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# Higher Order Perturbative QCD Calculation of Jet Cross Sections in $e^+e^-$ Annihilation

K. Fabricius[1], G. Kramer[2], G. Schierholz[2], I. Schmitt[1]*

1 Fakultät für Physik der Universität, D-5600 Wuppertal, Federal Republic of Germany

2 II. Institut für Theoretische Physik der Universität**, D-2000 Hamburg, Federal Republic of Germany

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**Abstract.** We present in detail the analytic calculation of the Sterman-Weinberg type 3-jet cross section to order $\alpha_s^2$. The fit to recent PLUTO data gives in the $\overline{\text{MS}}$ scheme $\alpha_s = 0.17$ which corresponds to $\Lambda = 0.24$ GeV in the 1-loop approximation.

## I. Introduction

Recently, 3-jet events have been observed in $e^+e^-$ annihilation at the higher PETRA energies [1]. This is interpreted as hard single-gluon bremsstrahlung off quarks and antiquarks at small distances before fragmentation into final state hadrons takes place and was first predicted in [2]. More lately, somewhat more unambiguous tests of perturbative QCD in $e^+e^-$ annihilation have been proposed [3–6]. So far one can say that the observed properties of the 3-jet events seem to be in good qualitative agreement with lowest order QCD perturbation theory [1, 7].

This agreement of theory and experimental data could, however, be fortuitous if higher-order corrections should turn out to be large. In such an event also the recent determinations of the strong coupling constant $\alpha_s$ from the analysis of $e^+e^-$ jets [8] had to be revised. Moreover, it is well known that the determination of $\alpha_s$ or, equivalently, of the scale parameter of the strong interactions $\Lambda$ and comparison with other processes (e.g., deep inelastic lepton scattering) is meaningful only if higher-order corrections are included [9]. The amount of higher-order contributions depends on the renormalization scheme which is used to define $\alpha_s$ or $\Lambda$. Only if the higher-order corrections are known can we try to fix the renormalization scheme in such a way that they are minimized in as many processes as possible.

A first step in this direction has been the calculation of the order $\alpha_s^2$ correction to the total $e^+e^-$ annihilation cross section $\sigma$. In the $\overline{\text{MS}}$ renormalization scheme this was found to be small [10]. The $\overline{\text{MS}}$ scheme also minimizes higher-order corrections to various other processes [11] which one might take as an indication that the perturbation expansion converges well in the $\overline{\text{MS}}$ scheme***.

In view of the small order $\alpha_s^2$ correction to $\sigma$ (it has an effect of less than half a percent on the zero'th order background), this seems favoured for determining $\alpha_s$. However, because of substantial statistical and systematic errors attending the measurement of $\sigma$, an accurate determination of $\alpha_s$ or $\Lambda$ is not feasible for the time being [13].

In this paper we shall calculate the order $\alpha_s^2$ corrections to 3-jet final states, which is the last step in our systematic study of order $\alpha_s^2$ contributions to $e^+e^-$ jet cross sections [4–6]. For that purpose we have to cut out the 2-jet region. In principle, this can be done in various ways used to from the order $\alpha_s$ analysis. But some caution is demanded.

We have chosen to use a Sterman-Weinberg type [14] angle and energy cut off [15] for a variety of reasons: (*i*) The Sterman-Weinberg type multi-jet cross sections are the only ones known to us which factorize**** [16] like the

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*** It appears, though, that this is not true for the total decay rate of heavy pseudoscalar quark-antiquark states [12]. This quantity is, however, sensitive to details of the bound state wave function

**** The 2-jet cross section slightly modified though

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total cross section. We emphasize that a cross section which does not factorize has a priori no physical meaning, in particular, as higher order contributions and hadronization effects may be large. (*ii*) The $C$ and ("bare") thrust ($T$) distributions calculated in [17, 18] (e.g.) are not defined at $C=\frac{3}{4}$ and $T=\frac{2}{3}$ while the Sterman-Weinberg 3-jet cross section is bounded to all orders by $x_{\max}\geqq\frac{2}{3}$, $x_{\max}$ being the scaled maximum jet energy. (*iii*) As we shall see, the (numerically) relevant pieces of the Sterman-Weinberg 3-jet cross section can be checked independently. (*iv*) At a later stage we want to include fragmentation which requires a clear separation between 3- and 4-jet final states.

In a recent letter we have reported some results of our calculation [15]. Here we shall give account of the details.

The outline of the paper is as follows. In Sect. II we calculate the order $\alpha_s^2$ virtual corrections to $q\bar{q}g$ production. The integration of the $q\bar{q}gg$ and $q\bar{q}q\bar{q}$ final state contributions over small energy and/or angular regions is done in Sect. III. In Sect. IV we present the Sterman-Weinberg 3-jet cross section in analytic form and compare it to experiment. Section V, finally, contains some concluding remarks.

## II. Loop Corrections to $q\bar{q}g$ Final States

In the following we will be dealing with the order $\alpha_s^2$ corrections to

$$e^+(p_+)+e^-(p_-)\to q(p_1)+\bar{q}(p_2)+g(p_3) \tag{2.1}$$

and the processes

$$e^+(p_+)+e^-(p_-)\to q(p_1)+\bar{q}(p_2)+g(p_3)+g(p_4), \tag{2.2}$$

$$e^+(p_+)+e^-(p_-)\to q(p_1)+\bar{q}(p_2)+q(p_3)+\bar{q}(p_4). \tag{2.3}$$

The symbols in brackets in (2.1)–(2.3) denote the momenta of the particles. All partons, quarks and gluons, are massless.

We shall regularize the infrared and collinear divergences in (2.1)–(2.3) as well as the ultraviolet divergences in (2.1) by continuing to space-time dimension $n=4-\lambda$ [19]. In the first place this modifies the phase space for $j$ massless final state particles so that

$$(\text{phase space})^{(j)}=\int\prod_{i=1}^{j}\left(\frac{d^{n-1}p_i}{(2\pi)^{n-1}2E_i}\right)(2\pi)^n\delta^{(n)}\left(q-\sum_{i=1}^{j}p_i\right) \tag{2.4}$$

which enters the cross section formula for the above processes:

$$d\sigma^{(j)}=\frac{e^4}{2q^6}(L^{\mu\nu}H_{\mu\nu})(\text{phase space})^{(j)}. \tag{2.5}$$

Here $L_{\mu\nu}$ is the lepton and $H_{\mu\nu}$ the hadron tensor. The latter contains summation over the final state spin, colour and flavour indices and includes the appropriate quark charge factors.

The cross section formula (2.5) contains the various angular correlations between the final state particles and the direction of the incoming beams [4, 6, 20]. In the following we are only interested in the jet cross sections with all angular correlations integrated out. This allows us to write for the leptonic tensor

$$L_{\mu\nu}\left(\equiv p_{+\mu}p_{-\nu}+p_{-\mu}p_{+\nu}-g_{\mu\nu}\frac{q^2}{2}\right)=-g_{\mu\nu}\frac{q^2}{3}. \tag{2.6}$$

For arbitrary $n$ the zeroth order total $e^+e^-$ annihilation cross section which occurs as a common factor in the cross section formulae of (2.1)–(2.3) becomes

$$\sigma^{(2)}=\sigma_0\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\left(1-\frac{\lambda}{2}\right)\frac{\Gamma\left(1-\frac{\lambda}{2}\right)}{\Gamma(2-\lambda)},\quad \sigma_0=\frac{4\pi\alpha^2}{3q^2}N_c\sum_{i=1}^{N_f}e_i^2, \tag{2.7}$$

where $N_c$ is the number of colours, $N_f$ the number of flavours and $e_i$ the quark charge in units of $e$. The parameter $\mu$ is an arbitrary scale parameter and the factor $\left(1-\frac{\lambda}{2}\right)$ is due to the $n$ dimensional contraction of $g^{\mu\nu}H_{\mu\nu}$.

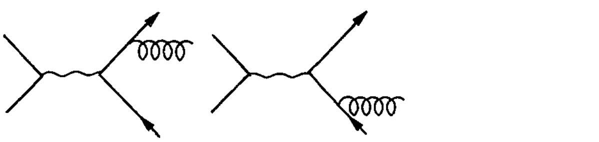

**Fig. 1.** Diagrams with three partons in the final state to order $\alpha_s$

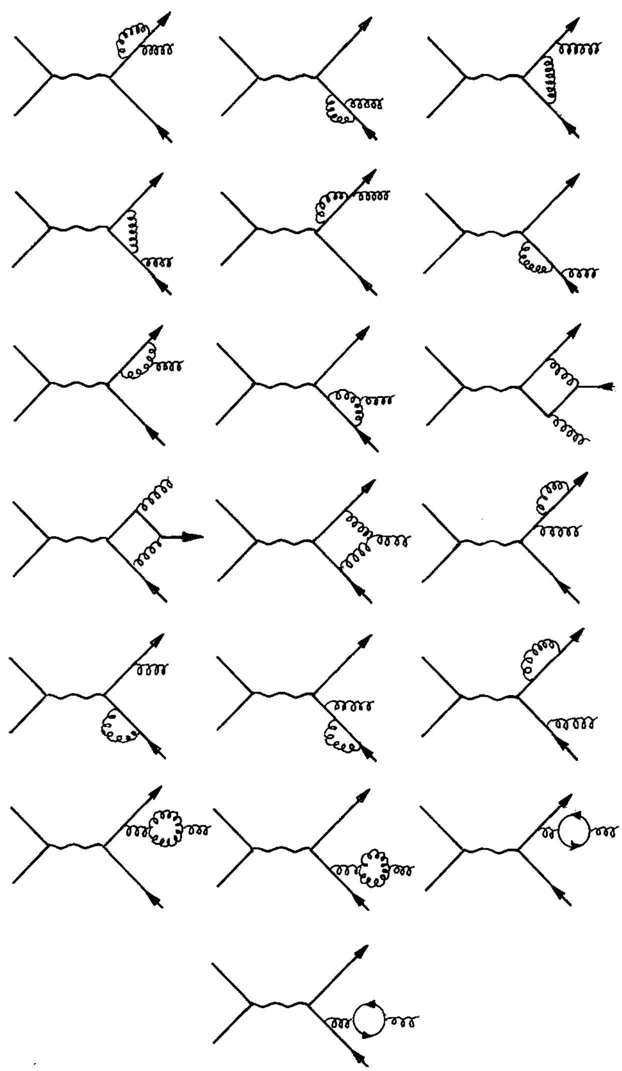

**Fig. 2.** Diagrams with three partons in the final state to order $\alpha_s^2$ interfering with the diagrams in Fig. 1

To lowest order the process (2.1) proceeds by the diagrams shown in Fig. 1. For the 3-jet cross section we obtain

$$d\sigma^{(3)}=\sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\cdot\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}dx_1dx_2[(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}\frac{\alpha_s(\mu^2)}{2\pi}C_FB^{V-\lambda/2S}(x_1,x_2), \tag{2.8}$$

where

$$B^{V-\lambda/2S}(x_1,x_2)=B^V(x_1,x_2)-\frac{\lambda}{2}B^S(x_1,x_2),$$

$$B^V(x_1,x_2)=\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)},$$

$$B^S(x_1,x_2)=\frac{x_3^2}{(1-x_1)(1-x_2)} \tag{2.9}$$

and $C_F=\frac{4}{3}$. The variables are $x_i=2E_i/\sqrt{q^2}$ and $y_{ij}=s_{ij}/q^2$, $s_{ij}=(p_i+p_j)^2$. They are related by $x_1=1-y_{23}$, $x_2=1-y_{13}$, $x_3=1-y_{12}$ and $x_1+x_2+x_3=2$. For $\lambda=0$ the three-jet cross section goes over into the familiar form. Note that $B^S$ is identical to the matrix element for the production of a scalar gluon.

After these preliminaries let us now come to the topic of this section which is the calculation of the order $\alpha_s^2$ corrections to the process (2.1). The processes (2.2) and (2.3) with four partons in the final state will be treated in the next section.

The loop corrections to (2.1) are shown in Fig. 2. The calculations are done in the Feynman gauge. The ultraviolet divergences appear as poles in $\lambda$. We perform renormalization in the minimal subtraction scheme (MS) [21] which corresponds to the subtraction of the ultraviolet poles. The counterterm in the MS scheme is given by (for the details see Appendix A).

$$\frac{d\sigma^{(ct)}}{dx_1 dx_2} = \sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2} \frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)} [(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}$$

$$\cdot \frac{\alpha_s(\mu^2)}{2\pi} C_F B^{V-\lambda/2S}(x_1,x_2) \frac{\alpha_s(\mu^2)}{2\pi} \left(\tfrac{1}{3}N_f - \tfrac{11}{6}N_c\right)\frac{2}{\lambda}. \tag{2.10}$$

Other renormalization schemes will be discussed later on.

The integrals over loop momenta can be expressed, using the reduction method of 'tHooft and Veltman [22], in terms of a few standard integrals corresponding to scalar 2-, 3-, and 4-point diagrams. These are given in Appendix B (and can be traced back in the final cross section formula given below). The traces and matrix elements throughout this paper have been calculated and processed with the help of Schoonship [24] and Reduce [25]. In intermediate stages of the calculation expressions become quite lengthy. The final result has, however, the relatively compact form:

$$\frac{d\sigma^{(3)}}{dx_1 dx_2} = \sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2} \frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)} ((1-x_1)(1-x_2)(x_1+x_2-1))^{-\lambda/2} \frac{\alpha_s(\mu^2)}{2\pi} C_F$$

$$\cdot \left[B^{V-\lambda/2S}(x_1,x_2) + \frac{\alpha_s(\mu^2)}{2\pi} \frac{\Gamma\left(1-\frac{\lambda}{2}\right)}{\Gamma(1-\lambda)} \left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2} \left\{ B^{V-\lambda/2S}(x_1,x_2)\left[-\frac{4}{\lambda^2}(2C_F+N_c)\right.\right.\right.$$

$$\left.\left.\left. + \frac{a}{\lambda} + b\right] + f(x_1,x_2)\right\}\right], \tag{2.11}$$

where

$$a = -6C_F + \tfrac{2}{3}N_f - \tfrac{11}{3}N_c + 2(2C_F - N_c)\ln y_{12} + 2N_c \ln(y_{13}y_{23}), \tag{2.12}$$

$$b = (2C_F+N_c)\frac{\pi^2}{3} - 8C_F + \tfrac{1}{2}N_c(\ln^2 y_{12} - \ln^2 y_{13} - \ln^2 y_{23}) - C_F \ln^2 y_{12}$$

$$+ \left(\tfrac{1}{3}N_f - \tfrac{11}{6}N_c\right)\left(\gamma - \ln(4\pi) - \ln\frac{\mu^2}{q^2}\right), \tag{2.13}$$

$$f(x_1,x_2) = C_F\left[\frac{y_{12}}{y_{12}+y_{13}} + \frac{y_{12}}{y_{12}+y_{23}} + \frac{4y_{12}}{y_{13}+y_{23}} - \frac{y_{12}}{y_{13}} - \frac{y_{12}}{y_{23}} - \frac{y_{13}}{y_{23}} - \frac{y_{23}}{y_{13}}\right]$$

$$+ N_c\left[\frac{y_{12}}{y_{13}} + \frac{y_{12}}{y_{23}} + \frac{y_{13}}{y_{23}} + \frac{y_{23}}{y_{13}} - \frac{2y_{12}}{y_{13}+y_{23}}\right]$$

$$+ \ln y_{13}\left[N_c \frac{y_{13}}{y_{12}+y_{23}} + C_F\left(4 - \frac{y_{13}y_{23}}{(y_{12}+y_{23})^2} + \frac{2y_{13}-4y_{23}}{y_{12}+y_{23}}\right)\right]$$

$$+ \ln y_{23}\left[N_c \frac{y_{23}}{y_{12}+y_{13}} + C_F\left(4 - \frac{y_{23}y_{13}}{(y_{12}+y_{13})^2} + \frac{2y_{23}-4y_{13}}{y_{12}+y_{13}}\right)\right]$$

$$+ 2(2C_F - N_c)\ln y_{12}\left[\frac{2y_{12}}{y_{13}+y_{23}} + \frac{y_{12}^2}{(y_{13}+y_{23})^2}\right] - N_c B^V(x_1,x_2) r(y_{13},y_{23})$$

$$-(2C_F - N_c)\left[\frac{y_{12}^2+(y_{12}+y_{13})^2}{y_{13}y_{23}} r(y_{12},y_{23}) + \frac{y_{12}^2+(y_{12}+y_{23})^2}{y_{13}y_{23}} r(y_{12},y_{13})\right]. \tag{2.14}$$

Herein $\gamma$ is the Euler constant and $r(x,y)$ is defined as

$$r(x,y)=\ln x \ln y-\ln x \ln(1-x)-\ln y \ln(1-y)+\frac{\pi^2}{6}-\mathcal{L}_2(x)-\mathcal{L}_2(y), \tag{2.15}$$

where $\mathcal{L}_2(x)$ is the Spence function

$$\mathcal{L}_2(x)=-\int_0^x dz \frac{\ln(1-z)}{z}. \tag{2.16}$$

In (2.11) we notice several expressions which are proportional to $(\frac{1}{3}N_f-\frac{11}{6}N_c)$ and, hence, can be absorbed into the definitions of the strong coupling constant. The large logarithmic term

$$-(\tfrac{1}{3}N_f-\tfrac{11}{6}N_c)\ln\frac{\mu^2}{q^2}B^{V-\lambda/2S}(x_1,x_2)$$

in (2.13) represents the explicit beginning of the renormalization group improvement of (2.8) and arranges that $\alpha_s(\mu^2)$ becomes the running coupling constant:

$$\begin{aligned}\alpha_s(q^2)&=\alpha_s(\mu^2)\left[1+\frac{\alpha_s(\mu^2)}{2\pi}(\tfrac{11}{6}N_c-\tfrac{1}{3}N_f)\ln\frac{q^2}{\mu^2}\right]^{-1}\\&\equiv\frac{2\pi}{(\frac{11}{6}N_c-\frac{1}{3}N_f)\ln\frac{q^2}{\Lambda^2}}+O(\alpha_s^3).\end{aligned} \tag{2.17}$$

For better convergence of the perturbation series it is customary to also subtract the expression $(\frac{1}{3}N_f-\frac{11}{6}N_c)(\gamma-\ln(4\pi))$ from (2.13) together with the ultraviolet poles (as we will do later on) which then is called the $\overline{\text{MS}}$ scheme [26]. As can easily be seen, this is equivalent to replacing $\Lambda$ by

$$\Lambda_{\text{MS}}\rightarrow\Lambda_{\overline{\text{MS}}}=e^{(\ln(4\pi)-\gamma)/2}\Lambda_{\text{MS}}. \tag{2.18}$$

Our result for the loop diagrams agrees with that of Ellis et al. [17].

## III. Reduction of Diagrams with Four Partons in the Final State

In this section we shall calculate the contributions of $q\bar{q}gg$ and $q\bar{q}q\bar{q}$ final states to the 3-jet cross section. The corresponding diagrams are shown in Fig. 3. The transition matrix element for the processes (2.2) and (2.3) have been given already in two of our earlier papers [5] and [6], respectively (for $n=4$ though).

By 3-jet cross section we shall understand more precisely now the cross section for events which have all but a fraction* $\varepsilon/2$ of the total energy distributed within three separated cones of (full) opening angle $\delta$. In other words, we call an event (on the parton level) a 3-jet event (with jet axis and $\varepsilon,\delta$ specified beforehand) if all the (parton)

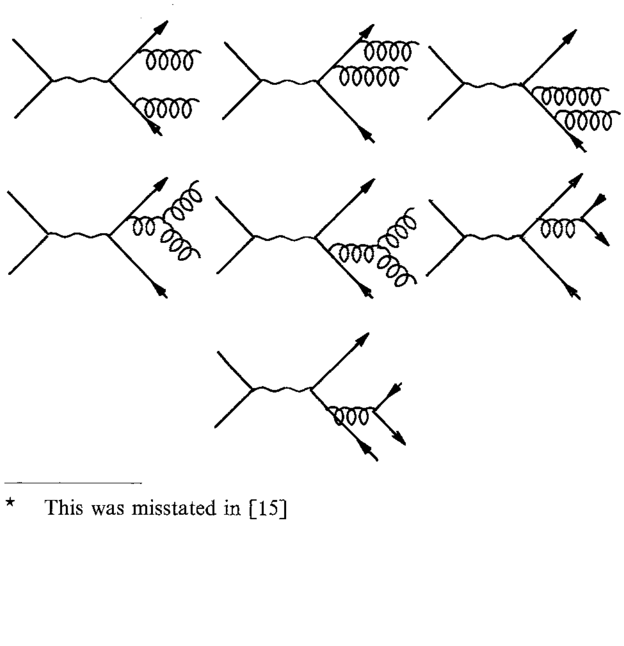

**Fig. 3.** Diagrams with four partons in the final state

* This was misstated in [15]

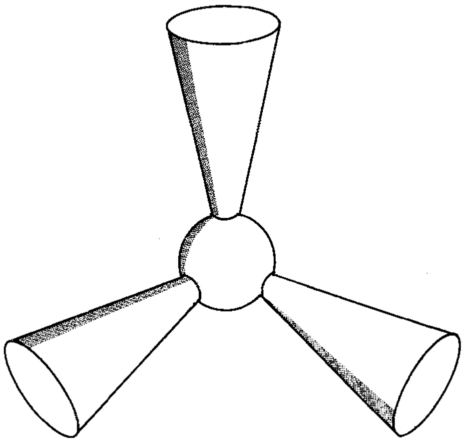

Fig. 4. Three-jet phase volume

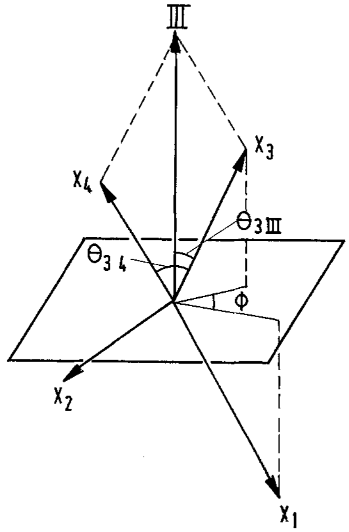


Fig. 5. Kinematics of 4-parton final state. The plane is perpendicular to the axis labelled III

momenta fall inside the phase volume shown in Fig. 4. By construction this includes the singular region associated with one of the gluons in (2.2) being soft and/or collinear with one of the quarks or the other gluon and one of the quarks in (2.3) being collinear with one of the antiquarks, respectively.

The 3-jet cross section is finite by virtue of the Kinoshita-Lee-Nauenberg theorem [27]. This is to say that the processes (2.2) and (2.3) must contribute the same pole terms as the loop corrections (2.11) but with opposite sign.

The first step will be now to set up the 4-particle phase space. For obvious reasons we shall decompose it into a quasi 3-body (3-jet) phase space times an integral in one of the parton momenta extending over soft and/or collinear configurations (collinear with one of the jets). Which parton momentum this concerns depends on the singularities of the particular contribution to the 4-parton cross section to be considered. We will restrict ourselves to the case where the parton with momentum $p_3$ is soft and/or collinear with the parton of momentum $p_4$.

We shall start from the 4-parton cross section (2.5) which we rewrite in the form

$$d\sigma^{(4)}=\sigma_0\frac{2\pi}{q^2}(\mu^3)^{4-n}\int\prod_{i=1}^{4}\frac{d^{n-1}p_i}{(2\pi)^{n-1}2E_i}(2\pi)^n\delta^{(n)}\left(q-\sum_{i=1}^{4}p_i\right)(-g^{\mu\nu}\tilde{H}_{\mu\nu}), \tag{3.1}$$

where the scale factor, which makes $\alpha, \alpha_s$ dimensionless, has been factored out explicity and

$$H_{\mu\nu}=N_c\sum_{i=1}^{N_f}e_i^2\tilde{H}_{\mu\nu}. \tag{3.2}$$

Note that the statistical factors are, different from [5, 6], now included in $\tilde{H}_{\mu\nu}$. As before, our notation will be $x_i=2E_i/\sqrt{q^2}$ and $s_{ij}=(p_i+p_j)^2=y_{ij}q^2$. The two sets of variables are related by $x_1=1-y_{23}-y_{24}-y_{34}$, $x_2=1-y_{13}-y_{14}-y_{34}$, $x_3=1-y_{12}-y_{14}-y_{24}$, $x_4=1-y_{12}-y_{13}-y_{23}$, and $x_1+x_2+x_3+x_4=2$ now. The angles between parton momenta $\mathbf{p}_i$ and $\mathbf{p}_j$ are denoted by $\Theta_{ij}$. The reference frame we shall use is shown in Fig. 5.

We now integrate out all variables but $x_1$, $x_2$, $x_3$, the polar angle $\Theta_{3\mathrm{III}}$ (between $\mathbf{p}_3$ and $\mathbf{p}_3+\mathbf{p}_4$) and the azimuthal angle $\phi$. Clearly, the hadronic tensor will only depend on the latter. This yields

$$d\sigma^{(4)}=\sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}dx_1dx_2[\tfrac{1}{2}x_1x_2\sin\Theta_{12}]^{-\lambda}$$
$$\cdot\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\frac{2^\lambda}{\Gamma\left(2-\frac{\lambda}{2}\right)}\frac{q^2}{128(2\pi)^5}dx_3x_3^{1-\lambda}\left(\frac{x_3+x_4}{x_4}\right)d\Theta_{3\mathrm{III}}\sin^{1-\lambda}\Theta_{3\mathrm{III}}d\phi(-g^{\mu\nu}\tilde{H}_{\mu\nu}), \tag{3.3}$$

where we have made use of the phase space formulae [19]

$$\int \frac{d^{n-1}p}{2E} = \frac{1}{2}\int dp\, p^{n-3} d\Theta \sin^{n-3}\Theta \ldots = \frac{\pi^{n/2-1}}{\Gamma\left(\frac{n}{2}-1\right)} \int dp\, p^{n-3} d\Theta \sin^{n-3}\Theta\,,$$

$$\int_0^\pi d\Theta \sin^{n-3}\Theta = 2^{n-3} \frac{\left(\Gamma\left(\frac{n}{2}-1\right)\right)^2}{\Gamma(n-2)}\,. \tag{3.4}$$

The factor $[\frac{1}{2}x_1 x_2 \sin\Theta_{12}]^{-\lambda}$ in (3.3) differs from its 3-jet limit by an expression of the form (see Appendix C)

$$[\tfrac{1}{2}x_1 x_2 \sin\Theta_{12}]^{-\lambda} - [(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2} = O(\lambda y_{34})\,. \tag{3.5}$$

It is obvious that a term proportional to $\lambda$ will only survive in conjunction with a singular term. Since $y_{34} g^{\mu\nu}\tilde{H}_{\mu\nu}$ is finite in the kinematical region we are considering, this means we can write

$$[\tfrac{1}{2}x_1 x_2 \sin\Theta_{12}]^{-\lambda} = [(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}\,.$$

The matrix element $g^{\mu\nu}\tilde{H}_{\mu\nu}$ is a function of the invariants $s_{ij}$, dominantly $s_{34}$ in the region under discussion, so that it is more convenient to use $\Theta_{34}$ rather than $\Theta_{3\mathrm{III}}$ as integration variable. The transformation $\Theta_{3\mathrm{III}}$ to $\Theta_{34}$ variables is given by

$$d\cos\Theta_{3\mathrm{III}} = d\cos\Theta_{34} x_4^2 (x_3\cos\Theta_{34}+x_4)(x_3^2+2x_3x_4\cos\Theta_{34}+x_4^2)^{-3/2}$$
$$\cdot \sin\Theta_{3\mathrm{III}} = x_4 \sin\Theta_{34}(x_3^2+2x_3x_4\cos\Theta_{34}+x_4^2)^{-1/2} \tag{3.6}$$

After this the cross section formula (3.3) assumes the form

$$d\sigma^{(4)} = \sigma^{(2)} \left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2} \frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)} dx_1 dx_2 [(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}$$
$$\cdot \left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2} \frac{2^\lambda}{\Gamma\left(2-\frac{\lambda}{2}\right)} \frac{q^2}{128(2\pi)^5} dx_3 (x_3x_4)^{1-\lambda}(x_3+x_4) d\Theta_{34} \sin^{1-\lambda}\Theta_{34}$$
$$\cdot (x_3\cos\Theta_{34}+x_4)(x_3^2+2x_3x_4\cos\Theta_{34}+x_4^2)^{\frac{\lambda-3}{2}} d\phi(-g^{\mu\nu}\tilde{H}_{\mu\nu})\,. \tag{3.7}$$

The first part of (3.7) is exactly the 3-jet phase space as it appears in (2.8) and (2.11) (with "composite" jet energy $2-x_1-x_2$), while the remainder represents the relative parton distribution inside a 3-jet event. The appropriate phase space formulae for the other regions of integration can be derived from (3.7) by interchange of momentum labels.

In the next step we shall integrate over the internal parton distribution holding the jet energies fixed. Our aim thereby is to calculate the 3-jet cross section analytically except for terms of order $\varepsilon$ and $\delta^2$. Since only the singular terms contribute to the nonvanishing part (for $\varepsilon, \delta \to 0$) of the 3-jet cross section we shall focus on these contributions here. The corrections of order $\varepsilon$ and $\delta^2$ will be calculated numerically.

The singular pieces have either one of the forms [5, 6]

$$(i)\quad \frac{s_{12}}{s_{13}s_{23}s_{14}s_{24}}\,, \tag{3.8}$$

$$(ii)\quad \frac{1}{s_{13}s_{34}}, \frac{1}{s_{14}s_{24}} + (1\leftrightarrow 2) + (3\leftrightarrow 4) + (1\leftrightarrow 2, 3\leftrightarrow 4)\,, \tag{3.9}$$

$$(iii)\quad \frac{1}{s_{34}}, \frac{1}{s_{13}} + (1\leftrightarrow 2) + (3\leftrightarrow 4) + (1\leftrightarrow 2, 3\leftrightarrow 4)\,. \tag{3.10}$$

The first term can be decomposed into a sum of two terms plus a nonsingular piece $\Delta$ which separate the singularities at $x_3=0$ and $x_4=0$:

$$\frac{s_{12}}{s_{13}s_{23}s_{14}s_{24}}=\frac{s_{\mathrm{I\,II}}}{s_{\mathrm{I\,III}}s_{\mathrm{II\,III}}}\left(\frac{1}{s_{13}s_{23}}+\frac{1}{s_{14}s_{24}}\right)+\Delta, \tag{3.11}$$

where*

$$s_{KL}=(p_K^{\mathrm{jet}}+p_L^{\mathrm{jet}})^2|_{3\text{-jet limit}};\, K,L=\mathrm{I,II,III}, \tag{3.12}$$

$p_K^{\mathrm{jet}}$ being the 4-momentum of the $K$-th jet, and the labels I, II and III refer to the jets having quark, antiquark and gluon quantum numbers, respectively. This leaves us with terms of the form (3.9) and (3.10).

Let us first consider (3.9)

$$\frac{1}{s_{13}s_{34}}=\frac{4}{(q^2)^2x_1x_4}\frac{1}{x_3^2(1-\cos\Theta_{13})(1-\cos\Theta_{34})} \tag{3.13}$$

which is singular at $x_3=0$ and $\Theta_{13}$, $\Theta_{34}=0$. This may be decomposed into separate pole terms

$$\frac{1}{x_3^2(1-\cos\Theta_{13})(1-\cos\Theta_{34})}=\frac{1}{1-\cos\Theta_{\mathrm{I\,III}}}\left[\frac{1}{x_3^2(1-\cos\Theta_{13})}\right.$$
$$\left.+\frac{1}{x_3^2(1-\cos\Theta_{34})}+D(x_3,\cos\Theta_{13},\cos\Theta_{34})\right], \tag{3.14}$$

where**

$$\Theta_{KL}=\sphericalangle(\mathbf{p}_K^{\mathrm{jet}},\mathbf{p}_L^{\mathrm{jet}})|_{3\text{-jet limit}};\, K,L=\mathrm{I,II,III},$$

$$\cos\Theta_{KL}=1-\frac{2}{x_Kx_L}(x_K+x_L-1),\; x_K=\left.\frac{2E_K^{\mathrm{jet}}}{\sqrt{q^2}}\right|_{3\text{-jet limit}},\; x_{\mathrm{I}}+x_{\mathrm{II}}+x_{\mathrm{III}}=2, \tag{3.15}$$

and $D(x_3,\cos\Theta_{13},\cos\Theta_{34})$ is singular only at $x_3=0$ as can easily be deduced. Noticing that the phase space (3.7) can be written

$$dx_3(x_3x_4)^{1-\lambda}(x_3+x_4)d\Theta_{34}\sin^{1-\lambda}\Theta_{34}(x_3\cos\Theta_{34}+x_4)(x_3^2+2x_3x_4\cos\Theta_{34}+x_4^2)^{\frac{\lambda-3}{2}}d\phi$$
$$=dx_3\left[x_3\left(1-\frac{x_3}{x_{\mathrm{III}}}\right)\right]^{1-\lambda}d\Theta_{34}\sin^{1-\lambda}\Theta_{34}d\phi+O(x_3(1-\cos\Theta_{34})),$$
$$x_{\mathrm{III}}=x_3+x_4, \tag{3.16}$$

and similarly for the other configurations, we obtain the following contribution to the (nonvanishing part of the) 3-jet cross section, suppressing factors,

$$H_{\mathrm{I\,III}}=2\pi\left\{\left(\int_0^\varepsilon dx_3\int_0^\pi d\Theta_{34}+\int_\varepsilon^{x_{\mathrm{III}}}dx_3\int_0^\delta d\Theta_{34}\right)\left(\left[x_3\left(1-\frac{x_3}{x_{\mathrm{III}}}\right)\sin\Theta_{34}\right]^{1-\lambda}\frac{1}{x_3^2(1-\cos\Theta_{34})}\right)+(4\leftrightarrow 1,\mathrm{III}\leftrightarrow\mathrm{I})\right\}$$
$$+\int_0^\varepsilon dx_3\int_0^\pi d\Theta_{34}\int_0^{2\pi}d\phi\left[x_3\left(1-\frac{x_3}{x_{\mathrm{III}}}\right)\sin\Theta_{34}\right]^{1-\lambda}D(x_3,\cos\Theta_{13},\cos\Theta_{34}) \tag{3.17}$$

etc. For the last integral it is the same if we integrate over $\Theta_{34}$, $\phi$ or $\Theta_{13}$, $\phi$. The result of (3.17) is given in Appendix D.

The single pole terms (3.8) have only a collinear singularity so that the relevant integral is

$$M_{\mathrm{III}}=2\pi\int_0^{x_{\mathrm{III}}}dx_3\int_0^\delta d\Theta_{34}\left[x_3\left(1-\frac{x_3}{x_{\mathrm{III}}}\right)\sin\Theta_{34}\right]^{1-\lambda}\frac{1}{x_{\mathrm{III}}x_3(1-\cos\Theta_{34})}, \tag{3.18}$$

etc. The result of (3.18), as well as that of some related integrals, is given in Appendix D.

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* E.g., $s_{\mathrm{I\,II}}|_{\cos\Theta_{13}=1}=q^2(1-x_4)$

** E.g., $x_{\mathrm{I}}|_{\cos\Theta_{13}=1}=x_1+x_3$

After these preliminaries we shall now compile the singular contributions of the 4-parton transition matrix elements. According to the various groups of diagrams we write

$$(-g^{\mu\nu}\tilde{H}_{\mu\nu})=(4\pi\alpha_s)^2 C_F\left\{C_F A^{(1)}+(2C_F-N_c)A^{(2)}+N_c A^{(3)}+\frac{N_f}{2}A^{(4)}\right\}, \tag{3.19}$$

where $A^{(1)}$, $A^{(2)}$, and $A^{(3)}$, corresponding for $q\bar{q}gg$ final states (2.2), receive contributions from the following diagrams, using the notation of [5],

$$\begin{aligned}
A^{(1)}&: A(1,1),\ A(2,1),\ A(2,2),\ A(3,1),\ A(3,2),\ A(3,3),\ A(4,4),\ A(5,4),\ A(5,5),\ A(6,4),\ A(6,5),\ A(6,6)\\
A^{(2)}&: A(4,1),\ A(4,2),\ A(4,3),\ A(5,1),\ A(5,2),\ A(5,3),\ A(6,1),\ A(6,2),\ A(6,3)\\
A^{(3)}&: A(7,1),\ A(7,2),\ A(7,3),\ A(7,4),\ A(7,5),\ A(7,6),\ A(7,7),\ A(8,1),\ A(8,2),\ A(8,3),\ A(8,4),\ A(8,5),\\
&\quad A(8,6),\ A(8,7),\ A(8,8)
\end{aligned} \tag{3.20}$$

while $A^{(4)}$ proceeds from $q\bar{q}q\bar{q}$ final states (2.3). Since the double poles (3.9) lead (after integration) to singularities $1/\lambda^2$ and the single poles (3.10) to $1/\lambda$, we have to compute the residues of the double-pole terms (3.9) to order $\lambda^2$ and those of the single poles (3.10) to order $\lambda$ which goes begond our earlier calculations [5,6]. The result can be written

$$A^{(i)}=\left(1-\frac{\lambda}{2}\right)(A_0^{(i)}-\lambda A_1^{(i)}), \tag{3.21}$$

where the factor $\left(1-\frac{\lambda}{2}\right)$ reflects the $n$ dimensional contraction of $g^{\mu\nu}H_{\mu\nu}$ as before and the $A_0^{(i)}$ and $A_1^{(i)}$ are given in Appendix E.

The (real) order $\alpha_s^2$ contributions to the 3-jet cross section can now be read off from the formulae in Appendices D and E. Let us define

$$B^G(x_\mathrm{I},x_\mathrm{II})=\frac{1-x_\mathrm{III}}{(1-x_\mathrm{I})(1-x_\mathrm{II})} \tag{3.22}$$

and (see Appendix D)

$$\begin{aligned}
R_{KL}^{V,S}&=B^{V,S}(x_\mathrm{I},x_\mathrm{II})H_{KL},\\
S_K^{V,S}&=B^{V,S}(x_\mathrm{I},x_\mathrm{II})M_K,\\
T_K^{V,S,G}&=B^{V,S,G}(x_\mathrm{I},x_\mathrm{II})N_K
\end{aligned} \tag{3.23}$$

($K,L=$ I, II, III). We then find

$$\frac{d^2\sigma^{(4)}(\varepsilon,\delta)}{dx_\mathrm{I}dx_\mathrm{II}}=\sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}[(x_\mathrm{I}+x_\mathrm{II}-1)(1-x_\mathrm{I})(1-x_\mathrm{II})]^{-\lambda/2}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\frac{2^\lambda}{4\pi\Gamma\left(1-\frac{\lambda}{2}\right)}\left(\frac{\alpha_s}{2\pi}\right)^2 C_F T, \tag{3.24}$$

where*

$$\begin{aligned}
T=&\,C_F[2(S_\mathrm{I}^{V-\lambda/2S}+S_\mathrm{II}^{V-\lambda/2S})+4(B^V+2B^S)]+(2C_F-N_c)2R_\mathrm{I\,II}^{V-\lambda/2S}+N_c[2(R_\mathrm{I\,III}^{V-\lambda/2S}+R_\mathrm{II\,III}^{V-\lambda/2S})\\
&+2T_\mathrm{III}^{V-\lambda/2S}+2(2B^G-\tfrac{11}{3}B^S)]+\frac{N_f}{2}[4(S_\mathrm{III}^{V-\lambda/2S}-T_\mathrm{III}^{V-\lambda/2S})+\tfrac{4}{3}B^S],\ U^{V-\lambda/2S}\equiv U^V-\frac{\lambda}{2}U^S,\ U=R,S,T.
\end{aligned} \tag{3.25}$$

Explicitly, this leads to the pole terms

$$\begin{aligned}
T=&\,4\pi 2^{-\lambda}C_F B^{V-\lambda/2S}(x_\mathrm{I},x_\mathrm{II})\left\{\frac{4}{\lambda^2}(2C_F+N_c)\right.\\
&\left.+\frac{1}{\lambda}(6C_F-\tfrac{2}{3}N_f+\tfrac{11}{3}N_c-2(2C_F-N_c)\ln y_\mathrm{I\,II}-2N_c\ln(y_\mathrm{I\,III}y_\mathrm{II\,III})\right\}\\
&(y_\mathrm{I\,II}=1-x_\mathrm{III},\ y_\mathrm{I\,III}=1-x_\mathrm{II},\ y_\mathrm{II\,III}=1-x_\mathrm{I}).
\end{aligned} \tag{3.26}$$

* Note that (3.8) is symmetric under $x_1=x_\mathrm{I}$, $x_2=x_\mathrm{II}\to x_K$, $x_L$, $K\neq L$ ($K,L=$ I, II, III)

The full 3-jet cross section will be the sum of (2.11) and (3.24) with $x_1 \equiv x_{\rm I}$, $x_2 \equiv x_{\rm II}$ and $x_3 \equiv x_{\rm III}$. Utilizing

$$\frac{\left(\Gamma\left(1-\frac{\lambda}{2}\right)\right)^2}{\Gamma(1-\lambda)} = 1 + O(\lambda^2), \tag{3.27}$$

it is readily seen that the pole terms cancel as they should. More generally, we can state that the differential cross section in any variable which is linear in the parton energies in the 3-jet limit is finite.

In the following we shall redefine the variables $x_K (K = {\rm I, II, III})$ to be twice the energy going into the $K$-th cone divided by the total energy going into all three jet cones. This definition of jet energies does not alter (3.24) as it effects only the order $\varepsilon, \delta^2$ contributions but is directly relevant to experiment. Note that $0 \leqq x_K \leqq 1$ and $x_{\rm I} + x_{\rm II} + x_{\rm III} = 2$ as before.

## IV. Final Results

After this the 3-jet cross section, which now is

$$d\sigma^{(3)}(\varepsilon, \delta) = d\sigma^{(3)} + d\sigma^{(4)}(\varepsilon, \delta), \tag{4.1}$$

can be written down explicitly. For ease of writing we shall label the jets by their arabic numbers (i.e., I, II, III→1, 2, 3) throughout this section. We obtain in the $\overline{\rm MS}$ scheme

$$\begin{aligned}\frac{d^2\sigma^{(3)}(\varepsilon,\delta)}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s(q^2)}{2\pi} C_F \Bigg[ B^V(x_1,x_2) \Bigg\{ 1 - \frac{\alpha_s(q^2)}{\pi} \Bigg[ \Bigg( C_F \ln\frac{\varepsilon}{x_1} + C_F \ln\frac{\varepsilon}{x_2} \\ + N_c \ln\frac{\varepsilon}{x_3} + \frac{18 C_F + 11 N_c}{12} - \frac{N_f}{6} \Bigg) \ln\left(\frac{1-\cos\delta}{2}\right) - \ln\varepsilon \left( N_c \ln\left(\frac{1-\cos\Theta_{13}}{2}\right) + N_c \ln\left(\frac{1-\cos\Theta_{23}}{2}\right) \right. \\ \left. + (2C_F - N_c) \ln\left(\frac{1-\cos\Theta_{12}}{2}\right)\right) - \left( C_F \frac{\varepsilon}{x_1} + C_F \frac{\varepsilon}{x_2} + N_c \frac{\varepsilon}{x_3} \right) \ln\left(\frac{1-\cos\delta}{2}\right) \\ + \left(\tfrac{11}{6} N_c - \tfrac{1}{3} N_f\right) \ln x_3 + R(x_1,x_2) \Bigg]\Bigg\} + \frac{\alpha_s(q^2)}{\pi} F(x_1,x_2) \Bigg] + O(\varepsilon) + O(\delta^2),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}R(x_1,x_2) = \frac{N_c}{2} \mathcal{L}_2\left(\frac{1-\cos\Theta_{13}}{2}\right) + \frac{N_c}{2} \mathcal{L}_2\left(\frac{1-\cos\Theta_{23}}{2}\right) + \tfrac{1}{2}(2C_F - N_c) \mathcal{L}_2\left(\frac{1-\cos\Theta_{12}}{2}\right) \\ - C_F \mathcal{L}_2(1-x_1) - C_F \mathcal{L}_2(1-x_2) - (2C_F - N_c) \mathcal{L}_2(1-x_3) + \tfrac{1}{4}(2C_F - N_c) \left[ \ln^2(1-x_3) - \ln^2\left(\frac{1-\cos\Theta_{12}}{2}\right) \right] \\ + \frac{N_c}{4} \left[ \ln^2(1-x_1) - \ln^2\left(\frac{1-\cos\Theta_{23}}{2}\right) + \ln^2(1-x_2) - \ln^2\left(\frac{1-\cos\Theta_{13}}{2}\right) \right] \\ - N_c \ln^2 x_3 - (2C_F - N_c) \ln x_3 \ln(1-x_3) \\ + \tfrac{1}{2}(2C_F - N_c) \ln(1-x_3)(\ln(1-x_1) + \ln(1-x_2)) + \frac{N_c}{2} \ln(1-x_1)\ln(1-x_2) \\ - C_F(\ln^2 x_1 + \ln^2 x_2 + \ln x_1 \ln(1-x_1) + \ln x_2 \ln(1-x_2)) \\ - \tfrac{3}{2} C_F (\ln x_1 + \ln x_2) + \tfrac{1}{2} C_F \pi^2 + \tfrac{13}{18} N_f - \tfrac{5}{2} C_F - \tfrac{137}{36} N_c,\end{aligned} \tag{4.3}$$

$$F(x_1,x_2)=B^S(x_1,x_2)\Big(\tfrac{1}{4}(2C_F-N_c)\Big[\ln x_1\ln(1-x_1)+\mathcal{L}_2(1-x_1)$$
$$+\ln x_2\ln(1-x_2)+\mathcal{L}_2(1-x_2)-\ln(1-x_3)\ln(1-x_1)-\ln(1-x_3)\ln(1-x_2)$$
$$+2\mathcal{L}_2(1-x_3)+2\ln x_3\ln(1-x_3)-\frac{\pi^2}{3}\Big]+C_F-N_c+\frac{N_f}{12}\Big)$$
$$+B^3(x_1,x_2)\Big(\tfrac{1}{2}(2C_F-N_c)\Big[\ln(1-x_3)\ln(1-x_1)-\ln x_1\ln(1-x_1)-\mathcal{L}_2(1-x_1)$$
$$+\ln(1-x_3)\ln(1-x_2)-\ln x_2\ln(1-x_2)-\mathcal{L}_2(1-x_2)-2\ln x_3\ln(1-x_3)-2\mathcal{L}_2(1-x_3)+\frac{\pi^2}{3}\Big]-\tfrac{1}{2}(C_F-N_c)\Big)$$
$$+(2C_F-N_c)\Big[\ln x_3\ln(1-x_3)+\mathcal{L}_2(1-x_3)+\tfrac{1}{2}\ln x_1\ln(1-x_1)+\tfrac{1}{2}\ln x_2\ln(1-x_2)$$
$$+\tfrac{1}{2}\mathcal{L}_2(1-x_1)+\tfrac{1}{2}\mathcal{L}_2(1-x_2)-\tfrac{1}{2}\ln(1-x_3)\ln(1-x_1)-\tfrac{1}{2}\ln(1-x_3)\ln(1-x_2)$$
$$-\frac{\pi^2}{6}\Big]+2C_F[\ln(1-x_1)+\ln(1-x_2)]+2C_F-\tfrac{5}{6}N_c$$
$$+\frac{x_1^2-x_2^2}{(1-x_1)(1-x_2)}\tfrac{1}{4}(2C_F-N_c)[\ln x_1\ln(1-x_1)+\mathcal{L}_2(1-x_1)-\ln(1-x_1)\ln(1-x_3)$$
$$-\ln x_2\ln(1-x_2)-\mathcal{L}_2(1-x_2)+\ln(1-x_2)\ln(1-x_3)]$$
$$-\tfrac{1}{2}C_F\left(\frac{1-x_1}{x_2}+\frac{1-x_2}{x_1}\right)-\tfrac{1}{2}C_F(1-x_1)(1-x_2)\left(\frac{\ln(1-x_1)}{x_1^2}+\frac{\ln(1-x_2)}{x_2^2}\right)$$
$$-(2C_F(1-x_2)-\tfrac{1}{2}(2C_F-N_c)(1-x_1))\frac{\ln(1-x_1)}{x_1}$$
$$-(2C_F(1-x_1)-\tfrac{1}{2}(2C_F-N_c)(1-x_2))\frac{\ln(1-x_2)}{x_2}+(2C_F-N_c)\frac{1-x_3}{x_3}\left(1+\frac{1+x_3}{x_3}\ln(1-x_3)\right) \tag{4.4}$$

and

$$B^3(x_1,x_2)=\frac{x_3}{(1-x_1)(1-x_2)}. \tag{4.5}$$

The cross section (4.2) was given already in the letter version of this work [15] (where, however, we have set $C_F=\frac{4}{3}$ and $N_c=3$ explicitly* right from the beginning).

In view of the contrasting statements in [17, 18], namely that the order $\alpha_s^2$ corrections to the 3-jet cross section are large, the central question is now how (4.2)–(4.4) can be checked (independently).

Let us first consider the logarithmic terms $\ln\varepsilon$, $\ln\frac{(1-\cos\delta)}{2}$, i.e., the leading contribution for $\varepsilon,\delta\to 0$. This has been calculated independently by Smilga and Vysotsky [28] who found**

$$\frac{d^2\sigma^{(3)}(\varepsilon,\delta)}{dx_1dx_2}=\sigma_0\frac{\alpha_s(q^2)}{2\pi}C_FB^V(x_1,x_2)\Big\{1-\frac{\alpha_s(q^2)}{\pi}\Big[\Big(C_F\ln\frac{\varepsilon}{x_1}+C_F\ln\frac{\varepsilon}{x_2}$$
$$+N_c\ln\frac{\varepsilon}{x_3}+\frac{18C_F+11N_c}{12}-\frac{N_f}{6}\Big)\ln\frac{\delta^2}{4}-\ln\varepsilon\Big(N_c\ln\Big(\frac{1-\cos\Theta_{13}}{2}\Big)$$
$$+N_c\ln\Big(\frac{1-\cos\Theta_{23}}{2}\Big)+(2C_F-N_c)\ln\Big(\frac{1-\cos\Theta_{12}}{2}\Big)\Big)\Big]\Big\}. \tag{4.6}$$

As can readily be seen, (4.2) agrees with*** (4.6).

* In (5) of [15] the last term, $-\frac{44}{3}$, should read $-\frac{59}{4}$ and in (7) of [15] the last term in the first curly bracket, $-\frac{11}{12}$, should read $-\frac{5}{3}$

** Note that [28] contains a misprint (A.V. Smilga, private communication) and that our $\delta$ is the full opening angle

*** Note that $\ln\left(\frac{1-\cos\delta}{2}\right)=\ln\frac{\delta^2}{4}+O(\delta^2)$

In order to check the remaining finite terms we proceed in two steps. First we set $N_c = N_f = 0$ in (4.2)–(4.4) (retaining $\sigma_0 \neq 0$ though). This makes the diagrams involving the triple-gluon coupling vanish ("abelian limit") and switches off the $q\bar{q}q\bar{q}$ production channel so that only the quark and antiquark jets are "composite". From the Ward identity we then derive the relation

$$\int_0^{1-\Delta} dx_1 \int_{1-x_1}^{1-\Delta} dx_2 \frac{d^2\sigma^{(3)}(\varepsilon,\delta)}{dx_1 dx_2} \underset{\Delta\to 0}{=} \ln^2\Delta \frac{\alpha_s(q^2)}{\pi} C_F \sigma^{(2)}(\varepsilon,\delta), \tag{4.7}$$

$\sigma^{(2)}(\varepsilon,\delta)$ being the well known ($O(\alpha_s)$) Sterman-Weinberg cross section for two jets [14, 29]:

$$\sigma^{(2)}(\varepsilon,\delta) = \sigma_0 \left\{1 - \frac{\alpha_s(q^2)}{\pi} C_F \left[(2\ln\varepsilon + \tfrac{3}{2})\ln\frac{\delta^2}{4} - 2\varepsilon\ln\frac{\delta^2}{4} + \frac{\pi^2}{3} - \tfrac{5}{2}\right]\right\}, \tag{4.8}$$

which must be satisfied by our 3-jet cross section.

We notice that only the most singular part (for $x_3 \to 0$) of the 3-jet cross section (4.2)–(4.4) contributes a factor $\ln^2\Delta$. This is*

$$\frac{d^2\sigma^{(3)}(\varepsilon,\delta)}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s(q^2)}{2\pi} C_F B^V(x_1,x_2) \left\{1 - \frac{\alpha_s(q^2)}{\pi} C_F \left[\left(\ln\frac{\varepsilon}{x_1} + \ln\frac{\varepsilon}{x_2} + \tfrac{3}{2}\right)\ln\left(\frac{1-\cos\delta}{2}\right) - \left(\frac{\varepsilon}{x_1} + \frac{\varepsilon}{x_2}\right)\ln\left(\frac{1-\cos\delta}{2}\right) + \mathcal{L}_2\left(\frac{1-\cos\Theta_{12}}{2}\right) - 2\mathcal{L}_2(1-x_3) + \frac{\pi^2}{2} - \tfrac{5}{2}\right]\right\}. \tag{4.9}$$

If we integrate (4.9) over $x_1$, $x_2$ we obtain

$$\int_0^{1-\Delta} dx_1 \int_{1-x_1}^{1-\Delta} dx_2 \frac{d^2\sigma^{(3)}(\varepsilon,\delta)}{dx_1 dx_2} \underset{\Delta\to 0}{=} \ln^2\Delta \frac{\alpha_s(q^2)}{\pi} C_F \sigma_0 \left\{1 - \frac{\alpha_s(q^2)}{\pi} C_F \left[(2\ln\varepsilon + \tfrac{3}{2})\ln\left(\frac{1-\cos\delta}{2}\right) - 2\varepsilon\ln\left(\frac{1-\cos\delta}{2}\right) - \mathcal{L}_2(1) + \frac{\pi^2}{2} - \tfrac{5}{2}\right]\right\} \tag{4.10}$$

which agrees with the right-hand side of (4.7), realizing that $\mathcal{L}_2(1) = \frac{\pi^2}{6}$. The untested less singular terms (giving rise to a single logarithm in $\Delta$ or a constant) are numerically small over the whole kinematical range.

In order to check the terms proportional to $N_c$ and $N_f$ we shall set $C_F^2 = 0$ (but retain $C_F N_c$, $C_F N_f \neq 0$) and consider the limit $x_1, x_3 \to 1$ with $(1-x_3)/(1-x_1) \to 1$. This (again) corresponds to a 2-jet configuration with one jet being a "composite" gluon jet and the other consisting of a single quark. In this limit we expect our 3-jet cross section to be proportional to the Sterman-Weinberg type cross section for gluon jets as derived in [30, 31].

The cross section in [30] cannot be directly compared to (4.2)–(4.4) since it is source dependent. In order to divide out the source dependence we have to calculate the equivalent of what is called the "total cross section" in [30]. We have done this and obtain, after a straightforward evaluation of our formulae,

$$\text{“}\sigma_{\text{tot}}\text{”} \underset{x_1,x_3\to 1}{=} \frac{1}{1-x_1} \frac{\alpha_s(q^2)}{2\pi} C_F \sigma_0 \left\{1 - \frac{\alpha_s(q^2)}{\pi}\left[-\left(\frac{\pi^2}{12} + \tfrac{19}{8}\right)N_c + \tfrac{7}{12}N_f\right]\right\} \tag{4.11}$$

which leads to (noticing that $\cos\Theta_{13} \to -1$, $\cos\Theta_{23} \to 0$, $\cos\Theta_{12} \to 0$)

$$\frac{1}{\text{“}\sigma_{\text{tot}}\text{”}} \frac{d^2\sigma^{(3)}(\varepsilon,\delta)}{dx_1 dx_2} \underset{x_1,x_3\to 1}{=} 1 - \frac{\alpha_s(q^2)}{\pi}\left[(N_c\ln\varepsilon + \tfrac{11}{12}N_c - \tfrac{1}{6}N_f)\ln\left(\frac{1-\cos\delta}{2}\right) + \left(\frac{\pi^2}{6} - \tfrac{49}{72}\right)N_c + \frac{N_c}{18}\right]. \tag{4.12}$$

This agrees with (3.7) in [30] taking into account that the square bracket in (4.12) has to be multiplied by a factor two for comparison as we have only one gluon jet here.

* Note that $\lim_{x_1,x_2\to 1} \cos\Theta_{12} = -1$

The "total cross section" (4.11) is partly hidden in the $O(\varepsilon)$, $O(\delta^2)$ contributions not written down explicitly. To be more explicit, we can proceed in an alternative way. From (3.24) and (3.26) we obtain

$$\left[\sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}[(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}\frac{\alpha_s(q^2)}{2\pi}C_F B^V(x_1,x_2)\right]^{-1}\frac{d^2\sigma^{(4)}(\varepsilon,\delta)}{dx_1 dx_2}\underset{x_1,\bar{x}_3\to 1}{=}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}$$
$$\cdot\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}\frac{\alpha_s(q^2)}{2\pi}\left\{N_c\left[\left(\frac{4}{\lambda^2}+\frac{11}{3\lambda}\right)-(2\ln\varepsilon+\tfrac{11}{6})\ln\left(\frac{1-\cos\delta}{2}\right)-\tfrac{5}{6}\pi^2+\tfrac{67}{9}\right]+\frac{N_f}{2}\left[-\frac{4}{3\lambda}+\tfrac{2}{3}\ln\left(\frac{1-\cos\delta}{2}\right)-\tfrac{26}{9}\right]\right\}, \tag{4.13}$$

where terms proportional to $B^S(x_1,x_2)$, which come from the $n$ dimensional traces over lines $x_1, x_2$ in conjunction with the poles in $\lambda$, have been discarded* in order to meet the assumptions of [30]. The right-hand side of (4.13) then is found to agree with (3.5) in [30] apart from the last term in the square bracket multiplying $N_f$, $-\frac{26}{9}$, which in [30] reads $-\frac{23}{9}$. This difference can be traced back to (2.1) in [30] which has also been criticized in [31] (but may be source dependent). Our result corresponds to setting $D=0$ in the second square bracket in (2.1). Note that in the ratio (4.12) (which is unambiguous) the difference drops out.

As before the untested terms are numerically very small with the exception of the term $\left(\frac{11}{6}N_c-\frac{N_f}{3}\right)\ln x_3$ [cf. (4.2)]. The reason is that they (also) are less singular on the 2-jet limit $x_3\to 0$.

We like to emphasize at this point that there are no *large* $\pi^2$ terms as found in [17]. The $\pi^2$ terms in (4.3) have the opposite sign and are nearly cancelled by the constant terms as in the Sterman-Weinberg cross section (4.8) (and the total cross section).

The coupling constant in (4.2) is evaluated at $q^2$ (see Sect. II) which provides the natural scale if the angles between the three jets are all large. In the 2-jet limit the 4-momentum squared, which determines the strength of the strong coupling constant, becomes, however, much smaller. This mismatch of scales results in the large logarithmic term**

$$-\sigma^{(2)}\frac{\alpha_s(q^2)}{2\pi}C_F B^V(x_1,x_2)\frac{\alpha_s(q^2)}{\pi}(\tfrac{11}{6}N_c-\tfrac{1}{3}N_f)\ln x_3 \tag{4.14}$$

in (4.1). As can readily be seen, (4.14) can be absorbed into $\alpha_s$ by the change of scales

$$q^2\to x_3^2 q^2, \alpha_s(q^2)\to\alpha_s(x_3^2 q^2). \tag{4.15}$$

This means, if the renormalization is performed at $x_3^3 q^2$ rather than $q^2$, the logarithmic term (4.14) is exactly cancelled. This mechanism is expected to repeat itself in higher orders so that better convergence of the perturbation series is obtained if the 3-jet cross section is expanded in powers of $\alpha_s(x_3^2 q^2)$.

We shall now present numerical results for the cross section

$$\frac{1}{\sigma}\frac{d\sigma^{(3)}(\varepsilon,\delta)}{dx_{\max}}, x_{\max}=\max\{x_1,x_2,x_3\}, \tag{4.16}$$

where [10]

$$\sigma=\sigma_0\left[1+\frac{\alpha_s(q^2)}{\pi}+(1.986-0.115N_f)\left(\frac{\alpha_s(q^2)}{\pi}\right)^2\right]. \tag{4.17}$$

This derives from (4.2) by integrating over one of the jet energies.

We expect (4.16) to be insensitive to hadronization and higher order corrections and, therewith, directly relevant to experiment as it factorizes [16] – in contrast to the $C$ and ("bare") thrust distributions [17, 18] which do not. Furthermore, (4.16) has the advantage that $x_{\max}$ extends from 2/3 to 1 independent of the number of partons/particles going into it.

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* It can easily be deduced from (3.21) and (E.1)–(E.8) that the terms proportional to $B^S(x_1,x_2)$ vanish if the traces are taken 4-dimensional

** Note that the remaining logarithms in (4.2) are finite in the limit $x_3\to 0$ or cancel against $-N_c\ln x_3\ln\left(\frac{1-\cos\delta}{2}\right)$

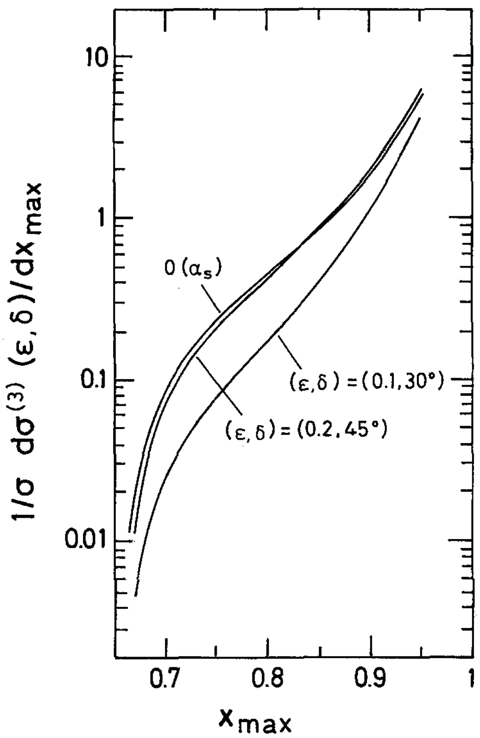


**Fig. 6.** Three-jet cross section for $(\varepsilon, \delta)=(0.2, 45°)$ and $(\varepsilon, \delta)=(0.1, 30°)$ together with the Born cross section as a function of $x_{max}$ for $\alpha_s=0.17$

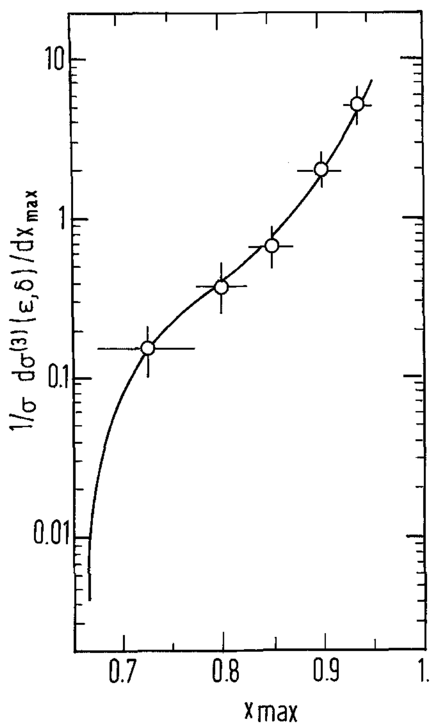


**Fig. 7.** Three-jet cross section fitted to the PLUTO data [33] with $(\varepsilon, \delta)=(0.2, 45°)$

In Fig. 6 we have shown the cross section (4.16) for $(\varepsilon, \delta)=(0.2, 45°)$ and $(\varepsilon, \delta)=(0.1, 30°)$ together with the Born distribution. We find that the order $\alpha_s^2$ corrections to the 3-jet cross section are small for not too small $\varepsilon$, $\delta$, as one would have expected from the total cross section [10], and negative. The latter one would also have expected from factorization plus the phase space cut off. In contrast to this, large positive corrections have been found for the $C$ and ("bare") thrust distributions [17, 18].

Experimentally, the Sterman-Weinberg 3-jet cross section can be determined via a cluster analysis [32]. This has been done by the PLUTO group [33]. We have fitted (4.16) to the PLUTO data ($Q=30$ GeV, $\varepsilon=0.2$, $\delta=45°$) and obtain

$$\alpha_s=0.17. \tag{4.18}$$

This corresponds to $\Lambda_{\overline{MS}}=0.24$ GeV using the 1-loop formula (2.17), while the 2-loop approximation [34]

$$\alpha_s(q^2)=\frac{2\pi}{b_0 \ln\frac{q^2}{\Lambda^2}+\frac{b_1}{b_0}\ln\left(\ln\frac{q^2}{\Lambda^2}\right)},$$

$$b_0=\tfrac{11}{6}N_c-\tfrac{1}{3}N_f,\ b_1=\tfrac{17}{6}N_c^2-\tfrac{5}{6}N_cN_f-\tfrac{1}{2}C_FN_f \tag{4.19}$$

gives $\Lambda_{\overline{MS}}=0.48$ GeV. The data and the fitted cross section curve are shown in Fig. 7. One should note, that the experimental data points have larger error bars and so has $\Lambda_{\overline{MS}}$ extracted from them.

## V. Conclusions

We have shown that the order $\alpha_s^2$ corrections to the 3-jet cross section, defined to be the cross section for events which have all but a fraction $\varepsilon/2$ of the total energy distributed within three separated cones of opening angle $\delta$, are small beyond all doubts.

## Appendix A

The renormalized current matrix element is given by

$$g_0\langle 0|T(\psi\bar{\psi}Aj)|0\rangle = \frac{1}{Z_2\sqrt{Z_3}} g_0\langle 0|T(\psi\bar{\psi}Aj)|0\rangle_0, \tag{A.1}$$

where the unrenormalized matrix element $\langle 0|T(\psi\psi Aj)|0\rangle_0$ is calculated from the diagrams in Figs. 1 and 2. The $Z$ factors, renormalizing the quark and the gluon wave functions, respectively, are (UV = ultraviolet, IR = infrared):

$$\begin{aligned} Z_2 &= 1 - \frac{g_0^2}{16\pi^2} C_F\left(\frac{2}{\lambda_{\mathrm{UV}}} - \frac{2}{\lambda}\right), \\ Z_3 &= 1 + \frac{g_0^2}{16\pi^2}\left(\tfrac{5}{3}N_c - \tfrac{2}{3}N_f\right)\left(\frac{2}{\lambda_{\mathrm{UV}}} - \frac{2}{\lambda}\right), \\ \lambda &= \lambda_{\mathrm{IR}} \end{aligned} \tag{A.2}$$

and the unrenormalized coupling constant $g_0$ reads in terms of the renormalized coupling constant $g$:

$$g_0 = g + \delta g_0 = \frac{Z_1}{Z_2\sqrt{Z_3}} g, \tag{A.3}$$

where

$$Z_1 = 1 - \frac{g_0^2}{16\pi^2}(N_c + C_F)\left(\frac{2}{\lambda_{\mathrm{UV}}} - \frac{2}{\lambda}\right). \tag{A.4}$$

In the minimal subtraction scheme (MS) only the ultraviolet poles in $g_0$ are subtracted which gives

$$\frac{1}{g}\delta g_0 = \frac{g^2}{16\pi^2}\left(\tfrac{2}{3}N_f - \tfrac{11}{3}N_c\right)\frac{1}{\lambda_{\mathrm{UV}}}. \tag{A.5}$$

This leads us to the counterterm (2.10).

## Appendix B

Here we shall list the scalar 2-, 3-, and 4-point function integrals which we encounter in calculating the loop corrections to (2.1).

(*i*) 2-point function

$$J_2(p) = \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2(l-p)^2} = \begin{cases} \dfrac{i}{8\pi^2}\left(\dfrac{4\pi}{-p^2}\right)^{\lambda/2} \dfrac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}\left(\dfrac{1}{\lambda_{\mathrm{UV}}} + 1\right), & \text{off shell: } p^2 \neq 0. \\ \dfrac{i}{8\pi^2}\left(\dfrac{1}{\lambda_{\mathrm{UV}}} - \dfrac{1}{\lambda}\right), & \text{on shell: } p^2 = 0. \end{cases} \tag{B.1}$$

(*ii*) 3-point function

$$\begin{aligned} J_3(p_1, p_2, p_3) &= \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2(l+p_1)^2(l-p_2-p_3)^2} \\ &= \frac{i}{8\pi^2}\frac{1}{s_{12}+s_{13}}\left(\frac{4\pi}{-p^2}\right)^{\lambda/2} \frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}\left(\ln y_{23}\frac{1}{\lambda} - \tfrac{1}{4}\ln y_{23}\right), \quad p_3 \neq 0, \end{aligned} \tag{B.2}$$

$$J_3(p_1,p_2,0)=\frac{i}{8\pi^2}\frac{1}{s_{12}}\left(\frac{4\pi}{-p^2}\right)^{\lambda/2}\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)}\left(\frac{2}{\lambda^2}-\frac{1}{\lambda}\ln y_{12}+\tfrac{1}{4}\ln^2 y_{12}\right), \tag{B.3}$$

where $p=p_1+p_2+p_3$ and $p_i^2=0$.
(*iii*) 4-point function (see also [23])

$$J_4(p_1,p_2,p_3)=\int\frac{d^n l}{(2\pi)^n}\frac{1}{l^2(l+p_1)^2(l-p_2)^2(l+p_1+p_3)^2}$$
$$=\frac{i}{8\pi^2}\frac{1}{s_{12}s_{13}}\left(\frac{4\pi}{-p^2}\right)^{\lambda/2}\frac{\Gamma\left(1+\frac{\lambda}{2}\right)\left(\Gamma\left(1-\frac{\lambda}{2}\right)\right)^2}{\Gamma(1-\lambda)}\left[\frac{4}{\lambda^2}-\frac{2}{\lambda}(\ln y_{12}+\ln y_{13})\right.$$
$$+\tfrac{1}{2}\ln^2 y_{12}+\tfrac{1}{2}\ln^2 y_{13}+\ln y_{12}\ln y_{13}-\ln y_{12}\ln x_3-\ln y_{13}\ln x_2$$
$$\left.-\mathscr{L}_2(y_{12})-\mathscr{L}_2(y_{13})+\frac{\pi^2}{6}\right]. \tag{B.4}$$

## Appendix C

The aim of this appendix is to derive (3.5). From $\sum_{i=1}^{4}\mathbf{p}_i=\mathbf{q}=0$ follows

$$x_1^2+2x_1x_2\cos\Theta_{12}+x_2^2=x_3^2+2x_3x_4\cos\Theta_{34}+x_4^2. \tag{C.1}$$

This leads to

$$[\tfrac{1}{2}x_1x_2\sin\Theta_{12}]^2=(x_1+x_2-1)(1-x_1)(1-x_2)+\frac{x_1x_2x_3x_4}{4}(1-\cos\Theta_{34})\left[2-\frac{4}{x_1x_2}(x_1+x_2-1)-\frac{x_3x_4}{x_1x_2}(1-\cos\Theta_{34})\right] \tag{C.2}$$

and

$$[\tfrac{1}{2}x_1x_2\sin\Theta_{12}]^{-\lambda}=[(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}\left\{1-\frac{\lambda}{2}\ln(1+y_{34}C)\right\}$$
$$C=x_1x_2\left[1-\frac{2}{x_1x_2}(x_1+x_2-1)-\frac{y_{34}}{x_1x_2}\right][(x_1+x_2-1)(1-x_1)(1-x_2)]^{-2} \tag{C.3}$$

which finally proves (3.5).

## Appendix D

In this appendix we give the results of the different integrals encountered in Sect. III.

We start with (3.17). First we shall perform integration of $D(x_3,\cos\Theta_{13},\cos\Theta_{34})$ over the azimuthal angle $\phi$. The result is

$$\int_0^{2\pi}d\phi D(x_3,\cos\Theta_{13},\cos\Theta_{34})=-\frac{4\pi}{x_3^2(1-\cos\Theta_{34})}\left[\Theta(\cos\Theta_{\mathrm{1III}}-\cos\Theta_{34})\right.$$
$$\left.+(\Theta(\cos\Theta_{\mathrm{1III}}-\cos\Theta_{34})-\tfrac{1}{2})\frac{|\cos\Theta_{\mathrm{1III}}-\cos\Theta_{34}|-|\cos\Theta_{\mathrm{1III}}-\cos\Theta_{\mathrm{3III}}|}{|\cos\Theta_{\mathrm{1III}}-\cos\Theta_{\mathrm{3III}}|}\right]. \tag{D.1}$$

The second term in the square brackets of (D.1) is not singular. As we will calculate only the nonvanishing part (for $\varepsilon,\delta\to 0$) of the 3-jet cross section analytically, we can omit this term from our further presentation. We then obtain for

$H_{\mathrm{I\,III}}$:

$$H_{\mathrm{I\,III}}=2\pi\,2^{-\lambda}\left\{\frac{\left(\Gamma\left(1-\frac{\lambda}{2}\right)\right)^2}{\Gamma(1-\lambda)}\frac{2}{\lambda}\left[\frac{\varepsilon^{-\lambda}}{\lambda}+\frac{\varepsilon^{1-\lambda}}{x_{\mathrm{III}}(1-\lambda)}+\lambda\left(\mathscr{L}_2\left(\frac{\varepsilon}{x_{\mathrm{III}}}\right)\right.\right.\right.$$
$$\left.\left.+\left(\frac{\varepsilon}{x_{\mathrm{III}}}-1\right)\ln\left(1-\frac{\varepsilon}{x_{\mathrm{III}}}\right)-\frac{\varepsilon}{x_{\mathrm{III}}}\right)\right]+2^{\lambda/2}\frac{2}{\lambda}(1-\cos\delta)^{-\lambda/2}\left[\frac{x_{\mathrm{III}}^{-\lambda}-\varepsilon^{-\lambda}}{\lambda}\right.$$
$$+\frac{x_{\mathrm{III}}^{1-\lambda}-\varepsilon^{1-\lambda}}{x_{\mathrm{III}}(1-\lambda)}+\lambda\left(\mathscr{L}_2\left(\frac{\varepsilon}{x_{\mathrm{III}}}\right)-\mathscr{L}_2(1)+\left(\frac{\varepsilon}{x_{\mathrm{III}}}-1\right)\ln\left(1-\frac{\varepsilon}{x_{\mathrm{III}}}\right)\right.$$
$$\left.\left.\left.+1-\frac{\varepsilon}{x_{\mathrm{III}}}\right)\right]\right\}+2\pi\,2^{-\lambda}\{x_{\mathrm{III}}\leftrightarrow x_{\mathrm{I}}\}$$
$$+2\pi\,2^{-\lambda}\left\{2^{\lambda/2}\frac{\varepsilon^{-\lambda}}{\lambda}\left[4\frac{(1-\cos\Theta_{\mathrm{I\,III}})^{-\lambda/2}-2^{-\lambda/2}}{\lambda}+\lambda\left(\mathscr{L}_2(1)-\mathscr{L}_2\left(\frac{1-\cos\Theta_{\mathrm{I\,III}}}{2}\right)\right)\right]\right\} \tag{D.2}$$
$$=2\pi\,2^{-\lambda}\left\{\frac{4}{\lambda^2}-\frac{2}{\lambda}[\ln(1-x_{\mathrm{II}})-2]-2\ln\varepsilon\left[\ln\left(\frac{1-\cos\delta}{2}\right)\right.\right.$$
$$\left.-\ln\left(\frac{1-\cos\Theta_{\mathrm{I\,III}}}{2}\right)\right]+\ln\left(\frac{1-\cos\delta}{2}\right)(\ln x_{\mathrm{I}}+\ln x_{\mathrm{III}}-2)$$
$$+\left(\frac{\varepsilon}{x_{\mathrm{I}}}+\frac{\varepsilon}{x_{\mathrm{III}}}\right)\ln\left(\frac{1-\cos\delta}{2}\right)-\mathscr{L}_2\left(\frac{1-\cos\Theta_{\mathrm{I\,III}}}{2}\right)+\tfrac{1}{2}\ln^2\left(\frac{1-\cos\Theta_{\mathrm{I\,III}}}{2}\right)$$
$$\left.+\ln x_{\mathrm{I}}(\ln x_{\mathrm{I}}-2)+\ln x_{\mathrm{III}}(\ln x_{\mathrm{III}}-2)-\tfrac{2}{3}\pi^2+8\right\}+O(n-4)+O(\varepsilon)+O(\delta^2), \tag{D.3}$$

where we have utilized $\mathscr{L}_2(1)=\frac{\pi^2}{6}$. The other integrals $H_{\mathrm{I\,II}}$ and $H_{\mathrm{II\,III}}$ follow from (D.3) by interchanging the jet labels.

The integral over the single terms (3.18) gives

$$M_{\mathrm{III}}=2\pi\,2^{-\lambda}\left\{2^{\lambda/2}\frac{2}{\lambda}(1-\cos\delta)^{-\lambda/2}\left[-x_{\mathrm{III}}^{-\lambda}\left(\frac{1}{1-\lambda}-\frac{1}{2-\lambda}\right)-\frac{\lambda}{4}\right]\right\}$$
$$=2\pi\,2^{-\lambda}\left[-\frac{1}{\lambda}+\tfrac{1}{2}\ln\left(\frac{1-\cos\delta}{2}\right)+\ln x_{\mathrm{III}}-2\right]+O(n-4)+O(\delta^2) \tag{D.4}$$

and similarly for $M_{\mathrm{I}}$, $M_{\mathrm{II}}$. Later on we shall encounter two more integrals of this sort:

$$\bar{M}_{\mathrm{III}}=2\pi\int_0^{x_{\mathrm{III}}}dx_3\int_0^{\delta}d\Theta_{34}\left[x_3\left(1-\frac{x_3}{x_{\mathrm{III}}}\right)\sin\Theta_{34}\right]^{1-\lambda}\frac{1}{x_{\mathrm{III}}x_4(1-\cos\Theta_{34})} \tag{D.5}$$

and

$$N_{\mathrm{III}}=2\pi\int_0^{x_{\mathrm{III}}}dx_3\int_0^{\delta}d\Theta_{34}\left[x_3\left(1-\frac{x_3}{x_{\mathrm{III}}}\right)\sin\Theta_{34}\right]^{1-\lambda}\frac{1}{x_{\mathrm{III}}^2(1-\cos\Theta_{34})}. \tag{D.6}$$

As can readily be checked $\bar{M}_{\mathrm{III}}=M_{\mathrm{III}}$, while for (D.6) we obtain

$$N_{\mathrm{III}}=2\pi\,2^{-\lambda}\left\{2^{\lambda/2}\frac{2}{\lambda}(1-\cos\delta)^{-\lambda/2}\left[-x_{\mathrm{III}}^{-\lambda}\left(\frac{1}{2-\lambda}-\frac{1}{3-\lambda}\right)-\tfrac{5}{36}\lambda\right]\right\}$$
$$=2\pi 2^{-\lambda}\left[-\frac{2}{3\lambda}+\tfrac{1}{6}\ln\left(\frac{1-\cos\delta}{2}\right)+\tfrac{1}{3}\ln x_{\mathrm{III}}-\tfrac{5}{9}\right] \tag{D.7}$$

and analogously for $N_{\mathrm{I}}$, $N_{\mathrm{II}}$.

## Appendix E

Here we shall list the singular matrix elements $A_0^{(i)}$ and $A_1^{(i)}$, $i=1, \ldots, 4$ of (3.21):

$$A_0^{(1)}=\frac{16}{q^2}\frac{1}{x_3(1-\cos\Theta_{13})}\left[\frac{(x_1+x_3)^2+x_2^2}{(1-x_1-x_3)(1-x_2)}\frac{x_3}{x_1(x_1+x_3)}+2\frac{(x_1+x_2+x_3-1)^2}{(1-x_1)(1-x_1-x_3)(1-x_2)(x_1+x_3)}\right.$$
$$\left.+2\frac{x_1+x_3}{(1-x_1)(1-x_1-x_3)}\right]+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4), \tag{E.1}$$

$$A_1^{(1)}=\frac{16}{q^2}\frac{1}{x_3(1-\cos\Theta_{13})}\left[\left(\frac{(2-x_1-x_2-x_3)^2}{(1-x_1-x_3)(1-x_2)}-2\right)\frac{x_3}{x_1(x_1+x_3)}+\frac{x_2x_3}{(1-x_1-x_3)(1-x_2)x_1}+\frac{2-x_1-x_2-x_3}{(1-x_1)(1-x_1-x_3)}\right]$$
$$+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4), \tag{E.2}$$

$$A_0^{(2)}=\frac{16}{q^2}\left\{\frac{x_{\mathrm{I}}^2+x_{\mathrm{II}}^2}{(1-x_{\mathrm{I}})(1-x_{\mathrm{II}})}\left[\frac{1}{x_3^2(1-\cos\Theta_{13})}+\frac{1}{x_3^2(1-\cos\Theta_{23})}\right.\right.$$
$$\left.+D(x_3,\cos\Theta_{13},\cos\Theta_{23})+(3\leftrightarrow 4)\right]-\left[\left(\frac{(x_1+x_2+x_3-1)^2}{(1-x_1)(1-x_1-x_3)(1-x_2)(x_1+x_3)}\right.\right.$$
$$\left.\left.\left.+\frac{x_1+x_3}{(1-x_1)(1-x_1-x_3)}\right)\frac{1}{x_3(1-\cos\Theta_{13})}+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4)\right]\right\}, \tag{E.3}$$

$$A_1^{(2)}=\frac{8}{q^2}\frac{1}{x_3(1-\cos\Theta_{13})}\left[\frac{(2-x_1-x_2-x_3)^2}{(1-x_1-x_3)(1-x_2)x_3}-\frac{2-x_1-x_2-x_3}{(1-x_1)(1-x_1-x_3)}\right.$$
$$\left.+\frac{2-x_1-x_2-x_3}{(1-x_2)x_1}\right]+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4), \tag{E.4}$$

$$A_0^{(3)}=\frac{16}{q^2}\left\{\frac{x_{\mathrm{I}}^2+x_{\mathrm{II}}^2}{(1-x_{\mathrm{I}})(1-x_{\mathrm{II}})}\left[\frac{1}{x_3^2(1-\cos\Theta_{13})}+\frac{1}{x_3^2(1-\cos\Theta_{34})}\right.\right.$$
$$\left.+D(x_3,\cos\Theta_{13},\cos\Theta_{34})\right]-\left[\frac{(x_1+x_2+x_3-1)^2}{(1-x_1)(1-x_1-x_3)(1-x_2)(x_1+x_3)}\right.$$
$$\left.\left.+\frac{x_1+x_3}{(1-x_1)(1-x_1-x_3)}\right]\frac{1}{x_3(1-\cos\Theta_{13})}+\frac{x_{\mathrm{I}}^2+x_{\mathrm{III}}^2}{(1-x_{\mathrm{I}})(1-x_{\mathrm{III}})}\frac{x_3}{2(2-x_1-x_2)^2}\frac{1}{x_3(1-\cos\Theta_{34})}\right\}$$
$$+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,3\leftrightarrow 4), \tag{E.5}$$

$$A_1^{(3)}=\frac{8}{q^2}\left\{\left[\frac{(2-x_1-x_2-x_3)^2}{(1-x_1-x_3)(1-x_2)x_3}-\frac{2-x_1-x_2-x_3}{(1-x_1)(1-x_1-x_3)}\right]\frac{1}{x_3(1-\cos\Theta_{13})}\right.$$
$$\left.+\left[\frac{(2-x_1-x_2)^2}{x_3^2}-\frac{2-x_1-x_2}{x_3}+\frac{x_1+x_2-1}{(2-x_1-x_2)^2}+1\right]\frac{1}{(1-x_1)(1-x_2)(1-\cos\Theta_{34})}\right.$$
$$+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4), \tag{E.6}$$

$$A_0^{(4)}=\frac{16}{q^2}\frac{1}{x_3(1-\cos\Theta_{34})}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}\frac{x_3^2+(2-x_1-x_2-x_3)^2}{(2-x_1-x_2)^2(2-x_1-x_2-x_3)}$$
$$+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4), \tag{E.7}$$

$$A_1^{(4)}=\frac{8}{q^2}\frac{1}{x_3(1-\cos\Theta_{34})}\frac{(2-x_1-x_2)^2}{(1-x_1)(1-x_2)(2-x_1-x_2-x_3)}$$
$$+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\ 3\leftrightarrow 4). \tag{E.8}$$

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