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First Results of a Calculation of the Long Range Quark–Antiquark Potential from Asymptotic QCD Dynamics

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Abstract. Starting from the asymptotic dynamics of the quark–gluon interaction for large distances we calculate the first two terms of the divergent phase of elastic quark–antiquark scattering. This phase in turn allows the determination of the long range behavior of the potential. In addition to a Coulomb force it turns out to have a repulsive barrier at large distances.

1. Introduction

We extend the method of Kulish and Faddeev [1] for the calculation of the relativistic Coulomb phase of QED to QCD. [2, 3]. This divergent phase reflects the infrared singularities being present in perturbation theory where massless quanta mediate the interaction between fermions. (In non-relativistic quantum mechanics the divergent phase in the radial wave function is proportional to ln r, r being the distance between the fermion and the scattering center.) It can be calculated if the evolution operator describing the asymptotic, i.e. large time, dynamics is known. This operator is explicitly given in terms of the asymptotic, i.e. large time, Hamiltonian of the system. The large time limit of the Hamiltonian is equivalent to a large distance limit, since this Hamiltonian describes fermions moving along a straight line. In the extension of the QED calculation we make use of Magnus' solution [4] for the equation of motion of the operator describing the asymptotic dynamics. Magnus' solution has been already exploited by two of the authors for the determination of the electromagnetic form factor of the quarks [3]. This solution is of exponential form with a series expansion *in the exponent*. Thus we sum up the contributions of an infinite number of Feynman graphs to the $q\bar{q}$ elastic scattering phase, even if we only calculate a few terms in the exponent. In this work we calculate the first two terms in the exponent of the Magnus solution, i.e. of the QCD phase of elastic $q\bar{q}$ scattering. Next we develop a method for the calculation of the non-relativistic potential following from the non-relativistic $q\bar{q}$ scattering phase.

The result we obtain consists of the Coulomb potential $-\frac{\alpha_S C_F}{r}$ and an additional term $C_F \frac{C_A}{3} \frac{\alpha_S^2}{\pi} \frac{\ln r}{r}$ which produces a repulsive barrier at large distances. In this first step in determining the QCD phase of $q\bar{q}$ scattering we leave out the gluon self interactions in order to simplify the calculation. The method employed here constitutes a consistent treatment of the radiation of gluons off a "classical" color current and correctly takes into account the non-Abelian character of the color charge.

2. The Picture of Asymptotic Dynamics in QCD

The asymptotic dynamics of quark–gluon interactions in QCD is described by the asymptotic limit for large times of the Hamiltonian which is given in the interaction picture by

$$\tilde{H}_I(t) = -g\,K(t), \quad K(t) = \int \tilde{j}^a_\mu(x)\, A^{a\mu}(x)\, d^3x \qquad (1)$$

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The gluon field couples to the asymptotic quark (antiquark) current

$$\tilde{j}^a_\mu(x) = \int d^3p\, \rho^a(\mathbf{p}) \frac{p_\mu}{\omega} \delta^{(3)}\left(\frac{\mathbf{p}}{\omega}t - \mathbf{x}\right) \tag{2}$$

where ρ^a is the color charge density, p_μ the quark (antiquark) four-momentum and ω is the corresponding energy. The particles are moving with the velocity $\mathbf{v} = \mathbf{p}/\omega$.

The color charge density is given in terms of quark/antiquark creation and annihilation operators by

$$\rho^a(\mathbf{p}) = \sum_{i,k}\sum_r \left[a_i^+(\mathbf{p},r)\frac{\lambda^a_{ik}}{2}a_k(\mathbf{p},r) + b_i^+(\mathbf{p},r)\frac{\bar{\lambda}^a_{ik}}{2}b_k(\mathbf{p},r)\right] \tag{3}$$

$\lambda^a, (\bar{\lambda}^a = -\lambda^{aT})$, being the Gell-Mann matrices in color space, r being spin indices. The asymptotic current $\tilde{j}^a_\mu$ is an operator due to the color it carries in contradistinction to the QED case where the corresponding current is equivalent to a c-number.

The transformation of the states $|\zeta>$ of the interaction picture into the asymptotic (coherent) states is given by*.

$$|\tilde{\zeta}> = U^+|\zeta> \tag{4}$$

The unitary operator U is a solution of the evolution equation in time t

$$\frac{dU(t)}{dt} = igK(t)U(t), \tag{5}$$

where g is the bare coupling constant.

As a consequence of the operator character of the current, the commutator of K with itself at different times is not a c-number. Therefore the T-product solution of (5) cannot be resolved into the simple structure of (9) of [1] which is of the form

$$U_{\mathrm{QED}}(t) = e^{R(t)}e^{i\phi(t)}. \tag{6}$$

Here $\phi(t)$ represents the logarithmically divergent Coulomb phase and R describes the emission and absorption of photons.

Instead we have to exploit the much more involved solution of Magnus [4] which is also of exponential type, however, with an infinite series in the exponent

$$U(t) = e^{\Omega(t)} \tag{7}$$

where the exponent fulfills the equation

$$\Omega(t) = ig\sum_{k=0}^{\infty}\beta_k\int_{t_0}^{t}\{K(t'),\Omega^k(t')\}\,dt' \tag{8}$$

Here the curly bracket stands for a repeated commutator of order k in Ω

$$\{K,\Omega^k\} \equiv [[\ldots[[K,\Omega],\Omega],\ldots],\Omega]$$
$$\{K,\Omega^o\} \equiv K \tag{9}$$

The coefficients β_k are related to the Bernoulli numbers B_{2m} [6]

$$\beta_0 = 1, \beta_1 = \tfrac{1}{2}, \beta_2 = \tfrac{1}{12}, \beta_{2m} = \frac{(-1)^{m-1}}{(2m)!}B_{2m};$$
$$m = 2, 3, \ldots$$

The implicit equation (8) can be solved with a power series expansion in g

$$\Omega(t) = \sum_{n=1}^{\infty}(ig)^n\Delta_n(t) \tag{10}$$

Comparing coefficients of equal powers in g after inserting (10) into (8) we get for the first four Δ_n

$$\Delta_1(t) = \int_{t_0}^{t}K(t_1)dt_1;\ \Delta_2(t) = \frac{1}{2}\int_{t_0}^{t}dt_2[K(t_2),\Delta_1(t_2)]$$
$$\Delta_3(t) = \frac{1}{2}\int_{t_0}^{t}dt_3[K(t_3),\Delta_2(t_3)]$$
$$+\frac{1}{12}\int_{t_0}^{t}dt_3[[K(t_3),\Delta_1(t_3)],\Delta_1(t_3)]$$
$$\Delta_4(t) = \frac{1}{2}\int_{t_0}^{t}dt_4[K(t_4),\Delta_3(t_4)]$$
$$+\frac{1}{12}\int_{t_0}^{t}dt_4[[K(t_4),\Delta_1(t_4)],\Delta_2(t_4)]$$
$$+\frac{1}{12}\int_{t_0}^{t}dt_4[[K(t_4),\Delta_2(t_4)],\Delta_1(t_4)] \tag{11}$$

The general expression for Δ_n can be given as a repeated commutator of K and expression Δ_m, $m<n$. Different expressions have been given by Wilcox [7] and Bialynicki et al. [8].

3. The Divergent QCD Phase of Elastic Quark–Antiquark Scattering and the Calculation of its First Two Terms

The asymptotic quark current, (2), conserves separately the numbers of quarks and antiquarks. We can therefore study the solution (7) of the evolution equation (5) in the sectors of fixed separate quark and antiquark numbers. For the investigation of the sector of one quark and one antiquark, it is useful to define projection operators Π_{ik} which project onto the subspaces of one quark of color i and an antiquark of color k and arbitrary numbers of gluons. Obviously the projection operators fulfill orthogonality relations of the type

$$\Pi_{ij}\Pi_{kl} = \delta_{ik}\delta_{jl}\Pi_{ij}. \tag{12a}$$

The projection operator for a quark-antiquark color singlet is simply given by

$$\Pi_S = \frac{1}{\sqrt{3}}\sum_{i=1}^{3}\Pi_{ii}. \tag{12b}$$

In the subspace of one quark and one antiquark the

* Coherent states have also been used by Greco and coworkers [5]

completeness relation for the projection operators

$$\sum_{ik} \Pi_{ik} = I_{q\bar{q}} \quad (12c)$$

holds, where $I_{q\bar{q}}$ is the identity within this subspace. The separate conservation of quark and antiquark number is then expressed in the commutation relation

$$[\tilde{H}_I, I_{q\bar{q}}] = 0. \quad (13)$$

With these projection operators we define the projection of the general Hamiltonian (1) onto the spaces defined by Π_{ij}

$$(K_{q\bar{q}})_{ijkl} = K_{ijkl} = \Pi_{ij} K \Pi_{kl} \quad (14)$$

describing the transition from a quark–antiquark state with colors k, l to a state with colors i, j.

The explicit form of the operator $K_{q\bar{q}}$ is then (summation over repeated indices understood)

$$K_{q\bar{q}} = \left[G^a(\xi)\frac{\Lambda^a}{2} + G^a(\eta)\frac{\bar{\Lambda}^a}{2} \right]. \quad (15)$$

Here

$$\frac{\Lambda^a}{2} = \frac{\lambda^a}{2} \otimes \mathbf{1}, \quad \frac{\bar{\Lambda}^a}{2} = \mathbf{1} \otimes \frac{\bar{\lambda}^a}{2} \quad (16)$$

are the matrix representations of the color matrices in $q\bar{q}$ space, and

$$G^a(\xi) = \frac{p_1^\mu}{\omega_1} A_\mu^a(\xi), \quad (17)$$

with p_1^μ, ω_1 being the four-momentum and energy resp. of the quark and $\xi^\mu = \left(t, \frac{\mathbf{p}_1}{\omega_1} t\right)$ describing the position of the quark at time t. Analogously we have

$$G^a(\eta) = \frac{p_2^\mu}{\omega_2} A_\mu^a(\eta) \quad (18)$$

p_2^μ, ω_2 being the four-momentum and energy resp. of the antiquark and $\eta^\mu = \left(t, \frac{\mathbf{p}_2}{\omega_2} t\right)$ the position of the antiquark at time t.

Instead of (15) we write, in matrix notation of color indices,

$$K_{ijkl} = G^a(\xi)\frac{\lambda^a_{ik}}{2}\delta_{jl} + G^a(\eta)\delta_{ik}\frac{\bar{\lambda}^a_{jl}}{2} \quad (19)$$

The evolution operator can also be projected onto the quark–antiquark sector

$$(U_{q\bar{q}})_{ijkl} = U_{ijkl} = \Pi_{ij} U \Pi_{kl}$$

The equation (5) for the evolution operator in time can now be reduced to the quark–antiquark sector

$$\frac{dU_{q\bar{q}}}{dt} = igK_{q\bar{q}} U_{q\bar{q}} \quad (20)$$

or, in explicit matrix notation for the color indices,

$$\frac{dU_{ijkl}}{dt} = igK_{ijmn} U_{mnkl} \quad (21)$$

where again summation over repeated indices is understood.

The phase $\varphi(t)$ of elastic color singlet quark–antiquark scattering is defined by

$$i\varphi(t) = \langle S, q\bar{q} | \Omega(t) | q\bar{q}, S \rangle, \quad (22)$$

where

$$|q\bar{q}, S\rangle = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} |q_i \bar{q}_i\rangle$$

refers to the color singlet state of the quark and antiquark. We have contributions only if all gluon operators in Ω are contracted since no gluon occurs in the states.

In a first step we have calculated the first two terms of the phase with the Hamiltonian $-gK(t)$ of (19):

$$i\varphi(t) = (ig)^2 \langle S, q\bar{q} | \Delta_2(t) | q\bar{q}, S \rangle + (ig)^4 \langle S, q\bar{q} | \Delta_4(t) | q\bar{q}, S \rangle \quad (23)$$

In the calculation of the right-hand-side of (23) we have neglected terms contributing to the renormalization of physical quantities. This will be investigated elsewhere.

The more explicit expression of the first term of (23) is (in Feynman gauge)

$$\begin{aligned}
&(ig)^2 \langle S, q\bar{q} | \Delta_2(t) | q\bar{q}, S \rangle \\
&= \frac{1}{2}(ig)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \langle S, q\bar{q} | [K(t_2), K(t_1)] | q\bar{q}, S \rangle \\
&= \frac{1}{2}(ig)^2 \sum_a \langle S, q\bar{q} \left| \frac{\lambda^a}{2}\frac{\bar{\lambda}^a}{2} \right| q\bar{q}, S \rangle \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \\
&\cdot \{ d(\xi_2 - \eta_1) + (\xi \leftrightarrow \eta) \}
\end{aligned} \quad (24)$$

where

$$\begin{aligned}
\delta^{ab} d(\xi_2 - \eta_1) &\equiv [G^a(\xi_2), G^b(\eta_1)] \\
&= \frac{p_1^\mu p_2^\nu}{\omega_1 \omega_2} [A_\mu^a(\xi_2), A_\nu^b(\eta_1)] = i\frac{p_1 \cdot p_2}{\omega_1 \omega_2} D(\xi_2 - \eta_1)\delta^{ab} \\
&= \frac{i}{2\pi} \frac{p_1 \cdot p_2}{\omega_1 \omega_2} \varepsilon(t_2 - t_1)\delta((\xi_2 - \eta_1)^2)\delta^{ab},
\end{aligned}$$

and $D(x)$ is the massless Pauli–Jordan function,

$$\xi_i^\mu = \left(t_i, \frac{\mathbf{p}_1}{\omega_1} t_i\right); \eta_j^\mu = \left(t_j, \frac{\mathbf{p}_2}{\omega_2} t_j\right).$$

For the quark–antiquark system in a color singlet state and for very large t, which is necessary for our approximation to be valid, we get

$$(ig)^2 \langle S, q\bar{q} | \Delta_2(t) | q\bar{q}, S \rangle = i\alpha_S c_F \frac{p_1 \cdot p_2}{\sqrt{(p_1 \cdot p_2)^2 - m^4}} \ln t \quad (25)$$

with $\alpha_S = \frac{g^2}{4\pi}$, m the quark mass and $c_F = 4/3$ the eigenvalue of the quadratic Casimir operator in quark color space.

The second contribution to the phase is, see (23),

$$
\begin{aligned}
&(ig)^4 \langle S, q\bar{q} | A_4(t) | q\bar{q}, S\rangle \qquad (26)\\
&= (ig)^4 \frac{c_A}{2} c_F \Bigg[-\frac{1}{8} \int_{t_0}^{t} dt_4 \int_{t_0}^{t_4} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \\
&\cdot \{ d(\xi_1 - \eta_2) d_1(\xi_3 - \eta_4) + d(\xi_1 - \eta_2) d_1(\eta_3 - \xi_4) \\
&+ d(\xi_1 - \eta_3) d_1(\xi_2 - \eta_4) - d(\xi_2 - \eta_3) d_1(\xi_1 - \eta_4) \\
&+ (\xi \leftrightarrow \eta)\} - \frac{1}{24} \int_{t_0}^{t} dt_4 \int_{t_0}^{t_4} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_3} dt_1 \\
&\cdot \{ d(\xi_3 - \eta_2) d_1(\xi_1 - \eta_4) + d(\xi_3 - \eta_2) d_1(\eta_1 - \xi_4) \\
&+ d(\xi_3 - \eta_1) d_1(\xi_2 - \eta_4) - d(\xi_2 - \eta_1) d_1(\xi_3 - \eta_4) \\
&+ (\xi \leftrightarrow \eta)\} - \frac{1}{24} \int_{t_0}^{t} dt_4 \int_{t_0}^{t_4} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_4} dt_1 \\
&\cdot \{ d(\xi_3 - \eta_2) d_1(\xi_4 - \eta_1) + d(\xi_3 - \eta_2) d_1(\eta_4 - \xi_1) \\
&+ d(\xi_3 - \eta_4) d_1(\xi_2 - \eta_1) - d(\xi_2 - \eta_4) d_1(\xi_3 - \eta_1) \\
&+ (\xi \leftrightarrow \eta)\} + \frac{1}{24} \int_{t_0}^{t} dt_4 \int_{t_0}^{t_4} dt_3 \int_{t_0}^{t_4} dt_2 \int_{t_0}^{t_2} dt_1 \\
&\cdot \{ d(\xi_4 - \eta_2) d_1(\eta_1 - \xi_3) - d(\xi_3 - \eta_2) d_1(\eta_1 - \xi_4) \\
&+ d(\xi_3 - \eta_1) d_1(\xi_4 - \eta_2) - d(\xi_4 - \eta_1) d_1(\xi_3 - \eta_2) \\
&+ (\xi \leftrightarrow \eta)\} \Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
d_1(\xi_i - \eta_j) &= -\frac{p_1 \cdot p_2}{\omega_1 \omega_2} D_1(\xi_i - \eta_j) \\
&= \frac{p_1 \cdot p_2}{\omega_1 \omega_2} \frac{1}{2\pi^2} P \frac{1}{(\xi_i - \eta_j)^2},
\end{aligned}
$$

P denotes the principal value, $c_A = 3$ is the eigenvalue of the quadratic Casimir operator in the gluonic color space.

The evaluation of the right-hand-side of (26) yields retaining only leading terms for large times,

$$
\begin{aligned}
&(ig)^4 \langle S, q\bar{q} | A_4(t) | q\bar{q}, S\rangle = i \frac{\alpha_S^2}{\pi} c_F \frac{c_A}{3} \left(\frac{A_+ + A_-}{2(A_+ - A_-)} \right)^2 \\
&\cdot \ln \frac{\left(A_+ \frac{\omega_1}{\omega_2} - 1\right)\left(A_+ \frac{\omega_2}{\omega_1} - 1\right) A_-^2}{\left(1 - A_- \frac{\omega_1}{\omega_2}\right)\left(1 - A_- \frac{\omega_2}{\omega_1}\right) A_+^2} \ln^2 t \\
&+ \text{non-leading terms} \qquad (27)
\end{aligned}
$$

where

$$m^2 A_\pm = p_1 \cdot p_2 \pm \sqrt{(p_1 \cdot p_2)^2 - m^4}.$$

The result for the first two terms is then:

$$
\begin{aligned}
&i\varphi(t) = i\alpha_S c_F \frac{A_+ + A_-}{A_+ - A_-} \ln t \\
&+ i \frac{\alpha_S^2}{\pi} c_F \frac{c_A}{3} \left(\frac{A_+ + A_-}{2(A_+ - A_-)} \right)^2 \\
&\cdot \ln \frac{\left(A_+ \frac{\omega_1}{\omega_2} - 1\right)\left(A_+ \frac{\omega_2}{\omega_1} - 1\right) A_-^2}{\left(1 - A_- \frac{\omega_1}{\omega_2}\right)\left(1 - A_- \frac{\omega_2}{\omega_1}\right) A_+^2} \ln^2 t \\
&+ \text{non-leading terms} \qquad (28)
\end{aligned}
$$

In the next section we shall use the non-relativistic limit of (28) in a system where one quark is at rest:

$$\varphi_{\text{n.r.}}\left(\frac{pt}{m}\right) = \alpha_S c_F \frac{m}{p} \ln \frac{pt}{m} - \frac{\alpha_S^2}{\pi} c_F \frac{c_A}{3} \frac{m}{2p} \ln^2 \frac{pt}{m} \qquad (29)$$

+ non-leading terms

where p is the momentum of the moving quark.

4. Derivation of the Asymptotic Potential of $q\bar{q}$ Scattering from the Phase

In order to determine the equivalent non-relativistic potential in the Schrödinger picture and in coordinate space we proceed in two steps:

i) We derive a relation between the non-relativistic phase operator $\varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)$ and the non-relativistic potential $V_I(t)$ of elastic $q\bar{q}$ scattering in the interaction picture and in momentum space.

ii) The equation obtained in step *i)* will be transformed into the Schrödinger picture and into coordinate space.

The first step can be carried out with the help of a relation in the paper of Magnus [4] which reexpresses the two-particle-potential in terms of the elastic $q\bar{q}$ scattering phase:

$$
\begin{aligned}
-iV_I\left(\frac{pt}{m}\right) &= \left\{ i \frac{d\varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)}{dt}, \frac{1 - e^{-i\varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)}}{i\varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)} \right\} \\
&= i\dot{\varphi}_{\text{n.r.}}\left(\frac{pt}{m}\right) + \frac{1}{2}\left[\dot{\varphi}_{\text{n.r.}}\left(\frac{pt}{m}\right), \varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)\right] \\
&- \frac{i}{6}\left[\left[\dot{\varphi}_{\text{n.r.}}\left(\frac{pt}{m}\right), \varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)\right], \varphi_{\text{n.r.}}\left(\frac{pt}{m}\right)\right] + \dots \qquad (30)
\end{aligned}
$$

Since the divergent phase is diagonal in momentum, all the commutators of the RHS vanish. We obtain the following relation between the asymptotic potential and the divergent phase

$$V_I\left(\frac{pt}{m}\right) = -\dot{\varphi}_{\text{n·r·}}\left(\frac{pt}{m}\right) \tag{31}$$

This means, knowing the divergent phase operator in the Schrödinger picture and in coordinate space we can simply read off the equivalent asymptotic potential.

The transformation of $\dot{\varphi}_{n\cdot r\cdot}$ into coordinate space yields

$$\dot{\varphi}_{\text{n·r·}}(\mathbf{r}', \mathbf{r}'', t) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\cdot\mathbf{r}}\,\dot{\varphi}\left(\frac{pt}{m}\right)e^{-i\mathbf{p}\cdot\mathbf{r}''}d^3\mathbf{p}$$
$$= \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{t}\right)^3\tilde{\dot{\varphi}}\left(\frac{m}{t}(\mathbf{r}' - \mathbf{r}'')\right) \tag{32}$$

Thus we have arrived at a time dependent potential in the interaction picture. The final step which still has to be carried out is the transformation into the Schrödinger picture. A local potential $V(r)$ is obtained in the Schrödinger picture if the phase (32) divergent for large times t is modified by a factor $e^{-i(m/2t)(r'^2 - r''^2)}$ leading to the replacement

$$\dot{\varphi}_{\text{n·r·}}(\mathbf{r}', \mathbf{r}'', t) \to e^{-i(m/2t)(r'^2 - r''^2)}$$
$$\cdot\frac{1}{(2\pi)^{3/2}}\left(\frac{m}{t}\right)^3\tilde{\dot{\varphi}}\left(\frac{m}{t}\left(\mathbf{r}' - \mathbf{r}''\right)\right) \tag{33}$$

The additional factor has oscillating behavior for finite values of t. However, it approaches unity for $t \to \infty$, thus it does not modify the leading behavior of the phase for large times. We note, that this non-leading modification is also needed in the reconstruction of the Coulomb potential from its divergent phase in QED.

The divergent phase in the Schrödinger picture is then

$$\dot{\varphi}_{\text{n·r·},S}(\mathbf{r}, \mathbf{r}''') = W^{-1}e^{-i(m/2t)(\mathbf{r}'^2 - \mathbf{r}''^2)}$$
$$\cdot\frac{1}{(2\pi)^{3/2}}\left(\frac{m}{t}\right)^3\tilde{\dot{\varphi}}\left(\frac{m}{2t}(\mathbf{r}' - \mathbf{r}'')\right)W$$
$$= \left(\frac{m}{2\pi t}\right)^3\int d^3\mathbf{r}'\,d^3\mathbf{r}''\,e^{i(m/2t)(\mathbf{r}-\mathbf{r}')^2}e^{-i(m/2t)(r'^2 - r''^2)}$$
$$\cdot\left(\frac{m}{t}\right)^3\frac{1}{(2\pi)^{3/2}}\tilde{\dot{\varphi}}_{\text{n.r.}}\left(\frac{m}{t}(\mathbf{r}' - \mathbf{r}'')\right)e^{-i(m/2t)(\mathbf{r}'' - \mathbf{r}''')^2} \tag{34}$$

Under the integral only the mixed terms $\mathbf{r}\cdot\mathbf{r}'$, $\mathbf{r}''\cdot\mathbf{r}'''$ survive in the exponent and we obtain

$$\dot{\varphi}_{\text{n.r.},S}(\mathbf{r}, \mathbf{r}''') = \left(\frac{m}{2\pi t}\right)^3\left(\frac{m}{t}\right)^3\frac{e^{i(m/2t)(r^2 - r'''^2)}}{(2\pi)^{3/2}}\cdot I$$

where

$$I = \int d^3\mathbf{r}'\,d^3\mathbf{r}''e^{-i(m/t)(\mathbf{r}\cdot\mathbf{r}' - \mathbf{r}''\cdot\mathbf{r}''')}\tilde{\dot{\varphi}}_{\text{n·r·}}\left(\frac{m}{t}(\mathbf{r}' - \mathbf{r}'')\right).$$

The calculation of the integral I proceeds in the following way

$$I = \int d^3(\mathbf{r}' - \mathbf{r}'')d^3\frac{\mathbf{r}' + \mathbf{r}''}{2}\exp\left[-\frac{i\,m}{t}\left[\frac{(\mathbf{r} + \mathbf{r}''')}{2}\cdot\right.\right.$$
$$\left.\left.\cdot(\mathbf{r}' - \mathbf{r}'') + \frac{(\mathbf{r}' + \mathbf{r}'')}{2}\cdot(\mathbf{r} - \mathbf{r}''')\right]\right]\tilde{\dot{\varphi}}_{\text{n.r.}}\left(\frac{m}{t}(\mathbf{r}' - \mathbf{r}'')\right)$$
$$= \left(\frac{t}{m}\right)^6(2\pi)^3(2\pi)^{3/2}\dot{\varphi}_{\text{n·r·}}\left(\left|\frac{\mathbf{r}' + \mathbf{r}'''}{2}\right|\right)\delta^{(3)}(\mathbf{r} - \mathbf{r}''')$$

Altogether we have for the divergent phase in the coordinate space representation

$$\dot{\varphi}_{\text{n·r·},S}(\mathbf{r}, \mathbf{r}''') = \dot{\varphi}_{\text{n·r·}}(|\mathbf{r}|)\delta^{(3)}(\mathbf{r} - \mathbf{r}''') \tag{35}$$

The diagonality of the phase operator and (31) give the asymptotic non-relativistic Schrödinger potential as multiplicative operator

$$V(|\mathbf{r}|) = -\dot{\varphi}_{\text{n·r·}}(|\mathbf{r}|). \tag{36}$$

5. The Non-Relativistic Long Range Potential from the Quark–Gluon Interaction and Concluding Remarks

In Sect. 3 we summed up an infinite number of Feynman graphs with multiple gluon exchanges in $q\bar{q}$ scattering. In the asymptotic limit, i.e. for large times one obtains divergent terms proportional to $\alpha_s^m(\ln t)^n$ in any finite order of perturbation theory. In the infrared limit these divergent terms exponentiate, which is elegantly obtained by using the Magnus solution, (7). This exponentiation leads to the phase factor exp $i\ \varphi(t)$, where $\varphi(t)$ is the generalization of the well-known Columb phase in QED [1]. In QED this phase is simply proportional to $\frac{m}{p}\alpha \ln t$, which gives $\dot{\varphi}(t) \sim \alpha/\left(\frac{p}{m}t\right)$ and therefore leads immediately to the Coulomb potential $-\alpha/r$ by means of (36). For the quark–antiquark system being in a color singlet state the QCD phase $\varphi(t)$ has an expansion in powers of $\alpha_s \ln t$; the first two terms have been calculated in Sect. 3 and are given in (28).

The physical interpretation of the phase $\varphi(t)$ has been given in Sect. 4, where we showed that the time derivative of the non-relativistic limit of $\varphi(t)$ defines the Schrödinger potential $V(r)$. We therefore obtain from eqs. (36) and (29) the first two terms in an expansion of the long range potential for a color singlet quark–antiquark pair

$$V(r) = -C_F\frac{\alpha_s}{r} + C_F\frac{C_A}{3}\frac{\alpha_s^2}{\pi}\frac{\ln(r/r_0)}{r} + \dots \tag{37}$$

We would like to emphasize that the potential (37) is not a standard perturbative result, since it has been obtained from an exponentiation of an infinite ladder of gluon exchanges.

In the derivation of (37) we have omitted all vertex corrections to bare gluon exchanges and have not included the modified gluon propagators due to the gluon self-interactions, although these terms are of course present in the Magnus solution (10). (Notice that only the three-gluon interaction enters in our calculation. The four-gluon interaction has to be taken into account for the first time in a claculation of the α_s^3-term). We expect, however, that the corrections do not change the α_s^2-term of the potential, since they should merely lead to a redefinition of the coupling strength $\alpha_s(\alpha_s \to \alpha_s(r))$, which would replace (37) by $(r > r_0)$

$$V(r) = -C_F \frac{\alpha_s(r)}{r} + C_F \frac{C_A}{3} \frac{\alpha_s^2(r)}{\pi} \frac{\ln(r/r_0)}{r} + \dots \qquad (38)$$

The first term of the potential is identical to the QED Coulomb potential (apart from a factor C_F), which is attractive also for long distances if the quark–antiquark pair is in a color singlet state. The second term, however, is a novel feature since it is positive for long distances $(r > r_0)$ and therefore represents a repulsive barrier.

The potential (38) contains two parameters, the QCD scale parameter Λ (via $\alpha_s(r)$) and the distance r_0. Since Λ is the only dimensional parameter in the theory, r_0 has to be proportional to Λ^{-1}, $r_0 = \rho_0/\Lambda$, ρ_0 being a dimensionless, as yet undetermined positive constant. The physical meaning of r_0 is clear: r_0 is that quark–antiquark separation where the long range potential $V(r)$ matches the asymptotic freedom potential $V_{AF}(r)$ [9]

$$V_{AF}(r) = -C_F \frac{\alpha_s(r)}{r} \qquad (39)$$

which is valid at short distances. Thus we may interpret r_0 as the length scale that determines the short distance region, $r < r_0 \ll 1/\Lambda$. Intuitively one expects that the short distance regime for heavy quarks (antiquarks) with mass m involves distances which are of the order of the Compton wave length of the quarks, and one therefore roughly expects $r_0 \cong 1/m$. For charmed quarks with $m = 1.5$ GeV this implies that the potential (38) should be reliable for distances larger than 0.1 fermi. On the other hand, since the quark–gluon Hamiltonian (1) does not allow the creation of additional quark–antiquark pairs—an effect which has nothing to do with the infrared behavior discussed in this paper—the potential (38) has physical meaning only in the infrared regime $r_0 < r < r_1$. Here r_1 is the distance, where the long range potential produces a chromoelectric field, which is strong enough to create a quark–antiquark pair. For a rough estimate of the screening length r_1 one could take the quark–antiquark separation at which the repulsive potential barrier exceeds the threshold energy for quark–antiquark production, $V(r_1) = 2m$.

In this paper we described a new method for obtaining the $q\bar{q}$ potential from QCD. In a first step we calculated the contributions coming from the Δ_2 and Δ_4 terms in the Magnus solution (10). Of course, what is required for a complete solution is a calculation of all the other Δ_n terms in (10). A first investigation of the next term in (10), i.e. the Δ_6 contribution shows that it behaves as $\alpha_s^3 \frac{\ln^2 r}{r}$. A computation of the higher terms in (10) is under way and will be published after completion.

Finally let us mention that our potential calculation differs from those existing in the literature [9] in the important fact, that we are considering *moving* quarks whereas the authors of [9] are dealing with *static* quarks. Furthermore we differ from those static potential calculations by having a repulsive potential barrier at long distances instead of a negative attractive potential. The α_s^2 contribution to $V(r)$ in [9] can be used for the definition of the running coupling strength $\alpha_s(r)$, which then turns out to coincide with the well-known asymptotic freedom $\alpha_s(r)$ at short distances. One concludes that the potential of [9] describes the short range behavior in contrast to our result (38) which is valid in the infrared regime.

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