) Green functions if $A = 0$, ineffective on the Dirichlet side.

In the separate calculation on the Dirichlet and Neumann sides, the difference in propagators actually has the effect that in (3.5) and (3.7), the coefficients of the counter terms are different on the two sides. Namely, the counter terms then stand, as they are used, infinitesimally off $\partial\Gamma$ inside Γ and inside Γ', respectively, and they do not commute with the decoupling generating term $\phi\partial_n\phi\delta(\sigma)$, in the interpretation (2.2). In particular, $\Delta L_{\partial\Gamma A}$ of (3.7) has to be set zero on the Neumann side.

On the Neumann side, the boundary condition (2.6b) cannot be upheld even to first order in perturbation theory, due to the necessity, for finiteness, of the c_1 term in (3.5). This terms, which destroys the Neumann property because of (2.7c), represents (to first order) the $\Delta^{-1}\delta(x)$ subtraction in (5.6a) below, or the corresponding subtraction in the regularization (6.16), the principal value being proportional to the renormalized first-order 2-point vertex in the sense of (B.1). However, since the Neumann part factors off and is A independent [for this the Neumann property (1.5b) of the bare propagators suffices], we need not discuss it further.

3.3. RENORMALIZATION GROUP PROPERTIES

The functional integral with action density (3.8) (setting $K\equiv 0$) is denoted by $\Psi(A\,|\,J)$. It factorizes into the Dirichlet part, depending on A and on J in Γ, and the Neumann part, depending on J in Γ', and we disregard the second factor until sect. 4.

From (3.8) one derives [7], by differentiation with respect to μ, the renormalization group equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g,\varepsilon)\frac{\partial}{\partial g}+\gamma(g)\int_{\Gamma}J\frac{\delta}{\delta J}-\sigma(g)\int_{\partial\Gamma}A\frac{\delta}{\delta A}+\eta(g)m^2\frac{\partial}{\partial m^2}\right]\Psi(A|J)=0\,. \tag{3.11}$$

Here, as usual,

$$\beta(g,\varepsilon)=-\varepsilon g+\beta(g)=-\varepsilon g\left[1+g\frac{\partial}{\partial g}\ln(Z_1Z_3^{-2})\right]^{-1}$$

$$=-\varepsilon g+(16\pi^2)^{-1}3g^2+\mathrm{O}(g^3)\,, \tag{3.17a}$$

$$\gamma(g)=\tfrac{1}{2}\beta(g,\varepsilon)\frac{\partial}{\partial g}\ln Z_3=(3\cdot 2^{10}\pi^4)^{-1}g^2+\mathrm{O}(g^3)\,, \tag{3.12b}$$

$$\eta(g)=\beta(g,\varepsilon)\frac{\partial}{\partial g}\ln(Z_2Z_3^{-1})=(16\pi^2)^{-1}g+\mathrm{O}(g^2)\,, \tag{3.12c}$$

while

$$\sigma(g)=\beta(g,\varepsilon)\frac{\partial}{\partial g}\ln(Z_5Z_3^{-1})$$

$$=(32\pi^2)^{-1}g+\mathrm{O}(g^2) \tag{3.12d}$$

is a new parametric function, ε-free under minimal choice of

$$Z_5=1-(32\pi^2\varepsilon)^{-1}g+\mathrm{O}(g^2)\,. \tag{3.13}$$

Note that the differentiation leading to (3.11) also produces the insertion

$$\Delta L = \beta(g, \varepsilon)\frac{\partial}{\partial g}\ln(Z_4 Z_3^{-1})\cdot Z_4\delta(\sigma)\phi\partial_n\phi \tag{3.14}$$

in the functional integral. However, this insertion gives zero due to the Dirichlet condition emphasized at the end of subsect. 3.2, and we may consider the $-A\partial_n\phi$ vertex in (3.8) as going to $\partial\Gamma$ from Γ latest, as we shall make explicit in subsect. 5.4.

To keep the notation simple, we shall now take as $\partial\Gamma$ the (euclidean) time plane $y=0$, as we did at the end of sect. 2. We write the sources $J(\boldsymbol{x}y)$ and $A(\boldsymbol{z})$. The Green functions

$$\begin{aligned} &G(\boldsymbol{z}_1\cdots\boldsymbol{z}_l|\boldsymbol{x}_1y_1\cdots\boldsymbol{x}_ny_n) \\ &\quad=\prod_{j=1}^{l}[\delta/\delta A(\boldsymbol{z}_j)]\prod_{i=1}^{n}[\delta/\delta J(\boldsymbol{x}_iy_i)]\ln\Psi(A|J)|_{A=J=0} \end{aligned} \tag{3.15}$$

are connected, and zero unless $n+l=\text{even}$. Eq. (3.11) becomes

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g,\varepsilon)\frac{\partial}{\partial g}+n\gamma(g)-l\sigma(g)+\eta(g)m^2\frac{\partial}{\partial m^2}\right]G(\boldsymbol{z}_1\cdots\boldsymbol{z}_l|\boldsymbol{x}_1y_1\cdots\boldsymbol{x}_ny_n)=0\,. \tag{3.16}$$

4. Casimir effect

4.1. GENERAL DISCUSSION

Renormalization renders all Green functions finite. It leaves untouched the quartically UV-divergent vacuum graphs, i.e. those without external lines, for which final subtraction (which would simply remove the whole graph) is not prescribed. It can be meaningful, however, to compare the vacuum energy, obtained (for the scalar field) by

$$E_{\bar\Gamma} = -\lim_{T\to\infty} T^{-1}\ln\int\mathscr{D}\phi\,\exp\left[\int_0^T \mathrm{d}t\int_{\bar\Gamma}\mathrm{d}\boldsymbol{x}\,L(\phi,\partial\phi)\right] \tag{4.1}$$

for different ν-1 dimensional regions $\bar\Gamma$ and with different boundary conditions on the ν-2 dimensional boundary $\partial\bar\Gamma$. (The relation to the notation so far is: $\Gamma=\bar\Gamma\times[0,T]$, $\partial\Gamma=\partial\bar\Gamma\times[0,T]$ plus the irrelevant $t=0$ and $t=T$ closures.)

One easily sees, however, that the quantity that is simple to compute is not the vacuum (i.e. ground-state) energy in $\bar\Gamma$, but the total energy, which is the ground-state energy in $\bar\Gamma$ with, e.g. Dirichlet boundary conditions, plus the one in the complementary region $\bar\Gamma'$ with Neumann boundary conditions. Then the boundary-independent free-space energy can be omitted, and the remainder is given by surface graphs only. In particular, in the simplest setting of Dirichlet conditions on the inner sides of two parallel plates, at distance L, the Neumann part is independent of L such that the L-dependence of the total energy is the same as if the two Neumann regions were absent, disregarding the free-space part.

As we shall see, the Casimir potential between disjoint surfaces is always well defined. That for a single surface, e.g. a sphere, in general is not, at least for the family of boundary conditions discussed in appendix A, due to divergence of the free-field part if taken absolutely and not, for example, relative to some other shape. In such a case, the physical problem requires a more complete formulation, which will then imply some other boundary conditions [8] than the (idealizing) Dirichlet (or Neumann) one.

4.2. FREE FIELD

Here, the surface graphs have merely the $\phi\partial_n\phi$ vertices on $\partial\Gamma$ shown in (2.2), and it is easy to derive the simple graphical expansion*

$$E_{\mathrm{D}\bar{\Gamma}}+E_{\mathrm{N}\bar{\Gamma}'}-\mathrm{const}=-\lim_{T\to\infty}\tfrac{1}{2}T^{-1}[2\,\mathrm{Tr}\,\overline{\partial_n G}+\tfrac{1}{2}2^2\,\mathrm{Tr}\,\overline{\partial_n G}\cdot\overline{\partial_n G}$$

$$+\tfrac{1}{3}2^3\,\mathrm{Tr}\,\overline{\partial_n G}\cdot\overline{\partial_n G}\cdot\overline{\partial_n G}+\cdots], \tag{4.2}$$

where Tr is the trace on the surface $\partial\Gamma=\partial\bar{\Gamma}\times[0,\,T]$ and $\overline{\partial_n G}$ is defined in (A.5b). The first term in the square bracket is zero under appropriate (e.g., Pauli–Villars) regularization for symmetry reasons. (This reflects the well-known fact that the strongest, cubically divergent part of the surface energy has the opposite sign in the Dirichlet and Neumann case.) The higher terms in (4.2) all vanish for flat $\partial\Gamma$ (i.e. $\partial\bar{\Gamma}$), since then $\overline{\partial_n G}$ is zero (see appendix A).

In the arrangement of two parallel plates in distance L, however, $\overline{\partial_n G}$ is not zero if its two arguments are on different plates. In the massless theory in ν dimensions, i.e. two ν-2 dimensional parallel plates in ν-1 dimensional space,

$$\overline{\partial_n G}=-\tfrac{1}{2}\pi^{-\nu/2}\Gamma(\tfrac{1}{2}\nu)[(\boldsymbol{x}-\boldsymbol{x}')^2+L^2]^{-\nu/2}L\,, \tag{4.3}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the $(\nu-2+1)$-dimensional arguments on the two plates extended in euclidean time. While on the r.h.s. of (4.2) the odd terms all vanish, the even ones are easily summed and lead to the well-known result

$$(E_{\mathrm{D}\bar{\Gamma}}+E_{\mathrm{N}\bar{\Gamma}'}-\mathrm{const})/\mathrm{area}$$

$$=-2^{-\nu}\pi^{-\nu/2}\Gamma(\tfrac{1}{2}\nu)\zeta(\nu)L^{-\nu+1}\qquad=-\tfrac{1}{1440}\pi^2L^{-3}\ \text{if}\ \nu=4\,. \tag{4.4}$$

Hereby in $\zeta(\nu)=\sum_{n=1}^{\infty}n^{-\nu}$ the nth term is obtained from the "one-loop" polygon with n vertices on each plate. Eq. (4.2) also shows that the Casimir potential decays exponentially in the massive case.

* Eq. (4.2) is a special case of a formula of Balian and Duplantier [9].

4.3. INTERACTING FIELD

The higher-order vacuum surface graphs are (for $\nu = 4$) cubically divergent [$E = 0$ in (3.3)] provided they can shrink to a point on $\partial\Gamma$. If, however, $\partial\Gamma$ consists of two disjoint pieces $\partial\Gamma_1$ and $\partial\Gamma_2$, the surface graphs on $\partial\Gamma_1$ are independent of the location of $\partial\Gamma_2$, and vice versa. Only the surface graphs with vertices on both $\partial\Gamma_1$ and $\partial\Gamma_2$ depend on the relative location, and since they cannot be shrunk to a point they are finite provided all subdivergences have been subtracted by counter terms. These are the usual free-space counter terms (3.2) and the ones on $\partial\Gamma_1$ and $\partial\Gamma_2$, as prescribed by (3.5), that render the surface graphs for these subsurfaces finite. Thus, the Casimir effect is finite and computable for any configuration of disjoint surfaces at each of which the theory has been made finite by counter terms as if the other surfaces did not exist.

Since we know (see appendix B) that the homogeneous Dirichlet condition can be implemented for the plane without need of a new renormalization parameter, the Casimir energy of a pair of parallel plates with these conditions on the two insides obeys, in massless $\phi^4_{4-\varepsilon}$ theory, according to (4.1) and (3.16),

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g,\varepsilon)\frac{\partial}{\partial g}\right]E(g,L,\mu,\varepsilon)=0\,. \tag{4.5a}$$

Thus

$$E(g,L,\mu,\varepsilon)/\text{area}\equiv L^{-3+\varepsilon}e(\bar g(g,(L\mu)^{-1},\varepsilon),\varepsilon)$$

$$=L^{-3+\varepsilon}\sum_{n=0}^{\infty}c_n(\varepsilon)[\bar g(g,(L\mu)^{-1},\varepsilon)]^n\,, \tag{4.5b}$$

where $\bar g$ is the usual sliding coupling constant and the $c_n(\varepsilon)$* are computable, with $c_0(\varepsilon)$ from (4.4).

Since the massless ϕ^4_4 theory is "asymptotically free in the infrared", $\bar g$ in (4.5b) vanishes, if $\varepsilon = 0$, proportional to $(\ln L)^{-1}$ as $L\to\infty$, such that then only the $c_0(0)$ term survives. This universality (i.e. independence of g) is a counterpart of the one observed by Lüscher [11] in a two-dimensional problem, and is, since insensitive to UV cutoffs, not affected by the fairly well established [12] non-existence of the continuum ϕ^4_4 theory in the UV.

4.4. FIRST-ORDER-CALCULATION

In terms of zeroth-order Dirichlet Green functions, we have, from (4.1), (4.5b)

$$L^{-3+\varepsilon}c_1(\varepsilon)\bar g(g,(L\mu)^{-1},\varepsilon)$$

$$=\tfrac{1}{8}g\mu^{\varepsilon}\Big[\int_0^L dy\,\{[G_D^L(\mathbf{0}y,\mathbf{0}y)-G(\mathbf{0}\mathbf{0})]^2-[G_D(\mathbf{0}y,\mathbf{0}y)-G(\mathbf{0}\mathbf{0})]^2$$

$$-[G_D(\mathbf{0}(L-y),\mathbf{0}(L-y))-G(\mathbf{0}\mathbf{0})]^2\}-2\int_L^{\infty}dy\,[G_D(\mathbf{0}y,\mathbf{0}y)-G(\mathbf{0}\mathbf{0})]^2\Big]$$

$$+\mathrm{O}(g^2)\,. \tag{4.6}$$

* We do not discuss here the IR problems [10] of the expansion in (4.5b) if $\varepsilon > 0$.

Here,

$$G_{\mathrm{D}}^{L}(\boldsymbol{x}y,\boldsymbol{x}'y')=\tfrac{1}{4}\pi^{-2+\varepsilon/2}\Gamma(1-\tfrac{1}{2}\varepsilon)\times\sum_{n=-\infty}^{\infty}\{[(\boldsymbol{x}-\boldsymbol{x}')^2+(y-y'+2nL)^2]^{-1+\varepsilon/2}-[(\boldsymbol{x}-\boldsymbol{x}')^2+(y+y'+2nL)^2]^{-1+\varepsilon/2}\}\tag{4.7}$$

is the Dirichlet Green function for a pair of $3-\varepsilon$ dimensional plates at $y=0$ and $y=L$, while G_{D} is given in (2.13) and $G(\cdots)$ is the free-space function. The cross term from the first square bracket in the curly bracket in (4.6) removes the free-space mass renormalization to the one-loop (i.e. zeroth-order) graph discussed in subsect. 4.2. The second and third terms in the curly bracket in (4.6) remove the two single-plate surface graphs to first order. The last integral in (4.6) extends these last two graphs to the whole half-space relevant for the single plate problem (otherwise the integration over $0\cdots L$ only would have given an unallowed L-dependence to the single-plate subtractions). Apart from this, in (4.6) the integration over $0\cdots L$ only is permitted since the energy contributions from the two outer regions are, after the free-space parts are subtracted, L-independent.

We now insert into (4.6)

$$\begin{aligned}G_{\mathrm{D}}^{L}(\mathbf{0}y,\mathbf{0}y)-G(\mathbf{00})&=\tfrac{1}{4}\pi^{-2+\varepsilon/2}(2L)^{-2+\varepsilon}\Gamma(1-\tfrac{1}{2}\varepsilon)\\&\quad\times[2\zeta(2-\varepsilon)-\zeta(2-\varepsilon,y/L)-\zeta(2-\varepsilon,1-y/L)]\\&=(48L^2)^{-1}[1-3(\sin(\pi y/L))^{-2}],\qquad\text{if }\varepsilon=0\,,\end{aligned}\tag{4.8}$$

where $\zeta(z,\alpha)$ is the generalized ζ-function, and find

$$\begin{aligned}c_1(\varepsilon)&=2^{-9+2\varepsilon}\pi^{-4+\varepsilon}\Gamma(1-\tfrac{1}{2}\varepsilon)^2\\&\quad\times[\zeta(2-\varepsilon)^2+(1-\cos\pi\varepsilon)B(-1+\varepsilon,3-2\varepsilon)\zeta(3-2\varepsilon)]\\&=2^{-11}\cdot3^{-2}\qquad\text{if }{}^{\star}\varepsilon=0\\&=-2^{-7}\pi^{-2}(2\gamma_1-C^2+\tfrac{1}{12}\pi^2)\qquad\text{if }\varepsilon=1\,,\end{aligned}$$

where C is Euler's constant and

$$\gamma_1=\frac{1}{2}\frac{\partial^2}{\partial s^2}[(s-1)\zeta(s)]\big|_{s=1}\,.\tag{4.9}$$

The UV finiteness of (4.6), independently of the counter terms in (3.5), generalizes to any disjoint pair of smooth surfaces: Let G_{D}^{1} be the Dirichlet Green function that vanishes on $\partial\Gamma_1$ and G_{D}^{12} the one that vanishes on $\partial\Gamma_1+\partial\Gamma_2$. Then

$$G_{\mathrm{D}}^{12}-G_{\mathrm{D}}^{1}=2G_{\mathrm{D}}^{1}\cdot[1-2\overline{\partial_nG_{\mathrm{D}}^{1}}]^{-1}\cdot\partial_nG_{\mathrm{D}}^{1}\,,$$

⋆ This result was obtained by Toms [13] using dimensional regularization.

where the surface integrations and inversion are on $\partial\Gamma_2$ only, vanishes of second order if the two arguments coalesce and go to $\partial\Gamma_1$ together. This secures the convergence of

$$\int_\Gamma \mathrm{d}x\, \{[G_D^{12}(x,x)-G(0)]^2-[G_D^1(x,x)-G(0)]^2-[G_D^2(x,x)-G(0)]^2\}$$

near $\partial\Gamma_1$ for $\varepsilon > -1$.

5. Behaviour at the boundary

5.1. CALCULATIONS TO FIRST ORDER

For the Green functions (2.13), we define Fourier transforms by

$$\int \mathrm{d}\boldsymbol{z}_1 \cdots \int \mathrm{d}\boldsymbol{z}_l \int \mathrm{d}\boldsymbol{x}_1 \cdots \int \mathrm{d}\boldsymbol{x}_n \exp\,[i\textstyle\sum \boldsymbol{k}_j\boldsymbol{z}_j + i\sum \boldsymbol{p}_i\boldsymbol{x}_i]$$
$$\times G(\boldsymbol{z}_1\cdots \boldsymbol{z}_l|\boldsymbol{x}_1y_1\cdots \boldsymbol{x}_ny_n)$$
$$=(2\pi)^{4-\varepsilon}\delta(\textstyle\sum \boldsymbol{p}+\sum \boldsymbol{k})\tilde{G}(\boldsymbol{k}_1\cdots \boldsymbol{k}_l|\boldsymbol{p}_1y_1\cdots \boldsymbol{p}_ny_n)\,.$$

Using the propagator (2.14), we find to first order in g

$$\tilde{G}(|\boldsymbol{p}y(-\boldsymbol{p})y') = \theta(y-y')\Big\{p^{-1}\,\mathrm{e}^{-py}\,\mathrm{sh}\,(py')$$
$$+2^{-5+\varepsilon}\pi^{-2+\varepsilon/2}p^{-2}g\mu^{\varepsilon}\Big[\int_0^{y'}\mathrm{d}t\,t^{-2+\varepsilon}(\mathrm{sh}\,(pt))^2\,\mathrm{e}^{-p(y+y')}$$
$$+\mathrm{e}^{-py}\,\mathrm{sh}\,(py')\int_{y'}^{y}\mathrm{d}t\,t^{-2+\varepsilon}\,\mathrm{sh}\,(pt)\,\mathrm{e}^{-pt}$$
$$+\mathrm{sh}\,(py)\,\mathrm{sh}\,(py')\int_y^\infty \mathrm{d}t\,t^{-2+\varepsilon}\,\mathrm{e}^{-2pt}\Big]\Big\}+(y\leftrightarrow y')\,, \quad (5.1)$$

where $p=|\boldsymbol{p}|$. While the Dirichlet condition $\tilde{G}((\boldsymbol{p}y(-\boldsymbol{p})0)=0$ is satisfied, $\partial_{y'}\tilde{G}((\boldsymbol{p}y(-\boldsymbol{p})y')$ has, if $\varepsilon>0$ but $y'=0$, an ε^{-1} singularity as $\varepsilon\searrow 0$, whereas, if $\varepsilon=0$, it blows up logarithmically as $y'\searrow 0$. In the first case, the remedy is the factor Z_5, given in (3.13) to this order, while at $\varepsilon=0$.

$$\lim_{y'\to 0}\,[1+(32\pi^2)^{-1}g\ln(\mu y')]\partial_{y'}\tilde{G}(|\boldsymbol{p}y(-\boldsymbol{p})y')$$
$$\equiv \tilde{G}(-\boldsymbol{p}\,|\boldsymbol{p}y)=\mathrm{e}^{-py}[1+(32\pi^2)^{-1}g(\psi(2)-\ln(2p\mu^{-1}))]$$
$$-(32\pi^2)^{-1}g\,\mathrm{e}^{py}\int_y^\infty \mathrm{d}t\,t^{-1}\,\mathrm{e}^{-2pt}\,. \quad (5.2)$$

The r.h.s. differs insignificantly from the corresponding minimal-subtraction function computed directly from (3.8) using the prescriptions of appendix B.

Letting $y \searrow 0$ in the $\varepsilon > 0$ formula for $\tilde{G}(-\boldsymbol{p}|\boldsymbol{p}y)$, again produces a $1/\varepsilon$ singularity, remedied by the factor $1+(32\pi^2\varepsilon)^{-1}g$, which is $Z_5^{-1}Z_3$ to this order. At $\varepsilon = 0$, we must again introduce a logarithmic factor,

$$\lim_{y\to 0}\{[1-(32\pi^2)^{-2}g(1+\ln(\mu y))]\tilde{G}(-\boldsymbol{p}|\boldsymbol{p}y)\} = 1\,. \tag{5.3}$$

If we form $\partial_y\tilde{G}(-\boldsymbol{p}|\boldsymbol{p}y)$, however, we must, to let $y\searrow 0$, make a subtraction, which we choose at zero momentum. Thereupon, at $\varepsilon = 0$, the same factor as in (5.2) is required:

$$\lim_{y\to 0}\{[1+(32\pi^2)^{-1}g\ln(\mu y)][\partial_y\tilde{G}(-\boldsymbol{p}|\boldsymbol{p}y)-\partial_y\tilde{G}(\boldsymbol{0}|\boldsymbol{0}y)]\}$$

$$\equiv \tilde{G}(\boldsymbol{p}(-\boldsymbol{p})|) = -p[1-(16\pi^2)^{-1}g(\ln(2p\mu^{-1})-\psi(2))]\,, \tag{5.4}$$

whereas at $\varepsilon > 0$ the factor Z_5 from (3.13) is again appropriate. As in (5.2), the r.h.s. in (5.4) differs insignificantly from the corresponding minimal-subtraction result. The functions in (5.1), (5.2), and (5.4) obey (3.16) with (3.12) to first order in g.

For computations in appendix D it is useful to observe that the r.h.s. of (5.2) and (5.4) are also given by

$$G(-\boldsymbol{p}|\boldsymbol{p}y) = e^{-py}+(64\pi^2p)^{-1}g\int_0^\infty dt\, e^{-pt}(e^{-p|y-t|}-e^{-p(y+t)})\mu^2P_+(t\mu)^{-2}\,, \tag{5.5a}$$

$$G((-\boldsymbol{p})\boldsymbol{p}|) = -p+(32\pi^2)^{-1}g\int_0^\infty dt\, e^{-2pt}\mu^2P_+(t\mu)^{-2}\,, \tag{5.5b}$$

where

$$P_+x^{-2} = \lim_{\Delta\to 0}[x^{-2}\theta(x-\Delta)-\Delta^{-1}\delta(x)-\delta'(x)\ln\Delta] \tag{5.6a}$$

is the one-sided principal value related by

$$P_+x^{-2} = [x_+^{-\lambda}-(\lambda-2)^{-1}\delta'(x)]|_{\lambda=2} \tag{5.6b}$$

to the function defined by Gelfand and Shilov [14].

The other first-order graph is the lowest-order contribution to the four-point function $G(|\boldsymbol{k}_1y_1\boldsymbol{k}_2y_2\boldsymbol{k}_3y_3(-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)y_4)$. One easily sees that it, and all first y-derivatives, are ordinary functions of the $\boldsymbol{k}$ and the y, and vanish if one or more y is set to zero without derivative.

5.2. GENERATING SURFACE ARGUMENTS

Eq. (3.8) shows that for $\varepsilon > 0$,

$$Z_5\partial_y\Phi(\boldsymbol{x}y) \xrightarrow[y\to 0]{} \delta/\delta A(\boldsymbol{x})\,. \tag{5.7}$$

Eq. (5.2) indicates that, for $\varepsilon = 0$, this needs to be replaced by

$$c(y)\partial_y \Phi(\boldsymbol{x}y) \xrightarrow[y\to 0]{} \delta/\delta A(\boldsymbol{x})\,. \tag{5.8a}$$

More generally, in view of (5.4),

$$\lim_{y\to 0}\{c(y)[\partial_y(\delta/\delta J(\boldsymbol{x}y)) - A(\boldsymbol{x})\partial_y\tilde{G}(\mathbf{0}\,|\,\mathbf{0}y)]\Psi(A\,|\,J)\} = (\delta/\delta A(\boldsymbol{x}))\Psi(A|J)\,. \tag{5.8b}$$

Here we recognize the subtraction corresponding to a particular choice of c_5 in (3.7), while the c_3 term does not contribute due to the Dirichlet property. In terms of Green functions (3.15), (5.8b) is

$$\begin{aligned}&\lim_{y\to 0}\{c(y)[\partial_y G(\boldsymbol{z}_1\cdots \boldsymbol{z}_l|\boldsymbol{x}y\boldsymbol{x}_1y_1\cdots \boldsymbol{x}_ny_n) - \delta_{n0}\delta_{l1}\delta(\boldsymbol{x}-\boldsymbol{z}_1)\partial_y\tilde{G}(\mathbf{0}|\mathbf{0}y)]\}\\ &\quad = G(\boldsymbol{z}_1\cdots \boldsymbol{z}_l\boldsymbol{x}|\boldsymbol{x}_1y_1\cdots \boldsymbol{x}_ny_n)\,,\end{aligned} \tag{5.8c}$$

valid, however, only in the sense of distributions with smooth test functions A and J. We can choose

$$c(y) = [\partial_y\tilde{G}(-\boldsymbol{p}|\boldsymbol{p}y) - \partial_y\tilde{G}(\mathbf{0}|\mathbf{0}y)]^{-1}\tilde{G}(\boldsymbol{p}(-\boldsymbol{p})|)|_{\boldsymbol{p}=\mathbf{0}} \tag{5.9}$$

which defines $c(y)$ up to a merely g (and, in the massive case, $\ln m\mu^{-1}$) dependent factor that depends on the convention used in computing the r.h.s. of (5.9). $c(y)$ as defined by (5.9) satisfies

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma(g) - \sigma(g) + \eta(g)m^2\frac{\partial}{\partial m^2}\right]c(y) = 0\,, \tag{5.10}$$

which renders (5.8c) consistent with (3.16). In the massless case, $c(y)$ is a power series in g with coefficients polynomial in $\ln \mu y$. In the massive case (5.9) also yields a dependence on m which can, however, be factored away under neglect also of $O(y)$ terms, since the UV effect to be achieved by $c(y)$ is m-independent. Note that, while $Z_5Z_3^{-1}$ in (3.10d) is, in minimal subtraction convention, uniquely determined by $\sigma(g)$, this is not so, except in the leading and next-to-leading logs, for $c(y)$ from (5.10), even if $m = 0$.

Eq. (5.8c) is easily interpreted: an external leg of G is upon normal differentiation bent to the boundary. Therefore, we need discuss only the cases of superficial divergence, $n = 1$, $l = 0$ and $n = 0$, $l = 1$, since the other cases are then covered by skeleton expansions (appendix B). In the first case,

$$\lim_{y\to 0}\{c(y)\partial_y G(|\boldsymbol{x}y\boldsymbol{x}_1y_1)\} = G(\boldsymbol{x}|\boldsymbol{x}_1y_1)\,, \tag{5.11}$$

we must remember (see appendix B) that the functions here are singular if any y goes to zero; $G(|\boldsymbol{x}y\boldsymbol{x}_1y_1)$ is comparable to a four-point function of the covariant theory, with $\boldsymbol{x}0$ and $\boldsymbol{x}_10$ the two suppressed arguments, and $G(\boldsymbol{x}|\boldsymbol{x}_1y_1)$ is comparable to a three-point (i.e. unamputated mass vertex) function of the covariant theory, with $\boldsymbol{x}_10$

the suppressed argument. In this sense, (5.11) means that the renormalized vertex function is gotten from the four-point function by binding two legs together with a point-split bare vertex, multiplying by a factor that depends logarithmically on the splitting distance (and on g and μ) only, and letting that distance go to zero.

In the covariant case, the reason for this factorization is a Wilson short-distance expansion [15], which to the order required here can be derived elementarily by manipulating algebraically the Bethe-Salpeter equation (see, e.g. [16]). In analogy, the reason for the validity of (5.11) is a small-y expansion of the "four-point function" on the l.h.s., the leading term being y times powers of $\ln(\mu y)$. The proof, on the basis of the Bethe–Salpeter equation, will not be given here. We merely note that the reason for $c(y)$ being merely logarithmic, in spite of superficial linear divergence, is the appearance of the factor y due to the Dirichlet property.

The other case,

$$\lim_{y\to 0}\left\{c(y)[\partial_y G(z|\boldsymbol{x}y)-\delta(\boldsymbol{x}-\boldsymbol{z})\int \mathrm{d}\boldsymbol{x}'\,\partial_y G(z|\boldsymbol{x}'y)]\right\}=G(z\boldsymbol{x}|)\,, \qquad (5.12)$$

is analogous to forming a two-mass-vertices correlation function in the covariant theory. Also there, an additive renormalization is required before the remaining divergence can be removed multiplicatively. Overlapping divergences are disentangled by the subtraction (or momentum differentiation, which corresponds to multiplication by $x_i - z_i$). Again, the proof, analogous to the one in the covariant case, will not be given here. In QED, an analogous procedure is applied when computing a quadratically divergent current correlation function or a photon self-energy part or a linearly divergent electron self-energy part.

5.3. BOUNDARY VALUE OF Φ

Taking Φ to the boundary in (3.8), where it vanishes, requires crossing the $-Z_5 A\partial_n\Phi$ term, and the canonical commutation relation gives

$$Z_5^{-1}Z_3\Phi(\boldsymbol{x}y)\xrightarrow[y\to 0]{} A(\boldsymbol{x})\,, \qquad (5.13)$$

for $\varepsilon>0$, while, as we saw in subsect. 5.1, for $\varepsilon=0$ this needs to be replaced by

$$a(y)\Phi(\boldsymbol{x}y)\xrightarrow[y\to 0]{} A(\boldsymbol{x}) \qquad (5.14a)$$

or, explicitly,

$$\lim_{y\to 0}\{a(y)[\delta/\delta J(\boldsymbol{x}y)]\Psi(A|J)\}=A(\boldsymbol{x})\Psi(A|J)\,, \qquad (5.14b)$$

or

$$\lim_{y\to 0}\{a(y)G(z_1\cdots z_l|\boldsymbol{x}y\boldsymbol{x}_1y_1\cdots\boldsymbol{x}_ny_n)\}=\delta_{l1}\delta_{n0}\delta(\boldsymbol{x}-\boldsymbol{z}_1)\,, \qquad (5.14c)$$

valid, again, only in the sense of distributions with smooth test functions A and J. We can choose

$$a(y) = \tilde{G}(\mathbf{0}|\mathbf{0}y)^{-1}\,, \tag{5.15}$$

which obeys, according to (3.16),

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma(g) + \sigma(g) + \eta(g)m^2\frac{\partial}{\partial m_2}\right]a(y) = 0\,, \tag{5.16}$$

and what was said after (5.10) on $c(y)$ applies also to $a(y)$.

Eq. (5.14c) is easily interpreted: if $l = 1$, $n = 0$, the l.h.s. vanishes due to the Dirichlet property if $\boldsymbol{x} \neq \boldsymbol{z}_1$. Thus, the r.h.s. if finite, and it is finite by the choice (5.15), must be a distribution with support on $\boldsymbol{x} = \boldsymbol{z}$, and by power counting, this must be a δ-function. In all other cases (i.e. unless $l = 1$, $n = 0$) power counting does not allow a singularity as strong as a δ-function on the r.h.s. of (5.1c) if $\boldsymbol{x}$ coalesces with any $\boldsymbol{z}$, the remaining arguments being understood not to coalesce with these or as being integrated over with smooth test functions.

5.4. A FORMULA FOR $\Psi(A|J)$

By repeating operation (5.8b) indefinitely, we can build up $\Psi(A|J)$ from $\Psi(0|J)$. However, due to the δ-singularity (2.15) of $\partial_y\partial_{y'}G(|\boldsymbol{x}y\boldsymbol{x}'y')$ at $\boldsymbol{x} = \boldsymbol{x}'$, $y = y'$, we must in the repetition employ a square-integrable smearing function $c_\Lambda(y)$ that, as $\Lambda \to \infty$, approaches $\delta(y)$, e.g.

$$c_\Lambda(y) = \Lambda^{-1}(n!)^{-1}(y/\Lambda)^n \exp(-y/\Lambda)\,, \qquad n \geq 1\,. \tag{5.17}$$

Then

$$\begin{aligned}\Psi(A|J) = \lim_{\Lambda\to\infty} \exp\Bigg[&\int \mathrm{d}\boldsymbol{z}\, A(\boldsymbol{z}) \int_0^\infty \mathrm{d}y\, c_\Lambda(y)c(y)\partial_y(\delta/\delta J(\boldsymbol{x}y)) \\ &-\tfrac{1}{2}\int_0^\infty \mathrm{d}y\, c_\Lambda(y)c(y)\int_0^\infty \mathrm{d}y'\, c_\Lambda(y')c(y')\partial_y\partial_{y'}\tilde{G}(|\mathbf{0}y\mathbf{0}y') \\ &\times \int \mathrm{d}\boldsymbol{z}\, A(\boldsymbol{z})^2\Bigg]\Psi(0|J)\,,\end{aligned} \tag{5.18}$$

which is, in the limit $\Lambda \to \infty$ and using (5.11), easily checked to be consistent with (5.8b). The subtraction in the exponent in (5.18) corresponds to the c_5 term in (3.7), and is needed even in the free-field case as remarked after (2.11). If $\Psi(0|J)$ satisfies (3.11) with $A = 0$, then, due to (5.10), $\Psi(A|J)$ also does with $A \neq 0$.

In view of (3.15), we write

$$\Psi(0|J) = \exp G(J)\,. \tag{5.19}$$

Then (5.18) becomes, in symbolic notation

$$\Psi(A|J) = \exp\left\{G(J + cA\partial_y) - \tfrac{1}{2}\int A^2[c\partial_y G_2 \overleftarrow{\partial}_{y'} c]\right\}, \tag{5.20}$$

where the Λ-smearing and limit is understood. The subtraction removes the singular part if the first term in the exponent is expanded in A, and effects $\tilde{G}(\mathbf{00}|)=0$. Only the J, A two-point function is operative in yielding (5.14b) from (5.20). The compact formula (5.20) will be useful in the following section.

6. Schrödinger equation

6.1. GENERALITIES

The Schrödinger functional $\Psi(A|J)$ is the scalar product $\langle A|e^{J\phi}\rangle$ of a state specified by the function A at (euclidean) time zero with the state obtained from the vacuum by operating on it with sources at various $y>0$. (More details will be given in sect. 7.) The dependence on the time of specification of A by shifting this time by $\Delta\tau$ in the negative y-direction is the same as when shifting the sources by $\Delta\tau$ in the positive y-direction, which means replacing $J(xy)$ by $J(xy-\Delta\tau)$, since the vacuum state is translation invariant. The Schrödinger equation expresses this dependence by an operation on the functional dependence on A alone. Infinitesimally,

$$\int dx\ \partial_y J(\boldsymbol{x}y)[\delta/\delta J(\boldsymbol{x}y)]\Psi(A|J)=H(A,\delta/\delta A)\Psi(A|J)\,, \tag{6.1}$$

provided J has support at $y>0$ only. $H(A,\delta/\delta A)$ is the hamiltonian as a functional differential operator. In particular, (6.1) implies

$$H(A,\delta A)\Psi(A|0)=0\,, \tag{6.2}$$

i.e. $H(A,\delta/\delta A)$ has the vacuum energy subtracted. We shall construct $H(A,\delta/\delta A)$ by analyzing the l.h.s. of (6.1) with the help of (5.20).

6.2. FREE FIELD AND COMBINATORICS

Disregarding renormalization problems, we can write, according to (2.3)

$$\Psi(0|J)=\exp[-P(\delta/\delta J)]\exp[\tfrac{1}{2}JG_{\mathrm{D}}J]\,, \tag{6.3}$$

where $P(\phi)$ is the interaction part of the action. (Not to have derivatives in $P(\phi)$, ϕ should be the unrenormalized field.) Then, from (2.11)

$$\Psi(A|J)=\exp[-P(\delta/\delta J)]\Psi^0(A|J)\,, \tag{6.4a}$$

where

$$\Psi^0(A|J)=\exp[\tfrac{1}{2}JG_{\mathrm{D}}J-A\partial_n G_{\mathrm{D}}J+\tfrac{1}{2}A\partial_n G_{\mathrm{D}}\overleftarrow{\partial}'_n A] \tag{6.4b}$$

with ∂_n the outer normal derivative $-\partial_y|_{y=0}$. Straightforward calculation gives

$$\begin{aligned}\partial_y J[\delta/\delta J]\Psi(A|J)=\exp[-P(\delta/\delta J)]\{&-\tfrac{1}{2}J(\partial_y G_{\mathrm{D}}+G_{\mathrm{D}}\overleftarrow{\partial}_{y'})J\\&+J\partial_y G_{\mathrm{D}}\overleftarrow{\partial}_n A-P'(\delta/\delta J)\partial_y\delta/\delta J\}\Psi^0(A|J)\,.\end{aligned} \tag{6.5}$$

The following equations hold (the dot means integration over the $y=0$ plane)

$$\partial_y G_D + G_D \overleftarrow{\partial}_{y'} = G_D \overleftarrow{\partial}_n \cdot \partial_n G_D\,, \tag{6.6a}$$

$$\partial_y G_D \overleftarrow{\partial}'_n = G_D \overleftarrow{\partial}_n \cdot \partial_n G_D \overleftarrow{\partial}'_n\,, \tag{6.6b}$$

$$\partial_n G_G \overleftarrow{\partial}'_n \cdot \partial'_n G_D \overleftarrow{\partial}''_n = \overleftarrow{\partial}_i \delta(\ ) \overrightarrow{\partial}_i + m^2 \delta(\ )\,. \tag{6.6c}$$

Namely, (6.6a) holds if (6.6b) does since, as a function of the right argument, both sides of (6.6a) are solutions of the Poisson equation with the same boundary value, due to (2.4), of the normal derivative at $y=0$, and vanish at infinity. (6.6b) holds since, as a function of the left argument, both sides are solutions of the Poisson equation with common boundary value, due to (2.7b), and vanish at infinity. (6.6c) follows from (6.6b) due to (2.4b) and (2.7b). Inversely, from (6.6a), (6.6b) follows. (6.6a) is actually a special case of the familiar formula for variation of G_D by boundary variation

$$[\delta/\delta\sigma(x)] G_D(x', x'') = G_D(x', \cdot\,) \overleftarrow{\partial}_n|_{xx} |\partial_n G_D(\cdot\,, x'')\,, \tag{6.6d}$$

with x' and x'' away from x, for smooth $\partial\Gamma$ with $x \in \partial\Gamma$.

Using (6.6a, b) in (6.5) we have

$$\begin{aligned}\partial_y J[\delta/\delta J]\Psi(A|J) = \exp[-P(\delta/\delta J)] \cdot \{-\tfrac{1}{2} J G_D \overleftarrow{\partial}_n \cdot \partial_n G_D J \\ + J G_D \overleftarrow{\partial}_n \cdot \partial_n G_D \overleftarrow{\partial}'_n A - P'(\delta/\delta J)\partial_y[\delta/\delta J]\}\Psi^0(A|J)\,.\end{aligned} \tag{6.7}$$

However, from (6.4)

$$[\delta/\delta A]\Psi(A|J) = \exp[-P(\delta/\delta J)] \cdot \{-\partial_n G_D J - \partial_n G_D \overleftarrow{\partial}'_n A\}\Psi^0(A|J)$$

and similarly for $[\delta^2/\delta A \delta A]$. Using this and (6.6c) in (6.7) and observing that the last term in the curly bracket in (6.7) contains a complete differential, allows one to obtain from (6.1)

$$\begin{aligned}H(A, \delta A)\Psi(A|J) = \int \mathrm{d}z\, [-\tfrac{1}{2}[\delta^2/\delta A \delta A] + \tfrac{1}{2}\partial_i A \partial_i A + \tfrac{1}{2} m^2 A^2 \\ + P(A) + \tfrac{1}{2}(\partial_n G_D \overleftarrow{\partial}_n) - \mathrm{const}]\Psi(A|J)\end{aligned} \tag{6.8}$$

where

$$\mathrm{const} = \Psi(A|J)^{-1} \lim_{y\to\infty} P(\delta/\delta J)(zy)\Psi(A|J)\,.$$

Eq. (6.8) is the expected result. We have given this combinatorial (i.e. "graphical") derivation since the actual calculations in renormalized perturbation theory, the results of which we shall present below, follow the same combinatorial pattern.

Even in the free unregularized theory, (6.8) is not usable as it stands, as seen from the meaningless subtraction term. One needs to introduce point splitting, i.e. with the first and (second to) last term as

$$\lim_{\boldsymbol{\Delta}\to 0} \int \mathrm{d}z\, \{-\tfrac{1}{2}[\delta^2/\delta A(z)\delta A(z+\boldsymbol{\Delta})] + \tfrac{1}{2}(\partial_n G_D \overleftarrow{\partial}_n)(z, z+\boldsymbol{\Delta})\}\,.$$

A local Schrödinger equation, with local hamiltonian density $\mathcal{H}(A, \delta/\delta A)$ instead of the hamiltonian $H(A, \delta/\delta A)$, holds for local deformation of a generally non-flat surface [cf. (6.6d)]. Calculations for the free theory in two-dimensional flat space with smooth $\partial\Gamma$ are given in appendix E of [17].

6.3. ANSATZ FOR THE INTERACTING THEORY

In the regularized interacting theory, in view of (3.8), (5.7), and (5.13), eq. (6.8) becomes

$$\begin{aligned} H(A, \delta/\delta A) = \int \mathrm{d}z\, [&-\tfrac{1}{2}Z_3 Z_5^{-2}(\delta^2/\delta A\delta A) \\ &+\tfrac{1}{2}Z_3^{-1}Z_5^2\,\partial_i A\partial_i A + \tfrac{1}{2}m^2 Z_3^{-2}Z_2 Z_5^2 A^2 \\ &+\tfrac{1}{24}g\mu^{\varepsilon}Z_1 Z_3^{-4}Z_5^4 A^4 + \text{const}]\,. \end{aligned} \tag{6.9}$$

To render this expression meaningful for the transition $\varepsilon \searrow 0$, we again must use point splitting and, with hindsight, add some terms that have no analog in (6.9) since they involve $\Delta = |\mathbf{\Delta}|$ also non-logarithmically:

$$\begin{aligned} H(A, \delta/\delta A) = \lim_{\mathbf{\Delta}\to 0} \int \mathrm{d}z\, [&-\tfrac{1}{2}\bar{K}_2(\Delta)(\delta^2/\delta A(z)\delta A(z+\mathbf{\Delta})) \\ &+\tfrac{1}{2}K_2(\Delta)\partial_i A\partial_i A + \tfrac{1}{24}gK_4(\Delta)A^4 + \bar{K}(\Delta)\Delta^{-1}A(\delta/\delta A) \\ &+\tfrac{1}{2}K_2'(\Delta)\Delta^{-2}A^2 + \tfrac{1}{2}K_2''(\Delta)\Delta^{-2}(\mathbf{\Delta}\cdot\boldsymbol{\partial}A)^2 + \tfrac{1}{2}K_2'''(\Delta)m^2A^2 \\ &+\Delta^{-4}K(\Delta) + m^2\Delta^{-2}K'(\Delta) + m^4K''(\Delta)]\,. \end{aligned} \tag{6.10}$$

Here all K-functions are logarithmic functions of $\Delta\mu$, and the form of (6.10) is guessed in parallel to the work on renormalized field equations [18]. Comparison of (6.10) with (6.9) suggests, in parallel to the relations between (3.8) on one hand and (5.10), (5.18) on the other, the renormalization group equations, with

$$\mu\frac{\partial}{\partial\mu} + \beta(g) = \Delta\frac{\partial}{\partial\Delta} + \beta(g)\frac{\partial}{\partial g} \equiv \mathcal{O}\!\!\!/ \,,$$

$$[\mathcal{O}\!\!\!/ + 2\sigma(g)]\bar{K}_2 = 0\,, \tag{6.11a}$$

$$[\mathcal{O}\!\!\!/ - 2\sigma(g)](K_2, K_2', K_2'') = 0\,, \tag{6.11b}$$

$$[\mathcal{O}\!\!\!/ - 2\sigma(g) + \eta(g)]K_2''' = 0\,, \tag{6.11c}$$

$$[\mathcal{O}\!\!\!/ - 4\sigma(g)]K_4 = 0\,, \tag{6.11d}$$

$$\mathcal{O}\!\!\!/\ \bar{K} = 0\,, \tag{6.11e}$$

$$\mathcal{O}\!\!\!/\ K = 0\,, \tag{6.11f}$$

$$[\mathcal{O}\!\!\!/ + \eta(g)]K' = 0\,, \tag{6.11g}$$

$$[\mathcal{O}_{\not{p}} + 2\eta(g)]K'' = 0\,. \tag{6.11h}$$

The $m = 0$ equations (i.e., except c, g, h) we shall verify to first order in g.

We now insert (5.20) and (6.10) into (6.1). With integrals suppressed in the notation, and setting $m = 0$ to simplify computations later on, the result* is

$$\begin{aligned}
&\tfrac{1}{2}\bar{K}_2 c\,\partial_y c\,\partial_{y'} G_{xx+\Delta}(J + Ac\,\partial_y)\\
&\quad - \bar{K}_2 Ac\,\partial_y G_x(J + Ac\,\partial_y)[c\,\partial_y G_2 \partial'_y c] + \tfrac{1}{2}\bar{K}_2 A^2[c\,\partial_y G_2 \overleftarrow{\partial}_{y'} c]\\
&\quad + \tfrac{1}{2}[c\,\partial_y G_x(J + Ac\,\partial_y)]^2 - \bar{K}\Delta^{-1} Ac\,\partial_y G_x(J + Ac\,\partial_y)\\
&\quad - \bar{K}A^2\Delta^{-1}[c\,\partial_y G_2 \overleftarrow{\partial}_{y'} c] - \Delta^{-4}K + \partial_y J G_x(J + Ac\,\partial_y)\\
&= \tfrac{1}{2}K_2 \partial_i A \partial_i A + \frac{1}{4!} gK_4 A^4 + \tfrac{1}{2}K'_2 \Delta^{-2} A^2 + \tfrac{1}{2}K''_2 \Delta^{-2}(\boldsymbol{\Delta}\cdot\boldsymbol{\partial} A)^2 + \mathrm{O}(\Delta \ln \Delta)\,,
\end{aligned} \tag{6.12}$$

where the last term means equality up to terms that disappear linearly (up to logarithms) as $\Delta \to 0$. Recalling that in the free-field case, (6.6b) follows from (6.6a) and (6.6c) from (6.6b), we shall here first show that (6.12) holds with $A = 0$, and then consider (6.12) successively in increasing powers of A. This corresponds to putting J-arguments in the $A = 0$ equation with a normal derivative successively on the boundary in the sense of the $\Lambda \to \infty$ limit (5.17), (5.18). We shall see that the r.h.s. of (6.12) hereby accounts for the superficial divergences that arise in this process.

In passing, we note that if the Schrödinger equation is generalized to a curved boundary as mentioned at the end of subsect. 6.2, the terms with factors Δ^{-n}, $n \geq 1$, in the hamiltonian density corresponding to (6.10) get further contributions involving the curvature of $\partial\Gamma$ explicitly, cf. (3.5) and (3.7).

6.4. $A = 0$ EQUATION AND $\bar{K}_2$ DETERMINATION

We rewrite (6.12) for $A = 0$:

$$-\int \mathrm{d}x\ \partial_y J G_x(J) = \int \mathrm{d}\boldsymbol{z}\ \{\tfrac{1}{2}\bar{K}_2(\Delta) c\,\partial_y c\,\partial_{y'} G_{zz+\Delta}(J) + \tfrac{1}{2}\bar{K}_2(\Delta)[c\,\partial_y G_z(J)]^2 - \Delta^{-4}K(\Delta)\}\,, \tag{6.13}$$

which is the extension of (6.6a) to the interacting theory. We denote the terms in (6.13) from left to right L, R_1, R_2, and R_3. $-\mathrm{R}_3$ is given by R_1 for $J = 0$, whereupon R_1 diverges quartically as $\Delta \to 0$. This yields K from $\bar{K}_2$. Next we decompose $G_x(J)$ in L into free-space and surface parts, all J-arguments being at $y > 0$. Since the free-space part is translation invariant, only the surface part contributes to L.

For two J-arguments, (6.13) becomes, with Fourier transform taken with respect to the space variables,

$$\begin{aligned}
(\partial_y + \partial_{y'})\tilde{G}(|\boldsymbol{p}y(-\boldsymbol{p})y') = \bar{K}_2(\Delta)\Big[&\tfrac{1}{2}(2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}\ \mathrm{e}^{i\boldsymbol{k}\boldsymbol{\Delta}}\tilde{G}(\boldsymbol{k}(-\boldsymbol{k})|\boldsymbol{p}y(-\boldsymbol{p})y')\\
&+ \tilde{G}(-\boldsymbol{p}|\boldsymbol{p}y)\tilde{G}(\boldsymbol{p}|(-\boldsymbol{p})y')\Big] + \mathrm{O}(\Delta \ln \Delta)\,.
\end{aligned} \tag{6.14}$$

* Functional differentiation with respect to J at $x = \boldsymbol{x}y$ is indicated by a subscript x to the functional.

The square bracket on the r.h.s. has the form of an unamputated unrenormalized vertex, defined by point splitting. The corresponding bare vertex is $\partial_y\partial_{y'}$ or, more precisely,

$$c(y)c_\Lambda(y)\partial_y c(y')c_\Lambda(y')\partial_{y'}\big|_{\Lambda\to\infty}$$

according to subsects. 5.2 and 5.4. The l.h.s. of (6.14) is the actual renormalized unamputated vertex. Although the superficial divergence of the vertex function is 2 [see (3.4)], $\bar{K}_2(\Delta)$ is only logarithmically divergent as $\Delta \to 0$: The amputated vertex requires the counter terms

$$A(\Delta)\partial_y\partial_{y'} + B(\Delta)(\partial_y^2 + \partial_{y'}^2) + C(\Delta)p^2$$
$$+ D(\Delta)(\partial_y + \partial_{y'})p + \Delta^{-1}E(\Delta)(\partial_y + \partial_{y'}) + \Delta^{-1}F(\Delta)p + \Delta^{-2}G(\Delta)\,,$$

however, only the first term of these contributes, due to the Dirichlet condition. $\bar{K}_2(\Delta)$ may, to logarithmic accuracy in Δ, be defined by the l.h.s. of (6.14) divided by the square bracket on the r.h.s., at some values of $\boldsymbol{p}$, y, y' or, rather, in Fourier variables with respect to y and y' (e.g., $\boldsymbol{p} = 0$, $q = q' = \mu$). That the $\bar{K}_2(\Delta)$ so defined is (to logarithmic accuracy in Δ) independent of that choice follows by an argument analogous to the one used in subsect. 5.2 to ascertain the independence of $c(y)$ of the other variables in (5.11) and (5.12): one forms superficially convergent differences and then uses skeleton expansions, which will be the same on both sides of (the Fourier transformed) (6.14), upon use of (6.13) and (6.14) to the appropriate lower order.

By Legendre transformation, one shows that in (6.13) one may restrict oneself to the amputated 1PI equations in the $J^4, J^6 \cdots$ case; the ambiguity in inverting the full Dirichlet propagator does not affect (in perturbation theory at least) the higher-point functions (cp. appendix B). The 1PI J^4 equation has a logarithmic superficial divergence, which means that the amputated function requires a counter term proportional to a product of δ-functions on the boundary. Such a counter term is, however, annihilated when undoing the amputation due to the Dirichlet condition. Therefore, the J^4 equation of (6.13) is, recursively, a consequence of the J^2 equation due to skeleton expansion (see appendix B), and so are the $J^6, J^8 \cdots$ equations.

The actual computation to order g is easily done using representation (5.5a) of subsect. 5.1. One finds from (6.13)

$$\bar{K}_2(\Delta) = 1 - (16\pi^2)^{-1}g \ln(\tfrac{1}{2}\mu\Delta) + O(g^2)\,, \tag{6.15}$$

and verifies the $\boldsymbol{p}$, y, y' independence. The logarithm in (6.15) stems from the δ' term in

$$[4y^2 + \Delta^2]^{-1} = \tfrac{1}{4}\mu^2 P_+(\mu y)^{-2} + \tfrac{1}{4}\Delta^{-1}\pi\delta(y)$$
$$+ \tfrac{1}{4}\delta'(y)\ln(\tfrac{1}{2}\mu\Delta) + O(\Delta \ln \Delta)\,, \tag{6.16}$$

with the principle value from (5.6b). The 1PI J^4 equation to (6.13) is to order g

trivial, requires only (5.6a) to order g^2, and to order g^3 requires (6.14) only to order g. Finally,

$$K(\Delta) = (2\pi^2)^{-1} + (32\pi^4)^{-1} g(-\tfrac{1}{2} + \ln 2) + O(g^2)\,. \tag{6.17}$$

6.5. $A \neq 0$ EQUATIONS

The equations obtained from (6.13) are identities in their arguments (as $\Delta \to 0$), if all $y > 0$ and, of course, also upon differentiating with respect to some y. Putting undifferentiated arguments on the boundary gives (unless there is already a $c(y)\partial_y|_{y=0}$ argument at the same point) zero on both sides in accordance with (5.14c). Putting a $c(y)\partial_y$ argument on the boundary gives, in general position, an A-argument as set free by differentiating (6.12) with respect to A.

However, that (5.14c) only holds in the sense of distributions has the effect that at coincident such A-arguments, in general δ-type singularities and their derivatives appear, as the only possible singularities with pointlike support. While the coincidence of two (or more) A-arguments is taken care of by the distribution character of (6.12) in A, the coincidence of the extra surface arguments on the r.h.s. with A-arguments leads to precisely those pointlike singularities as arise from the polynomial terms on the r.h.s. as power counting and invariance considerations show, with K_2, K_4, K'_2 and K''_2 being logarithmic in Δ. (Note that J has support only at $y > 0$.) The $\bar{K}$ terms on the l.h.s., stemming from the $A[\delta/\delta A]$ term in (6.10), subtract $\Delta \to 0$ divergences arising upon application of $[\delta/\delta A(z)K_2(\Delta)\delta/\delta A(z+\boldsymbol{\Delta})]$ to expressions $(1/3!)\int AAAG(\cdots|xy)$ by acting on a corresponding $\int AG(\cdot|xy)$ in $\Psi(A|J)$. The $A[\delta/\delta A]$ term does not spoil the symmetry (as a functional differential operator) of $H(A, \delta/\delta A)$ since this operator is defined with the limit $\Delta \to 0$ performed first.

Again, with the help of (6.16) the order-g calculations are straightforward. The JA equation from (6.12) again yields (6.15) and in addition

$$\bar{K}(\Delta) = -(32\pi)^{-1} g + O(g^2)\,, \tag{6.18}$$

and the AA equation yields

$$K_2(\Delta) = 1 + (16\pi^2)^{-1} g \ln(\tfrac{1}{2}\Delta\mu) + O(g^2)\,. \tag{6.19}$$

To order g, K'_2 and K''_2 are zero. All these results verify the PDEs (6.11) to order g.

7. Completeness and unitarity

7.1. FREE FIELD

The free-field Schrödinger functional for a Dirichlet region Γ is, according to (2.11),

$$\psi^0(A|J) = \text{const}\exp[\tfrac{1}{2}A\partial_n G_D \overleftarrow{\partial}'_n A - A\partial_n G_D J + \tfrac{1}{2}JG_D J]\,.$$

Let Γ be divided by a surface $\overline{\partial\Gamma}$ into two subregions Γ_1 and Γ_2, such that $\partial\Gamma_{1,2} = \overline{\partial\Gamma_{1,2}} + \overline{\partial\Gamma}$. Let $\bar{A}$ denote the common boundary value of ϕ on $\overline{\partial\Gamma}$. Denote the restrictions of A to $\overline{\partial\Gamma_{1,2}}$ by $\bar{A}_{1,2}$, and the restrictions of J to $\Gamma_{1,2}$ by $J_{1,2}$. Then from composition formulae for Green functions, which are easy to derive, in obvious notation,

$$\text{const} \int \mathscr{D}\bar{A}\,\Psi^0_{\Gamma_1}(\bar{A}_1\bar{A}|J_1)\,\Psi^0_{\Gamma_2}(\bar{A}_2\bar{A}|J_2) = \Psi^0_{\Gamma}(A|J) \tag{7.1}$$

follows, the $\bar{A}$-integral being the obvious gaussian one. This formula is a consequence of the Markov property of the gaussian random field involved. Upon letting Γ be the infinite space, and choosing $\overline{\partial\Gamma}$ flat, a Wick rotation from the euclidean to the minkowskian frame can be performed, and (7.1) becomes equivalent to the ordinary completeness relation for the free-particle Fock space.

Note that if $\overline{\partial\Gamma}$ is infinite, the integral (7.1) will have a volume divergence which reflects Haag's theorem [19]. This can be handled by introducing a space cutoff in such a case, and extracting from the integral a, in the limit divergent, $J_{1,2}$ independent factor absorbed in the constant on the l.h.s. We will in the following tacitly understand this device to be employed where necessary.

7.2. INTERACTING FIELDS

If Pauli–Villars regularization is introduced, (7.1) can be extended* to fields in local non-derivative polynomial interaction, whereby if N regulator fields are used or, more directly, derivatives up to the $N+1$st occur in the kinetic part, the $\mathscr{D}\bar{A}$ integration must be replaced by one over $\bar{A}$ and the first N normal derivatives [20]. We are interested in the renormalized theory, however, where cutoffs are removed. Since hereby, A undergoes only multiplicative renormalization, we expect that (7.1) can be upheld for the interacting theory. For Γ again the infinite space and $\overline{\partial\Gamma}$ flat, (7.1) then becomes

$$\text{const} \int \mathscr{D}A\,\Psi(A|J)\,\Psi(A|J') = \left\langle \left(\exp \int \hat{J}\Phi\right)_+ \right\rangle, \tag{7.2}$$

where the r.h.s. is the generating functional of the ordinary covariant Schwinger functions, where

$$\hat{J}(\boldsymbol{x}y) = \begin{cases} J'(\boldsymbol{x}(-y)), & y<0, \\ J(\boldsymbol{x}y), & y>0, \end{cases}$$

$$\Phi(\boldsymbol{x}y) = e^{yH}\Phi(\boldsymbol{x}0)\,e^{-yH},$$

$(\ )_+$ denotes y-ordering (increasing y from right to left), and H is the Hamilton operator of the Schrödinger equation. The verification of (7.2) to first order in g proceeds in parallel to the computation in appendix D.

* The functional-integration concept used in the proof [20] is the one of Friedrichs and Shapiro [21].

By letting $y \to zy$ and analytic continuation from real z to $z = i$, one obtains from $\Psi(A|J)$ the minkowskian Schrödinger wave functional, in terms of the Volterra expansion in A and J if (3.15) is used. Then (7.2) becomes

$$\text{const} \int \mathscr{D}A\, \Psi(A|J)_{y\to it} \Psi(A|J')_{y\to it} = \left\langle \left(\exp\left[i \int \Phi \hat{J} \right] \right)_+ \right\rangle \tag{7.3}$$

with $\hat{J}$ as before (writing t in place of y), and $\Phi(\boldsymbol{x}t) = \mathrm{e}^{itH} \Phi(\boldsymbol{x}0)\, \mathrm{e}^{-itH}$, $(\ )_+$ being now the usual time ordering. Eq. (7.3) expresses the completeness of the states with diagonal A. Similarly,

$$\text{const} \int \mathscr{D}A\, \Psi(A|J)_{y\to it} \Psi(A|J')_{y\to -it} = \left\langle \left(\exp\left[i \int_0^\infty \Phi \bar{J} \right] \right)_+ \left(\exp\left[-i \int_0^\infty \Phi \bar{J}' \right] \right)_- \right\rangle, \tag{7.4}$$

where $(\cdot)_-$ denotes anti-time ordering and

$$\bar{J}(\boldsymbol{x}t) = J(\boldsymbol{x}y)|_{y=t}, \qquad \bar{J}'(\boldsymbol{x}t) = J'(\boldsymbol{x}y)|_{y=t}.$$

If $J = J'$ the r.h.s. of (7.4) is independent of J, and (7.4) expresses that $(\exp[-i \int_0^\infty \Phi \bar{J}])_-$ has unit norm. Eqs. (7.3) and (7.4) can be verified to first order in g in the same way as (7.2).

With the help of the usual asymptotic conditions, one can from $\Psi(A|J)_{y\to\pm it}$ obtain matrix elements $\Psi(A|\text{in-state})$ and $\Psi(A|\text{out-state})$, and (7.4) becomes the completeness relation of the diagonal states in Minkowski space.

7.3. COMPUTATION OF EXPECTATION VALUES

Eq. (5.13) has the consequence that, in regularized form, (7.2) leads to

$$\text{const} \int \mathscr{D}A\, \Psi(A|J) F(Z_5 Z_3^{-1} A) \Psi(A|J') = \left\langle \left(F(\Phi) \exp \int \Phi \bar{J} \right)_+ \right\rangle, \tag{7.5}$$

where $F(\Phi)$ is, e.g. a polynomial in time-zero smeared fields, such that the r.h.s. is finite. Since $Z_5 Z_3^{-1}$ diverges upon regularization removal (already to first order in g) this factor must be absorbed, similarly as such factors were absorbed by point-splitting and split-dependent factors in the transition from (6.9) to (6.10). Here, the necessary splitting is easily seen to be one in time:

$$\begin{aligned} Z_5 Z_3^{-1} A(\boldsymbol{z}) \Psi(A|J) &\to \lim_{y\to 0} [\delta/\delta J(\boldsymbol{z}y)] \Psi(A|J) \\ &= \Psi(A|J) \lim_{y\to 0} \sum_{n=0}^\infty \sum_{l=0}^\infty (n!)^{-1} (l!)^{-1} \\ &\quad \times \int \mathrm{d}\boldsymbol{z}_1 \cdots \mathrm{d}\boldsymbol{z}_l \int \mathrm{d}\boldsymbol{x}_1\, \mathrm{d}y_1 \cdots \mathrm{d}\boldsymbol{x}_n\, \mathrm{d}y_n J(x_1) \cdots J(x_n) A(\boldsymbol{z}_1) \cdots A(\boldsymbol{z}_l) \\ &\quad \times G(\boldsymbol{z}_1 \cdots \boldsymbol{z}_l | \boldsymbol{x}_1 y_1 \cdots \boldsymbol{x}_n y_n \boldsymbol{z} y), \end{aligned} \tag{7.6}$$

using (3.15), where this relation is meant in the sense of a replacement on the l.h.s. of (7.5). Higher powers of $Z_5 Z_3^{-1} A$ must be replaced by the expression obtained by repeated application of the functional differentiation in (7.6) in the obvious way, whereby some $Z_5 Z_3^{-1} A$ may be generated by acting on $\Psi(A|J)$ and the others by acting on $\Psi(A|J')$. We verify (7.5) with the replacement (7.6) to first order in g in appendix D.

Note that the limit $y \searrow 0$ in (7.6) cannot be taken under the integral sign in (7.5), since this would formally yield $\Psi(A|J) \lim a(y)^{-1} A(z) = \infty$ according to (5.14c). Eq. (5.14c) implies, however, that the replacement (7.6) in (7.5) introduces an infinite series in A and J, the coefficients of which become the smaller the smaller y is, apart from the term linear in A which approaches a δ-function with diverging coefficient. In a "non-asymptotically-free" theory, where ultraviolet logarithms are not under control, they must, if they appear, be cancelled identically (and in perturbation theory, they always must), which makes it unlikely that then the fairly complicated limit prescription given here can be replaced by a simpler one at this stage. In an "asymptotically free" theory, however, under the usual assumptions one can "sum the logarithms up", and then the factors $Z_5 Z_3^{-1}$ in (7.5) could presumably be removed in a simpler manner than by the replacement (7.6).

8. Discussion

8.1. OTHER MODELS

The techniques of this paper are applicable to any renormalizable bosonic theory, according to the following prescription (in the free or dimensionally regularized theory): (i) Choose a first-order formulation of the field equation (i.e., in the scalar case, the Kemmer [22] representation). (ii) Find that local linear transformation of the field components that corresponds to time reversal. (It is not necessary that the interaction part allows time reversal as an invariance, provided it involves no time derivatives. This will always be so in a renormalizable theory in four dimensions.) The field components that change sign under time reversal are called Neumann ones, the others Dirichlet ones. (iii) Construct that integral over (three) space that, if commuted with the field components, causes them to be transformed into the time reversed ones. Hereby, place the Dirichlet components on one layer and the Neumann components on an infinitesimally (in time) neighbouring one. (iv) This integral, inserted in the lagrangian at some time, implements Dirichlet/Neumann boundary conditions at that time. [We shall demonstrate steps (i)–(iv) in appendix E in the spin-$\frac{1}{2}$ case.] (v) In order to implement inhomogeneous Dirichlet conditions, distribute the Neumann components over space, multiplied by the Dirichlet-component valued source function [analogous to A in (2.5)] and let the space integral approach the time-reversal one from the Dirichlet side as in subsects. 3.2 and 5.4. (vi) Go from the minkowskian to the euclidean frame and, if desired, replace the plane surface by a curved one in the obvious way. Since the operator density on the surface has mass dimension three, power counting of superficial divergences is essentially the same as described in subsect. 3.2. While, under interaction, the terms already present on the surface will need logarithmically divergent factors; there may also arise new ones as counter terms. (The possible counter terms are restricted by the invariance not broken by the surface interaction.)

In gauge theory, the natural gauge to choose is the obvious (time) axial one. In the Schrödinger equation, in order to preserve invariance under time-independent gauge transformations, it may be useful to introduce ordered exponentials similar to those considered by Brandt [23] in the construction of point-split renormalized field equations in QED. However, the effect of such exponentials can also be obtained [23] from polynomial terms, in the limit $\Delta \to 0$.

Computable large-momenta behaviour ("asymptotic freedom") allows one to obtain the "precise" small-y behaviour of the analog of the functions $a(y)$ and $c(y)$ of sect. 5. The leading factor is a, in general broken, power of $\ln \mu y$. Knowledge of this kind may allow one in these theories to replace the complicated limit procedure of subsect 7.3 by a simpler one as remarked there.

In a fermion theory, the only new feature is that the "Dirichlet-component valued" source function is an anticommuting function, as is the ordinary space (time) source function. Therefore, the Green functions analogous to the ones in (3.15) are antisymmetric rather than symmetric in the two groups of arguments, whereby the components located on the boundary are Neumann ones. That the Dirichlet components of the fermion field take the anticommuting source function as boundary "value" is expressed by the validity of the analog of (5.14c). The point-split Schrödinger operator involves, of course, the fermionic (Dirichlet) source field and functional derivatives with respect to it, but the corresponding c-number equations are analogous to (6.6) in the free-field case, and to (6.14) etc. in the interacting one. For clarity, we verify the construction of the Dirichlet-Neumann surface interaction for the spin-$\frac{1}{2}$ case in appendix E. There is an arbitrariness in which components are called Dirichlet and which Neumann ones. This requires, in view of the discussion at the end of subsect. 3.2, to show that homogeneous and inhomogeneous boundary conditions can be upheld under interaction.

In appendix E, we find in the Majorana theory the boundary terms

$$B_{\pm} = \int \mathrm{d}\boldsymbol{x} \bar{\psi}(\boldsymbol{x}, +0)\tfrac{1}{2}(\gamma^0 \pm i\gamma_5)\psi(\boldsymbol{x}, -0) \tag{8.1}$$

to decouple positive from negative times and to induce the boundary conditions

$$(1 \pm i\gamma^0\gamma_5)\psi(\boldsymbol{x}, +0) = 0 = \bar{\psi}(\boldsymbol{x}, +0)(1 \pm i\gamma^0\gamma_5)\,. \tag{8.2}$$

In analogy to the first two terms on the r.h.s. of (3.5), the possible rotationally invariant counter terms on the $x^0 = +0$ side are proportional to

$$\int \mathrm{d}\boldsymbol{x}\bar{\psi}(\boldsymbol{x}, +0)\rho\psi(\boldsymbol{x}, +0)\,, \tag{8.3a}$$

with

$$\rho = 1,\ \gamma^0,\ \gamma_5,\ \gamma^0\gamma_5\,. \tag{8.3b}$$

Of these, γ^0 and γ_5 do not contribute due to (8.2).

Consider now the γ_5 transformation

$$\psi \to i\gamma_5\psi\,, \qquad \bar{\psi} \to \bar{\psi} i\gamma_5 \tag{8.4}$$

and a space reflection ($i = 1, 2$ or 3)

$$\psi \to \gamma^i\gamma_5\psi\,, \qquad \bar{\psi} \to \bar{\psi}\gamma^i\gamma_5\,, \tag{8.5}$$

and assume that the (here alone relevant) massless theory with interaction is invariant under these two transformations, supplemented by appropriate transformations of the other fields. Under (8.4), $B_+ \leftrightarrow B_-$ such that the precise form of possible counter terms must be

$$\Delta L_{\partial\Gamma} = \pm c \int \mathrm{d}\boldsymbol{x}\bar{\psi}(\boldsymbol{x}, +0)\psi(\boldsymbol{x}, +0) + d \int \mathrm{d}\boldsymbol{x}\bar{\psi}(\boldsymbol{x}, +0)\gamma^0\gamma_5\psi(\boldsymbol{x}, +0)\,, \tag{8.6}$$

with c and d logarithmically divergent, and the sign going with the sign in (8.1). Similarly, (8.5) allows only the form

$$\Delta L_{\partial\Gamma} = c' \int \mathrm{d}\boldsymbol{x}\bar{\psi}(\boldsymbol{x}, +0)\psi(\boldsymbol{x}, +0) \pm d' \int \mathrm{d}\boldsymbol{x}\bar{\psi}(\boldsymbol{x}, +0)\gamma^0\gamma_5\psi(\boldsymbol{x}, +0)\,. \tag{8.7}$$

Eqs. (8.6) and (8.7) allow only $c = c' = d = d' = 0$. Thus, there is no counter term possible playing the role of the c_1 term in (3.5), as discussed at the end of subsect 3.2. This shows the possibility of upholding homogeneous boundary conditions, under the stated assumption on the interaction, covering QED, Yukawa-type theories and QCD. Inhomogeneous "Dirichlet" conditions are implemented by adding to the action

$$\int \mathrm{d}\boldsymbol{x}\bar{\psi}(\boldsymbol{x}, +0)\tfrac{1}{2}(\gamma^0 \pm i\gamma_5)\eta(\boldsymbol{x})\,, \tag{8.8}$$

which, in the free case, yields

$$(1 \pm i\gamma^0\gamma_5)\psi(\boldsymbol{x}, +0) = (1 \pm i\gamma^0\gamma_5)\eta(\boldsymbol{x})\,,$$

$$\bar{\psi}(\boldsymbol{x}, +0)(1 \pm i\gamma^0\gamma_5) = \bar{\eta}(\boldsymbol{x})(1 \pm i\gamma^0\gamma_5)\,.$$

It immediately follows that, with interaction, the only rotationally covariant counter term in addition to (8.8), linear in η and the "Neumann" components of ψ, is a multiple of (8.8) such that, as in the scalar case, the "Dirichlet" components undergo only multiplicative renormalization. By the same argument as used in subsect. 5.2, a corresponding statement holds then for the behaviour of the "Neumann" components, as functional derivatives, upon approach to the boundary.

There are models that are not renormalizable in perturbation theory but in other expansions, notably the $O(N)$ non-linear σ-model in less than four dimensions is in a $1/N$ expansion [24]. We expect that, following the prescription outlined before, homogeneous and inhomogeneous Dirichlet conditions (for the fundamental field) can be obtained, and renormalized in a $1/N$ expansion.

8.2. INTERACTION REPRESENTATION

Expanding $\Psi(A|J)$, as a functional of A, in terms of hermite functionals, which are the $\Psi^0(A|J)$ of (6.4b) (e.g. for one-time sources J, and which are well known [1] to form a complete orthonormalizable system in Fock space) yields the euclidean analog of the interaction representation. This is seen most easily from (7.1) or its special case, for flat $\partial\Gamma$, (7.2). If $\Psi(A|J')$ is replaced by $\Psi^0(A|J')$, in the resulting euclidean functional integral on the r.h.s. the coupling constant is set zero in that part of four-space, such that there "free propagation" takes place.

The divergences that arise are the infrared ones (if $\partial\Gamma$ is infinite) already mentioned in subsect. 7.1, and the ultraviolet ones pointed out by Stückelberg [2] (see also [25]). Their nature was clarified by Bogoliubov and Shirkov [3]: If the coupling constant is space-time dependent, renormalization requires, beyond the counter terms in (3.2), where g is to be replaced by $g(x)$, also the terms $U(g(x))\Phi(x)\partial_\mu g(x)\partial_\mu\Phi(x)$ and $V(g(x))\Phi(x)^2\partial_\mu g(x)\partial_\mu g(x)$, with U and V logarithmically divergent (e.g. in dimensional regularization, power series in ε^{-1}). The first counter term becomes ambiguous and the second one meaningless for $g(x)$ a step function, and neither counter term is provided for in the original definition of the interaction representation.

A similar conclusion is obtained from a discussion of the UV divergences arising in the integration in (6.2) with $\Psi(A|J')$ replaced by $\Psi^0(A|J')$: The only (linearly) superficially divergent surface diagrams not yet subtracted have two J, or two J', or one J and one J' argument. The original definition of the interaction representation has no compensating terms for these, and these divergences are not related to the ones removed by Z_5 in the Schrödinger representation. That in the Dirichlet case, the Dirichlet property suppresses divergences brought about by the sharp boundary we showed in appendix C.

8.3. APPLICATIONS

We do not expect the results of this paper to have interesting applications in the conventional renormalizable theories, e.g. QED or QCD. Rather, our starting point was the lagrangian formulation of string theory [4] as an approximate model of

Wilson loop behaviour [26]. The expectation of the Wilson loop is approximated by the Schrödinger functional of a four-component field, in non-renormalizable self-interaction, with continuous boundary values $x_\mu(s)$ prescribed on the circumference of a two-dimensional domain. In this approximation, the QCD string tension is obtained [17, 11] from the Casimir potential* for two parallel lines.

We found in sect. 4 that the Casimir effect is finite (at least in perturbation expansion) provided the theory in the half-space is made finite (in that expansion) by appropriately chosen counter terms. The difficulty here is that, although one can make any polynomial theory "finite" in perturbation expansion by counter terms [3, 27], this expansion is most likely meaningless for a non-renormalizable theory since strong arguments have been given [28] that a non-renormalizable theory has, it it makes sense at all, in its correct perturbation expansion also terms involving the logarithm of the coupling constant.

A systematic construction of such "improved" perturbation expansion succeeds so far only in cases where it can be derived from an alternative expansion with stable power counting, e.g. from the $1/N$ expansion in the non-linear σ-model in less than four dimensions [24]. Of course, with such well-behaved expansion at hand, one will attempt to apply the considerations of this paper directly, since, e.g. the mentioned $1/N$ expansion is again one in terms of graphs. Therefore, the prerequisite for an application of the methods of this paper to the string lagrangians and similar models for extended structures is to find expansions in which these models become renormalizable in infinite space. So far, attempts by the author to find such expansions for the Nambu and Eguchi lagrangians [4] have failed.

9. Conclusions

We have shown that, in every renormalizable theory, (a) the Schrödinger representation exists, (b) in this representation, a Schrödinger equation (with point splitting as already needed in the free theory) holds, (c) the field operator that is being diagonalized is not the renormalized (nor the unrenormalized) one, but differs from it by a factor that diverges logarithmically if the distance from the boundary (which, if non-zero, acts like a cutoff) goes to zero, (d) this last feature requires a limit process to be employed in the calculation of expectation values, (e) the Casimir effect, for disjoint surfaces, is computable to all orders in renormalized perturbation theory.

For simplicity, we gave details only for the Φ_4^4 theory, but we explained the principles of the extension to other models, in particular, also those with fermions. Our reasoning was heuristic at times, and explicit calculations were given only for the first (already non-trivial), and in one case second order in the perturbation expansion. The author is convinced, however, that the conclusions hold to all orders.

Our motivation was the intended application to string lagrangian models of the Wilson loop. Unfortunately, this application requires one first to find a renormaliz-

* The connection with the Casimir effect was pointed out to the authors of [17] by Y. Nambu.

able expansion for such models in infinite space, which has not yet been done. Meanwhile, we recall that Dirac regretted [29] the lack of a Schrödinger equation in quantum electrodynamics.

The author is indebted to M. Lüscher and T.T. Wu for discussions.

Note added in proof

After submitting this paper for publication, the author learnt about related work on critical phenomena in the n-vector model in $4-\varepsilon$ dimensional semi-infinite space. Hereby, our homogeneous Dirichlet boundary condition in massless $\phi^4_{4-\varepsilon}$ (or $(\boldsymbol{\phi}^2)^2_{4-\varepsilon}$) theory corresponds to the "ordinary transition", and our (in zeroth order) homogeneous Neumann boundary condition to the "special transition". The most recent papers are: on the ordinary transition, ref. [32] (the second paper of Diehl and Dietrich being the most systematic paper so far); on the special transition, ref. [33]. These papers give, respectively, calculations, and results, to second order, i.e. two loops and, in contrast to our approach, all use only the half-space. The author is indebted to E. Brézin for having pointed out this relationship, and to H.W. Diehl and S. Dietrich for a discussion. Further references (cf. [8]) on realistic boundary conditions to render finite the Casimir effect for single surfaces are refs. [34]. For other recent papers on the Casimir effect see refs. [35].

Appendix A

A FAMILY OF BOUNDARY CONDITIONS

The gaussian integral (2.3) is evaluated using the Φ field equation, which becomes a linear integral equation for the correlation function. Hereby we must distinguish whether, seen from Γ, an argument goes to $\partial\Gamma$ first, like Φ, or second, like $\partial_n\Phi$, and we denote this order by a subscript to a vertical bar which means putting that argument on $\partial\Gamma$: $|_1$ means approach from Γ' and $|_2$ approach from Γ. It is instructive to give to the boundary interaction in (2.2) the general coefficient c rather than 1. Denoting by G the free-space Green function

$$G(x-x') = (2\pi)^{-\nu}\int \mathrm{d}^\nu p\ \mathrm{e}^{ip(x-x')}(p^2+m^2)^{-1}$$

$$= \tfrac{1}{4}\pi^{-\nu/2}\Gamma(\tfrac{1}{2}\nu-1)|x-x'|^{-\nu+2}\,, \qquad \text{if } m=0\,, \tag{A.1}$$

and by G_c the correlation function with $c \neq 0$, we find

$$G_c = G + cG\overleftarrow{\partial}_n|\cdot{}_1|G_c + cG|\cdot{}_2|\partial_n G_c\,, \tag{A.2}$$

where the dot means integration over $\partial\Gamma$. Herefrom

$$ {}_2|G_c = |G + c_2|G\overleftarrow{\partial}_n|_1 \cdot {}_1|G_c + c|G| \cdot {}_2|\partial_n G_c\,, \tag{A.3a}$$

$$ {}_1|G_c = |G + c_1|G\overleftarrow{\partial}_n|_2 \cdot {}_1|G_c + c|G| \cdot {}_2|\partial_n G_c\,, \tag{A.3b}$$

and similarly

$$ {}_2|\partial_n G_c = |\partial_n G + c|\partial_n G\overleftarrow{\partial}_n| \cdot {}_1|G_c + c_2|\partial_n G|_1 \cdot {}_2|\partial_n G_c\,, \tag{A.4a}$$

$$ {}_1|\partial_n G_c = |\partial_n G + c|\partial_n G\overleftarrow{\partial}_n| \cdot {}_1|G_c + c_1|\partial_n G|_2 \cdot {}_2|\partial_n G_c \tag{A.4b}$$

due to

$$ {}_2|G|_1 = {}_1|G|_2\,, \qquad {}_2|\partial_n G\overleftarrow{\partial}_n|_1 = {}_1|\partial_n G\overleftarrow{\partial}_n|_2\,. \tag{A.5a}$$

We now note the discontinuity relation

$$ {}_{\substack{1\\2}}|\partial_n G|_{\substack{2\\1}} = \mp\tfrac{1}{2}1 + \overline{\partial_n G}\,, \tag{A.5b}$$

where 1 is the δ-function on $\partial\Gamma$, and $\overline{\partial_n G}$, defined by this equation, is an integral kernel on $\partial\Gamma$ well known in potential theory [30], of the form

$$ \overline{\partial_n G} = -\tfrac{1}{4}\pi^{-\nu/2}\Gamma(\tfrac{1}{2}\nu)R^{-1}|x-x'|^{-\nu+2} + \text{less singular terms}\,, \tag{A.6}$$

where R^{-1} is the signed curvature of $\partial\Gamma$ along $x-x'$. $\overline{\partial_n G}$ vanishes on flat portions of $\partial\Gamma$, and for smooth compact $\partial\Gamma$, $\overline{\partial_n G}^k$ is a Fredholm kernel if $\nu \leqslant 2k$. Using (A.5b) in (A.3a, b) gives

$$ {}_2|G_c = (1-c)_1|G_c\,, \tag{A.7a}$$

and from (A.4a, b)

$$ (1-c)_2|\partial_n G_c = {}_1|\partial_n G_c\,. \tag{A.7b}$$

These homogeneous jump relations, where the right argument may be in Γ or Γ', completely characterize the effect of the surface interaction. With the help of Green's formula, (A.7a, b) lead back to (A.2).

To solve (A.2), we insert (A.7a) and obtain

$$ G_c = G + (1-c)^{-1}cG\overleftarrow{\partial}_n| \cdot {}_2|G_c + cG| \cdot {}_2|\partial_n G_c\,. \tag{A.8}$$

If both arguments are in Γ, Green's formula gives

$$ G_c = G - G\overleftarrow{\partial}_n| \cdot {}_2|G_c + G| \cdot {}_2|\partial_n G_c\,. \tag{A.9}$$

Herefrom and from (A.8)

$$ G_c = G + (2c-c^2)G| \cdot {}_2|\partial_n G_c\,, \tag{A.10a}$$

$$ G_c = G + (1-c)^{-2}c(2-c)G\overleftarrow{\partial}_n| \cdot {}_2|G_c\,. \tag{A.10b}$$

The solution of (A.10a), using (A.5b), is

$$(\Gamma\Gamma)\colon\ G_c = G + 2f(c)G| \cdot [1 - 2f(c)\overline{\partial_n G}]^{-1} \cdot |\partial_n G\,, \tag{A.11}$$

where

$$f(c) = [1 + (1-c)^2]^{-1}[1 - (1-c)^2]\,. \tag{A.12}$$

Similarly, (A.10b) yields the transpose of (A.11), which verifies the symmetry of G_c [which also follows from known properties of the functions on the r.h.s. of (A.11)].

In potential theory, the convergence of the iteration solution of the inverse in (A.11) is proven for $|f(c)| < 1$. G_1 is in $(\Gamma\Gamma)$ the Dirichlet function and $G_{\pm\infty}$ the Neumann function, as follows from (A.7).

For the left argument in Γ', the right argument in Γ, (A.9) is replaced by

$$\begin{aligned} 0 &= G - G\overleftarrow{\partial}_n| \cdot {}_2|G_c + G| \cdot {}_2|\partial_n G_c \\ &= G_c - G\overleftarrow{\partial}_n| \cdot {}_1|G_c + G| \cdot {}_1|\partial_n G_c\,. \end{aligned}$$

Herefrom, and from (A.7), we obtain the two forms

$$\begin{aligned} (\Gamma'\Gamma)\colon\ G_c &= (1-c)\{G + 2f(c)G\overleftarrow{\partial}_n| \cdot [1 - 2f(c)\overline{G\overleftarrow{\partial}_n}]^{-1} \cdot |G\} \\ &= (1-c)^{-1}\{G + 2f(c)G| \cdot [1 - 2f(c)\overline{\partial_n G}]^{-1} \cdot |\partial_n G\} \end{aligned} \tag{A.13}$$

which show that in this case, $G_c \equiv 0$ if and only if $c = 1$ or $c = \pm\infty$.

Finally, with both arguments in Γ', one derives in a similar way

$$(\Gamma'\Gamma')\colon\ G_c = G + 2f(c)G| \cdot [1 - 2f(c)\overline{\partial_n G}]^{-1} \cdot |\partial_n G\,, \tag{A.14}$$

which, upon comparison with (A.11) and noting the direction of the normal, shows that effectively a change of sign of $f(c)$ occurred, i.e. $c \to c' = c/(c-1)$ as also obtainable directly from (A.2) and (A.7). Thus, G_1 is, in $(\Gamma'\Gamma')$, the Neumann and $G_{\pm\infty}$ the Dirichlet function. This proves (2.3).

Interchanging in (2.2) the two layers, i.e. posing the Φ-layer inside the $\partial_n\Phi$ one, yields the correlation function G_c' which obeys

$$G_c' = G_{c/(1+c)}\,. \tag{A.15}$$

Thus, the order of the two layers does matter.

From (A.11) one easily derives by differentiation

$$\begin{aligned} (\Gamma\Gamma)\colon\ (\partial/\partial c)G_c &= 2(1-c)^{-1}G_c|_2 \cdot {}_2|\partial_n G_c \\ &= (1-c)^{-1}[G_c|_2 \cdot {}_2|\partial_n G_c + G_c\overleftarrow{\partial}_n|_2 \cdot {}_2|G_c]\,, \end{aligned} \tag{A.16}$$

which due to (A.7a) agrees with the general formula obtained from (2.3),

$$(\partial/\partial c)G_c = G_c\overleftarrow{\partial}_n|_2 \cdot {}_1|G_c + G_c|_1 \cdot {}_2|\partial_n G_c\,.$$

It is obvious that from (A.2) on, c could be taken to be a function of the point on $\partial\Gamma$.

All these boundary conditions have in common that they can be implemented by a bilinear interaction local on $\partial\Gamma$, with dimensionless coefficient.

Appendix B

PERTURBATION EXPANSION

The derivation of the action density (3.8), which provides all the possibly needed counter terms, also prescribes how to compute.

First, we set $A=0$. For a graph, one notes the 1PI parts as usual. Keeping full Dirichlet, Neumann, or zero lines [cf. (2.3)] for the connecting links and external legs, one decomposes the other propagators into free-space parts and surface parts as described in subsect. 3.1. This yields free-space and surface 1PI parts. One must now revoke the one-particle reducibility in any chain of $E=2$ subgraphs, if the two subgraphs at the ends are surface ones. For interpretation of this prescription one visualizes in these cases $\partial\Gamma$ (i.e. the plane $y=0$) as analogous to one line of a covariant Φ_4^4 graph: if two $E=2$ surface graphs are connected by one (possibly covariantly corrected) line "and by the surface", they are connected one-particle irreducibly. Thus, a chain of $E=2$ surface graphs, each one 1PI in the usual sense, must be computed like a covariant four-point vertex function that is the same number of times two-particle reducible in some channel. This means that subtractions for each possible sub-four-point vertex must be made in order to treat the overlapping divergences correctly. (One possible way to do this is by suitably arranged subtractions on a Bethe-Salpeter equation; see, e.g. [16].) In the present case, in each step the divergence is linear rather than logarithmic which requires one to make two subtractions rather than one.

Consider, e.g. an unsubtracted two-point surface vertex $\Gamma(|\boldsymbol{x}y\boldsymbol{x}'y')$, 1PI in the usual sense. Its Fourier transform $\tilde{\Gamma}(|\boldsymbol{k}y(-\boldsymbol{k})y')$ is, in general, if $\varepsilon=0$, not a distribution in y or y'; however,

$$\lim_{\varepsilon\to 0}\,[\tilde{\Gamma}_{\varepsilon>0}(|\boldsymbol{k}y(-\boldsymbol{k})y')-A(\varepsilon)(\delta(y)\delta'(y')+\delta'(y)\delta(y'))]\equiv\tilde{\Gamma}_{\mathrm{ren}}(|\boldsymbol{k}y(-\boldsymbol{k})y') \quad \text{(B.1)}$$

is, with the subtraction provided by a part of Z_4-1 in (3.8). If two Dirichlet or two Neumann lines are attached to the expression in the square bracket, with the other argument of that line strictly positive, the subtraction term is annihilated. This means that in this case, the renormalized unamputated two-point function is insensitive to the subtraction convention for the amputated function, which determines the finite part of $A(\varepsilon)$.

Let $\tilde{\Gamma}_{\mathrm{sub}}(|\boldsymbol{k}q(-\boldsymbol{k})q')$ be the Fourier transform of the square bracket in (B.1) with respect to y and y'. If, with (2.18)

$$(2\pi)^{-2}\int \mathrm{d}q''\int \mathrm{d}q'''\tilde{\Gamma}_{\mathrm{sub}}(|\boldsymbol{k}q(-\boldsymbol{k})q'')G(k,q''q''')\tilde{\Gamma}_{\mathrm{sub}}(|\boldsymbol{k}q'''(-\boldsymbol{k})q')$$

is formed, as $\varepsilon \searrow 0$ its singular part has the form $-iA'(\varepsilon)(q+q')$, and if it is removed, the limit $\varepsilon \searrow 0$ exists. More generally, one subtracts from integrals of this kind the Taylor expansion to first order in $\boldsymbol{k}$, q, q' at $\boldsymbol{k}=q=q'=0$, and adds $ic(q+q')$ with some c to satisfy the renormalization condition. The unamputated two-point function on the Dirichlet side (cf. below) is independent of the choice of that condition.

Skeleton expansions of 1PI $E \geq 4$ graphs are obtained as follows. One first identifies 1PI $E \leq 4$ subgraphs as usual. The one-particle links between these are again left as Dirichlet, Neumann, or zero propagators. The other propagators are decomposed into free-space and surface parts, and 1PI $E=2$ surface subgraphs are identified as described before. $E=4$ surface subgraphs are then further decomposed until the original graph is obtained as a sum of graphs with 1PI $\mathbb{D} \geq 0$ subgraphs connected by Dirichlet, Neumann, or zero propagators. For these subgraphs the appropriate subtractions must be made.

A consequence of this construction is: while the convolution inverse of the full two-point function is not unique since the Dirichlet propagator vanishes on the boundary, the amputated 4-point, 6-point, etc. functions can be defined (in perturbation theory) directly in terms of graphs contributing to them. This we make use of in sect. 6.

Consider now $A \neq 0$ in (3.8), which gives rise to one-leg surface vertices. If one leg of a two-point function is attached to such an A-vertex, the graph is a surface one, apart from the other-end propagator and a possible free-space propagator correction there. The surface graph must be computed as in the covariant theory an ordinary (e.g. mass) vertex is, with all possible subtractions [cf. (3.6)] to linearly divergent subgraphs, of which the subvertex is a prominent one. The subtraction convention for this subgraph does matter, and possible choices would be $[\partial/\partial q]\Gamma(|\boldsymbol{k}q)|_{q=0,|\boldsymbol{k}|=\mu} = i$ or minimal Z_5 as in (3.13). The relation between Green functions with and without A-arguments is discussed in subsect. 5.2.

In the description so far, both the Dirichlet and the Neumann regions were used. If the graphs from all splittings of the lines into free-space and surface ones are added up, the two regions again decouple. The line separation was needed only to show the sufficiency for finiteness of the counter terms in (3.8). The actual computation can, conveniently in $\boldsymbol{k}y$ space, be carried out using either the Dirichlet or the Neumann side alone, with the latter possessing no A-vertices. On the Dirichlet side, power counting shows that the Dirichlet condition is satisfied for each unamputated two-point function, and this renders all unamputated Green functions (and also all vacuum graphs, see subsect. 4.3) insensitive to the subtraction prescription that fixes Z_4-1. In fact, in computing these functions, the subtraction terms in (3.5) are inoperative due to preservation of the Dirichlet condition in every step. To illustrate the role of the Dirichlet condition we discuss in appendix C the computation to second order of the two-point function in some detail.

The circumstances on the Neumann side are described at the end of subsect. 3.2.

Appendix C

TWO-POINT FUNCTION AND DIRICHLET CONDITION

The graphs for the two-point function to second order are shown in fig. 1. In the one-vertex loops in B and C, only the surface propagator, with result proportional to

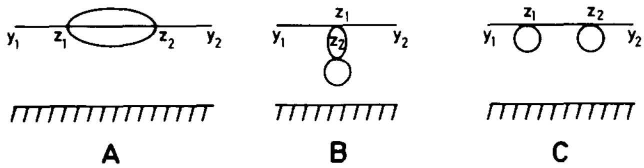

Fig. 1. Second-order contributions to the two-point function. Lines are Dirichlet Green functions.

$z^{-2+\varepsilon}$, is to be used since the free-space one is absorbed in the covariant mass renormalization. The two-point graph in B, which transmits $\boldsymbol{k}=0$, needs the ordinary free-space subtraction. The general two-point loop, with factor $\frac{1}{2}$ included, is

$$\begin{aligned}
&\tfrac{1}{32}\pi^{-4+\varepsilon}\mu^{\varepsilon}\int \mathrm{d}\boldsymbol{x}\ \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\{[\boldsymbol{x}^2+(z_1-z_2)^2]^{-1+\varepsilon/2}-[\boldsymbol{x}^2+(z_1+z_2)^2]^{-1+\varepsilon/2}\}^2\\
&=(32\pi^2)^{-1}\{\mathrm{e}^{-k|z_1-z_2|}P_\mu|z_1-z_2|^{-1}\\
&\quad-(kz_1z_2)^{-1}(\mathrm{e}^{-k|z_1-z_2|}-\mathrm{e}^{-k(z_1+z_2)})+(z_1+z_2)^{-1}\,\mathrm{e}^{-k(z_1+z_2)}\\
&\quad+2\delta(z_1-z_2)[\varepsilon^{-1}+\ln 2+1+\tfrac{1}{2}\ln\pi+\tfrac{1}{2}\psi(1)]\}+\mathrm{O}(\varepsilon)\,,
\end{aligned} \tag{C.1}$$

where P_μ denotes the two-sided principal value,

$$P_\mu|z|^{-1}=[\mu(\mu|z|)^{-\lambda}+2(\lambda-1)^{-1}\delta(z)]_{\lambda=1}\,, \tag{C.2}$$

denoted by $\mu(\mu|z|)^{-1}$ by Gelfand and Shilov [14]. The part proportional to ε^{-1} is absorbed by the usual covariant coupling constant renormalization. The integration in B, with $\boldsymbol{k}=0$, yields a term proportional to z_1^{-2} and one to $z_1^{-2}\ln(z_1\mu)$ due to the principal value in (C.1). The remaining z_1 integration is possible since, for $y_1>z_1$, $y_2>z_1$, the external-leg Dirichlet functions give rise to a factor z_1^2.

Graph C yields a sum of nine double integrals corresponding to the possible time orderings, all of them unproblematic.

A contains the free-space propagator part

$$\begin{aligned}
&\tfrac{1}{6}2^{-6}\pi^{-6+3\varepsilon/2}\mu^{2\varepsilon}\int \mathrm{d}\boldsymbol{x}\ \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}[\boldsymbol{x}^2+(z_1-z_2)^2]^{-3+3\varepsilon/2}\\
&=\tfrac{1}{3}\mu^{2\varepsilon}\Gamma(3-\tfrac{3}{2}\varepsilon)^{-1}\Gamma(\tfrac{3}{2}-\varepsilon)2^{-7}\pi^{\varepsilon-9/2}\\
&\quad\times[|z_1-z_2|^{-3+2\varepsilon}-\tfrac{1}{2}k^2(1-2\varepsilon)^{-1}|z_1-z_2|^{-1+2\varepsilon}+\text{non-sing.}]\,.
\end{aligned}$$

Use herein of (C.2) and [14]

$$P_\mu|z|^{-3}=[\mu^3(\mu|z|)^{-\lambda}+(\lambda-3)^{-1}\delta''(z)]_{\lambda=3}\,, \tag{C.3}$$

gives the $\varepsilon\searrow 0$ singular part

$$(3\cdot 2^{10}\pi^4\varepsilon)^{-1}[\delta''(z_1-z_2)-k^2\delta(z_1-z_2)]\,, \tag{C.4}$$

which is absorbed by the covariant amplitude renormalization using (3.10). (The remaining finite part is euclidean invariant, which verifies explicitly to this order the absence of a renormalization of the speed of light, cf. subsect. 3.1). The term in A with two free-space propagators and one surface propagator involves the loop (C.1), the $1/\varepsilon$ term of which is again absorbed by covariant coupling constant renormalization.

Collecting the finite parts we find, after some calculation,

$$\begin{aligned}\text{F.P. of A}&=(3\cdot 2^9\pi^4)^{-1}P_\mu[\mathrm{e}^{-k\Delta}(\Delta^{-3}(1+k\Delta)\\&\quad-12\Delta^{-1}(s^2-\Delta^2)^{-1}+48k^{-1}(s^2-\Delta^2)^{-2})-(\Delta\leftrightarrow s)]\\&=P_\mu(3\cdot 2^9\pi^4)^{-1}(s-\Delta)^3\{\Delta^{-3}s^{-3}(s+\Delta)^{-2}(s^2+5s\Delta+\Delta^2)\\&\quad-\tfrac{1}{2}k^2\Delta^{-1}s^{-1}(s+\Delta)^{-2}+\mathrm{O}(k^4)\}\,,\end{aligned} \tag{C.5}$$

where $\Delta=|z_1-z_2|$ and $s=z_1+z_2$. P_μ prescribes the two-sided principal value with respect to Δ of (C.2), (C.3). (C.5) gives in A for $y_1\geqslant 0$, $y_2\geqslant 0$ a well defined integral, due to the factor z_1z_2 from the two Dirichlet lines if $y_1>z_1$, $y_2>z_2$:

$$\begin{aligned}&\tfrac{1}{4}k^{-2}\int_0^\infty \mathrm{d}z_1\int_0^\infty \mathrm{d}z_2\,[\mathrm{e}^{-k|y_1-z_1|}-\mathrm{e}^{-k(y_1+z_1)}][\mathrm{e}^{-k|y_2-z_2|}-\mathrm{e}^{-k(y_2+z_2)}]P_\mu|z_1-z_2|^{-3}\\&=(2\pi)^{-1}\int_{-\infty}^\infty \mathrm{d}p(k^2+p^2)^{-2}p^2[\ln(|p|\mu^{-1})-\psi(3)](\mathrm{e}^{ipy_1}-\mathrm{e}^{-ky_1})(\mathrm{e}^{-ipy_2}-\mathrm{e}^{-ky_2})\,,\end{aligned} \tag{C.6}$$

where we used the Fourier representation of the principal value [14] and

$$\tfrac{1}{2}k^{-1}\int_0^\infty \mathrm{d}z[\mathrm{e}^{-k|y-z|}-\mathrm{e}^{-k(y+z)}]\,\mathrm{e}^{ipz}=(k^2+p^2)^{-1}(\mathrm{e}^{ipy}-\mathrm{e}^{-ky})\,.$$

The p-integral in (C.6) is absolutely convergent, and vanishes if $y_1\searrow 0$ or $y_2\searrow 0$. In higher orders in perturbation theory, there appears in (C.6) $P_\mu[(\ln|z_1-z_2|)^n|z_1-z_2|^{-3}]$ which also gives a well-defined integral. Also the formula

$$\begin{aligned}&\int_0^\infty \mathrm{d}z_1z_1\,\mathrm{e}^{iq_1z_1}\int_0^\infty \mathrm{d}z_2z_2\,\mathrm{e}^{iq_2z_2}P_\mu|z_1-z_2|^{-3}\\&=\{-2\psi(3)+\ln[\mu^{-2}(q_1+i0)(q_2+i0)]\}\\&\quad\times(q_1+q_2+i0)^{-3}iq_1q_2+\tfrac{1}{2}i(q_1+q_2+i0)^{-1}\end{aligned} \tag{C.7}$$

is instructive; for convergence again the factor z_1z_2 is needed.

In the interaction representation, as discussed in subsect. 8.2, the integral (C.6) appears but with the second terms in the square brackets missing since only the interaction, not also the kinetic part is switched off at the boundary (see also [25]). Then, that the principal value is integrable only with a $\mathbb{C}^2$ test function leads to a divergence.

Appendix D

COMPLETENESS CHECK TO FIRST ORDER

We write the Schrödinger functional (5.20) as

$$\Psi(A|J) = \exp\left[-\tfrac{1}{2}AKA - A\partial_n GJ + \tfrac{1}{2}JG_{\mathrm{D}}J + P(A) + Q(A|J) + R(J)\right], \quad \text{(D.1)}$$

where the first three terms on the right are the zero-order ones [cf. (6.4b)] and the (connected) $Q(A\,|\,J)$ shall have no term depending only on A or on J. For brevity, however, we will set $J=0$, i.e. consider in (7.5) the vacuum expectation values, since the calculation with $J=0$ offers no new feature. To first order,

$$P(A) = -\tfrac{1}{24}g\int \mathrm{d}x[(A\partial_n G_{\mathrm{D}})(x)]^4$$
$$+\tfrac{1}{2}g(32\pi^2)^{-1}\int \mathrm{d}\boldsymbol{x}\int_0^\infty \mathrm{d}y'[(A\partial_n G_{\mathrm{D}})(\boldsymbol{x}y')]^2\mu^2 P_+(\mu y')^{-2}\,, \quad \text{(D.2)}$$

according to (5.5b).

By gaussian integration, we find

$$\mathrm{const}\int \mathscr{D}AF(A)\Psi(A|0)\Psi(A|0)$$
$$= \mathrm{const}'\,F(\delta/\delta B)\exp[2P(\delta/\delta B)]\exp[\tfrac{1}{4}BK^{-1}B]|_{B=0}$$
$$= \exp[S(\tfrac{1}{2}K^{-1}[\delta/\delta A]) - S(0)]\exp(\tfrac{1}{4}[\delta/\delta A]K^{-1}[\delta/\delta A])F(A)|_{A=0}\,, \quad \text{(D.3)}$$

where we introduced

$$\exp(\tfrac{1}{4}[\delta/\delta A]K^{-1}[\delta/\delta A])\exp[2P(A+B)]|_{A=0} =: \exp S(B)\,. \quad \text{(D.4)}$$

Like P, Q and R, S is connected due to the linked-cluster theorem. In (D.3), $\exp S(0)$ is the infrared (by Haag's theorem) and ultraviolet divergent factor absorbed (see subsect. 7.1) in the constant in (7.2) and (D.3).

We choose $F(A) = A(z_1)A(z_2)$, which suffices to display all difficulties. The correct two-point vacuum expectation value to first order is the free one, $\tfrac{1}{2}K^{-1}_{12} = \tfrac{1}{2}\mathrm{FT}|\boldsymbol{k}|^{-1}$. From (D.3), however, we find a divergent result, formally (or, under regularization) proportional to K^{-1}_{12}, since it involves the last term in (5.5b) minus the same integral with the principal-value sign omitted. This second term stems from the first term on the r.h.s. of (D.2) and the evaluation

$$\tfrac{1}{4}[\delta/\delta A]K^{-1}[\delta/\delta A]\tfrac{1}{2}[(A\partial_n G_D)(\boldsymbol{x}y)^2]|_{A=0}$$
$$=\tfrac{1}{4}(2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}\, k^{-1}\,\mathrm{e}^{-2ky}=(32\pi^2y^2)^{-1}\,. \tag{D.5}$$

This remaining divergence is in accord with (7.5), where Z_5 diverges to first order; see (3.13).

The prescription of subsect. 7.3 gives, for approach of both operators in $F(A)$ to $y=0$ from the same side,

$$\begin{aligned}
&F(Z_5Z_3^{-1}A)\to F(A,\,y_1y_2)\\
&\quad=\Big\{-(A\partial_n G_\mathrm{D})(\boldsymbol{z}_1y_1)-g(32\pi^2)^{-1}\int \mathrm{d}\boldsymbol{x}\int_0^\infty \mathrm{d}y'(A\partial_n G_\mathrm{D})(\boldsymbol{x}y')G_\mathrm{D}(\boldsymbol{x}y',\,\boldsymbol{z}_1y_1)\\
&\qquad\times\mu^2P_+(\mu y')^{-2}+\tfrac{1}{6}g\int \mathrm{d}\boldsymbol{x}\int_0^\infty \mathrm{d}y'[(A\partial_n G_\mathrm{D})(\boldsymbol{x}y')]^3G_\mathrm{D}(\boldsymbol{x}y',\,\boldsymbol{z}_1y_1)\Big\}\\
&\qquad\times\{\text{same expression with } \boldsymbol{z}_1\to\boldsymbol{z}_2,\ y_1\to y_2\}\\
&\qquad+G_\mathrm{D}(\boldsymbol{z}_1y_1,\,\boldsymbol{z}_2y_2)+g\int \mathrm{d}\boldsymbol{x}\int_0^\infty \mathrm{d}y'\{(32\pi^2y'^2)^{-1}\\
&\qquad-\tfrac{1}{2}[(A\partial_n G_\mathrm{D})(\boldsymbol{x}y')]^2G_\mathrm{D}(\boldsymbol{x}y',\,\boldsymbol{z}_1y_1)G_\mathrm{D}(\boldsymbol{x}y',\,\boldsymbol{z}_2y_2)\}\,,
\end{aligned} \tag{D.6}$$

where only the terms up to first order in g are to be kept, and (5.5a) has been used. At fixed function A, the limit $y_1\searrow 0$, $y_2\searrow 0$ does not exist on the r.h.s. of (D.6), the factor $Z_5^{-2}Z_3^2$ missing. The A-independent terms, however, vanish in this limit.

Working out (D.3), with insertion of (D.6) and (D.2), term by term is trivial, the characteristic integration being again (D.5). Divergences do appear, however, in the y' integrations in separate terms. They are made unambiguous by dimensional regularization, whereby it is advantageous to replace the principal values on the r.h.s. of (D.6) by (4.6b). Then, all divergences are found to cancel, and the sum of finite terms to order g vanishes linearly, as $y_1\searrow 0$, $y_2\searrow 0$, as it should.

The same result is obtained if in (7.5), $A(\boldsymbol{z}_1)$ and $A(\boldsymbol{z}_2)$ are shifted into different factors $\Psi(A|0)$ in the sense (7.6). Some fewer terms then appear, but otherwise the calculations are identical to the ones just described.

Appendix E

DIRICHLET CONDITIONS FOR THE MAJORANA FIELD

The prescription given in subsect. 8.1 is simplest to illustrate for the spin-$\tfrac{1}{2}$ Majorana field, wherefrom the transition to the Dirac field is obvious. The Majorana lagrangian with surface term is

$$L = \tfrac{1}{2} i\psi\alpha^{\mu}\partial_{\mu}\psi - \tfrac{1}{2} m\psi\beta\psi + i\delta(x^0)\psi_1\chi\psi_2 , \tag{E.1a}$$

where, with two independent sets $\sigma.$ and $\tau.$ of Pauli matrices, we may choose

$$\alpha^0 = 1 , \qquad \alpha^1 = \sigma_3\tau_1 , \qquad \alpha^2 = \sigma_3\tau_3 , \qquad \alpha^3 = \sigma_1 ,$$
$$\beta = \sigma_2 , \qquad \psi = \psi^+ . \tag{E.1b}$$

In (E.1a), χ is an as yet unspecified matrix, and subscripts 1 and 2 indicate approach to the time-zero plane from positive and negative times, respectively.

From the field equations the integral equations for the Feynman Green functions follow:

$$G = G^0 - G^0.\chi_2 G + G^0.\chi_1^{\mathrm{T}} G = G^0 - G_1.\chi G^0 + G_2.\chi^{\mathrm{T}} G^0 , \tag{E.2}$$

where

$$G^0 = -i[(m - i0)\beta - i\alpha^{\mu}\partial_{\mu}]^{-1} , \tag{E.3}$$

and the dot indicates integration on the time-zero plane. The integral operators on the boundary

$${}_1G_2^0 =: P_+ , \qquad -{}_2G_1^0 =: P_- \tag{E.4a}$$

obey

$$P_+.P_+ = P_+ , \qquad P_-.P_- = P_- ,$$
$$P_+.P_- = P_-.P_+ = 0 , \qquad P_+ + P_- = 1 . \tag{E.4b}$$

Going in (E.3) to the boundary and using (E.4) yields

$$(1-\chi^{\mathrm{T}})_1 G = (1-\chi)_2 G , \qquad G_1(1-\chi) = G_2(1-\chi^{\mathrm{T}}) . \tag{E.5}$$

This allows us to rewrite (E.2) as

$$G = G - G^0.{}_2G + G^0.{}_1G = G - G_1.G^0 + G_2.G^0 , \tag{E.6}$$

and herefrom

$$G = G^0 + G^0.X.G^0 , \qquad X = {}_{11}G_2 - {}_{11}G_1 - {}_{22}G_2 + {}_{22}G_1 \tag{E.7}$$

easily follows. The subscript 11 indicates that the argument goes to the boundary from the 1 direction but later than the ones denoted by 1 or 2. The consequences of (E.2) or (E.7),

$${}_{11}G_1 - {}_1G_{11} = 1 = -{}_{22}G_2 + {}_2G_{22} ,$$

are also consequences of the canonical anticommutation relations. We set

$$\chi = S + A , \qquad S = S^{\mathrm{T}} , \qquad A = -A^{\mathrm{T}} . \tag{E.8}$$

With the notation $P_+ - P_- = Q$, (E.5) is, using (E.7), equivalent to

$$(1 - S1 + AQ).X = -2A1 = X.(1 - S1 + QA)\,. \tag{E.9}$$

Consistency requires

$$[S, A] = 0 = [\chi, \chi^{\mathrm{T}}]\,, \tag{E.10}$$

which, apparently, is the condition that the symmetry property of the Green function, expressed by the simultaneous validity of the two integral equations in (E.2), is not in contradiction to the lagrangian (E.1a) which, in general, is not hermitian. The solution of (E.10) $A = 0$ yields $X = 0$ for generic S, and by continuity also if S has eigenvalue 1. This corresponds to the surface term in (E.1a) being ineffective.

For the Green function to vanish if the two arguments are on opposite sides of the boundary,

$${}_1G_2 = 0 = {}_2G_1 \tag{E.11a}$$

is necessary, which with (E.7) becomes

$$P_+.X.P_+ = -P_+\,, \qquad P_-.X.P_- = P_-\,. \tag{E.11b}$$

That (E.11b) is also sufficient follows from the frequency property

$${}_{>0}G^0_1.P_- = {}_{<0}G^0_1.P_+ = P_+.{}_1G^0_{>0} = P_-.{}_2G^0_{<0} = 0\,.$$

If $S = 0$, every A with $A^2 = 1$ (which excludes real A) and such that $A + Q$ is invertible for all momenta leads to a solution $X = -2(A + Q)^{-1}$ of (E.9), (E.11b). This excludes all rotationally non-invariant A,

$$A = (a_1\gamma^1 + a_2\gamma^2 + a_3\gamma^3)\gamma^0\gamma_5\,, \qquad a_1^2 + a_2^2 + a_3^2 = 1\,,$$

and excludes among the rotationally invariant ones,

$$A = a\gamma^0 + b\gamma_5 + ci\gamma^0\gamma_5\,, \qquad a^2 + b^2 + c^2 = 1\,, \tag{E.12}$$

if $m > 0$ those with

$$(1 - b^2)^{-1}(-a \pm ibc) \in [1, +\infty)\,,$$

and if $m = 0$, those with $b = \pm 1$.

For all these solutions with $S = 0$, in (E.1a) the "1" and the "2" components in the surface term do not anticommute among themselves. Comparison with (1.2) suggests choosing χ such that the two groups of components do anticommute among themselves. This requires

$$\chi^{\mathrm{T}}\chi = 0 = \chi\chi^{\mathrm{T}}\,. \tag{E.13}$$

Then necessarily $S = \frac{1}{2}$ and $\chi = \frac{1}{2}(1 + A)$ with A the matrices just described, which yields the same X as before. X and G become simple only for, in (E.12), $c = \pm 1$,

$a=b=0$ which yields our final choice

$$\chi=\tfrac{1}{2}(1\pm i\gamma^0\gamma_5)\,. \tag{E.14}$$

Hereby, the time reversal matrix

$$A_t=\pm i\gamma^0\gamma_5=\pm\sigma_3\tau_2\,,$$

obeys

$$A_tP_+=P_-A_t\,,\qquad A_tP_-=P_+A_t$$

and yields $X=-A_t-Q$. The surface term in (E.1a) has now the properties described in subsect. 8.1. The boundary conditions (E.5) become

$$(1\pm\sigma_3\tau_2)_1G_{>0}=0={}_{>0}G_1(1\mp\sigma_3\tau_2)\,,$$

$$(1\mp\sigma_3\tau_2)_2G_{<0}=0={}_{<0}G_2(1\pm\sigma_3\tau_2)\,. \tag{E.15}$$

The decoupling of positive from negative times is possible since L from (E.1a) is (for no choice of A) hermitian.

For a simple characterization of Dirichlet and Neumann components one would, in view of (E.5), have to go to a representation where χ is diagonal. This is not possible in a Majorana representation since this contradicts $\chi=A$ or $\chi=\frac{1}{2}(1+A)$ with $A=-A^{\mathrm{T}}$, $A^2=1$. However, if the computation of this appendix is performed with a surface interaction on the timelike plane $\sum_{i=1}^{3}n_ix^i=0$, with the condition of decoupling of the two sides for all times, the matrix $\sigma_3\tau_2=i\gamma^0\gamma_5$ in (E.13) becomes replaced by*

$$n_i\gamma^i\gamma_5=-n_1\tau_3+n_2\tau_1-n_3\sigma_2\tau_2\,,$$

which can be made diagonal in a Majorana representation. The lagrangian is hermitian in this case.

For $x^0>0$, $x^{0'}>0$, the decoupling solutions (E.12) or (E.15) are

$$\begin{aligned}G&=\tfrac{1}{2}(1\pm\sigma_3\tau_2)[(m\beta-i\alpha^i\partial_i)G_{\mathrm{D}}+i\alpha^0\partial_0G_{\mathrm{N}}]\\&\quad+\tfrac{1}{2}(1\mp\sigma_3\tau_2)[(m\beta-i\alpha^i\partial_i)G_{\mathrm{N}}+i\alpha^0\partial_0G_{\mathrm{D}}]\\&=\tfrac{1}{2}(1\pm i\gamma^0\gamma^5)[(m+i\gamma^i\partial_i)G_{\mathrm{D}}+i\gamma^0\partial_0G_{\mathrm{N}}]\beta^{-1}\\&\quad+\tfrac{1}{2}(1\mp i\gamma^0\gamma_5)[(m+i\gamma^i\partial_i)G_{\mathrm{N}}+i\gamma^0\partial_0G_{\mathrm{D}}]\beta^{-1}\end{aligned} \tag{E.16a}$$

and (note $\partial_0G_{\mathrm{D}}=-G_{\mathrm{N}}\overleftarrow{\partial}_0'$, $\partial_0G_{\mathrm{N}}=-G_{\mathrm{D}}\overleftarrow{\partial}_0'$)

$$\begin{aligned}G&=[G_{\mathrm{D}}(m\beta+i\overleftarrow{\partial}_i'\alpha^i)-G_{\mathrm{N}}i\overleftarrow{\partial}_0'\alpha^0]\tfrac{1}{2}(1\mp\sigma_3\tau_2)\\&\quad+[G_{\mathrm{N}}(m\beta+i\overleftarrow{\partial}_i'\alpha^i)-G_{\mathrm{D}}i\overleftarrow{\partial}_0'\alpha^0]\tfrac{1}{2}(1\pm\sigma_3\tau_2)\\&=[G_{\mathrm{D}}(m-i\overleftarrow{\partial}_i'\gamma^i)-G_{\mathrm{N}}i\overleftarrow{\partial}_0'\gamma^0]\tfrac{1}{2}(1\pm i\gamma^0\gamma_5)\beta^{-1}\\&\quad+[G_{\mathrm{N}}(m-i\overleftarrow{\partial}_i'\gamma^i)-G_{\mathrm{D}}i\overleftarrow{\partial}_0'\gamma^0]\tfrac{1}{2}(1\mp i\gamma^0\gamma_5)\beta^{-1}\,.\end{aligned} \tag{E.16b}$$

* That at a timelike plane the boundary conditions $(1\pm n_i\gamma^i\gamma_5)\psi=0$ are meaningful has been noted by Wu [31] on the basis of different considerations.

Here,

$$G_{\mathrm{N}}^{\mathrm{D}}=(2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}(2k_0)^{-1}\,\mathrm{e}^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}[\mathrm{e}^{-ik_0|x^0-x^{0\prime}|}\mp\mathrm{e}^{-ik_0(x^0+x^{0\prime})}],$$

with $k_0=(\boldsymbol{k}^2+m^2)^{1/2}$ the Minkowski-space versions of (2.14). Note that the first line of (E.16a) does not equal the first line of (E.16b).

If $m=0$, the upper-sign solution for G is transformed into the lower-sign one by $\psi\to i\sigma_1\tau_2\psi=i\gamma_5\psi$. This means that there is, concerning UV behaviour, no qualitative distinction between the Dirichlet region and the Neumann one since the sign of the fermion mass term is then not essential. To call that region the "Dirichlet" one in which, by a boundary source that has no effect on the other region (step 5) in subsect. 8.1, inhomogeneous "Dirichlet" boundary conditions are imposed, is therefore, for the spin-$\frac{1}{2}$ field, merely a convention.

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