A DISORDER PARAMETER THAT TESTS FOR CONFINEMENT IN GAUGE THEORIES WITH QUARK FIELDS*

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We propose to use a suitably defined vortex free energy as a disorder parameter in gauge field theories with matter fields. It is supposed to distinguish between the confinement phase, massless phase(s) and Higgs phase where they exist. The matter fields may transform according to an arbitrary representation of the gauge group. We compute the vortex free energy by series expansion for a Z_2 Higgs model and for SU(2) lattice models with quark or Higgs fields in the fundamental representation at strong coupling (confinement phase), and for the Z_2 Higgs model in the range of validity of low-temperature expansions (Higgs phase). The results are in agreement with the expected behavior.

1. Introduction

In this paper we investigate a (non-local) disorder parameter for gauge theories with matter fields such as, for instance, quantum chromodynamics in euclidean formulation. It is supposed to distinguish between confinement phase, massless phase(s), and Higgs phase (in the sense of 't Hooft [1]) where they exist**. As is well known, the Wilson loop cannot be used for this purpose when the matter fields transform according to a faithful representation of the gauge group G. We propose to use the vortex free energy instead. It was introduced for pure Yang–Mills theories by 't Hooft [1] and by Mack and Petkova [2]. It was computed by the Monte Carlo method in [3, 4] and can also be defined for theories with matter fields, e.g. quark fields, as was pointed out in [5]. It can be defined both in the continuum and on the lattice, and for matter fields that transform according to arbitrary representations of the gauge group.

One considers a system on a finite lattice Λ whose boundary $\partial\Lambda$ is not simply connected. *Free* boundary conditions are imposed for the matter fields (i.e. matter fields $\psi(x)$ with $x \in \partial\Lambda$ are integrated over independently). A vortex free energy is the change in free energy in Λ when the boundary conditions (b.c.) for the gauge

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** Cautionary remark: Since the disorder parameter is non-local, one cannot be sure that a change in its asymptotic behavior is always associated with a phase transition in the sense that the free energy is non-analytic.
We dedicate this paper to the memory of our student and friend Tizian Maren, deceased 22 August 1981.

field are twisted:

$$\nu_\gamma(\Lambda, \text{b.c.}) = -\tfrac{1}{2}\ln\left[Z(\Lambda, \text{twisted b.c.})/Z(\Lambda, \text{untwisted b.c.})\right]. \qquad (1.1)$$

γ labels the possible twists. The twist is introduced by the action of a singular gauge transformation on the boundary values $U|_{\partial\Lambda}$ of the gauge field. Free boundary conditions for the matter fields are used, rather than periodic ones, because the action of the singular gauge transformation on matter fields which transform according to a faithful representation of the simply connected gauge group G could not be defined. A vortex free energy can be defined for any gauge group G with non-trivial center Γ. (When G has trivial center the problem of confinement does not pose itself.)

Λ may be a parallelopiped in $D \geq 2$ dimensions [1], or a vortex container [2], and the untwisted boundary conditions may be periodic b.c. (p.b.c.), or boundary conditions $U(b) = 1$ for $b \in \partial\Lambda$, etc. To be definite, we will consider lattice gauge theories with gauge group G = SU(2) or Z_2 whose center $\Gamma = Z(2)$, and we take for Λ a parallelopiped of sides $d_1 \times d_2 \times d_3 \times \cdots \times d_D$ with $d_3, \ldots, d_D \gg d_1, d_2$. We impose periodic boundary conditions on the gauge fields in directions 1 and 2, and free b.c. in the other directions. There is then only one way of introducing a twist. If the path C is the intersection of a plane $x_3, \ldots, x_D = \text{const.}$ with $\partial\Lambda$, then the singular gauge transformation which twists the boundary conditions changes $U(C)$ into $-U(C)$. Explicit definitions for the 3-dimensional case are given in fig. 1.

The expected asymptotic behavior of the vortex free energy in the limit d_1, $d_2 \to \infty$ is given in table 1. In the next sections we will compute the vortex free energy for several models, for ranges of parameters where expansion methods (high- or low-temperature expansions) are applicable. The results are in agreement with the list in table 1. The Z_2 Higgs model possesses a Higgs phase, which can be studied with the help of low-temperature expansions, besides its confining high-temperature phase.
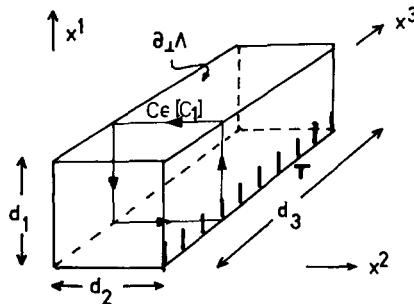

Fig. 1. Twisted boundary conditions on a 3-dimensional box. The twist is introduced by changing $U(b)$ into $U_\gamma(b)$; $U_\gamma(b) = U(b)\gamma$ for $b \in T$, and $= U(b)$ for $b \in \partial\Lambda - T$. $\gamma = -1$ for G = SU(2) or Z_2.

TABLE 1

Expected asymptotic behavior of the vortex free energy

	Without matter fields, or with matter fields that transform trivially under Γ	With matter fields that transform faithfully under Γ (e.g. quark fields)
Confinement phase	$c_1 e^{-\alpha d_1 d_2}$ α = string tension	$c_1' e^{-2m(d_1 + d_2)}$ m = "quark mass"
Massless phase(s)	$c_2 (d_1 d_2)^{-1}$	$c_2' (d_1 d_2)^{-1}$ (Coulomb phase) or similar power law
Higgs phase	$c_3 \neq 0$	$c_3' \neq 0$

The entries give the expected asymptotic behavior of $\lim_{d_3 \cdots d_D \to \infty} (d_3 \cdots d_D)^{-1} \nu$ when $d_1, d_2 \to \infty$.

The massless phases are not well understood in the presence of matter fields. Some may be characterized further by the presence of photons and Coulomb forces (example: QED), and others by massless quarks and spontaneously broken chiral symmetry.

For gauge theories *without* matter fields, the vortex free energy has the advantage over the Wilson loop expectation value [6] that no static quarks are introduced to probe the system, and their self-energy (which produces a perimeter-law behaved factor in the Wilson loop expectation value) therefore does not enter. Münster has verified [7] that the following asymptotic relation holds for a 4-dimensional pure SU(2) lattice Yang–Mills theory in the high-temperature regime, to all orders of the high-temperature expansion, in the same limit as in table 1:

$$(d_3 d_4)^{-1} \nu(\Lambda, \text{p.b.c.}) = a^{-2} e^{-\alpha d_1 d_2} , \qquad (1.2)$$

$$\alpha = \text{string tension} , \qquad a = \text{lattice spacing} .$$

A similar relation was predicted to hold outside the high-temperature regime [5]. This was later verified by Monte Carlo computation [4]. There is no perimeter-law behaved factor in either case, in contrast with the Wilson loop. In the presence of matter fields which transform non-trivially under Γ, the area law is lost, but a perimeter law decay persists in the high-temperature regime (confinement phase). This is in contrast with the Higgs phase in which ν does not decay to zero when d_1, d_2 increase to infinity.

The notion of a Higgs phase as used here and in the work of 't Hooft [1] is much more restrictive and should not be confused with what is customarily called a "Higgs mechanism". Several years ago, one of us argued that mass generation in the confining phase of a pure Yang–Mills theory could be understood as a dynamical Higgs mechanism (which leaves the center of the gauge group "unbroken") [8, 9]. This view was supported by later work of Fradkin and Shenker [10]. The Higgs
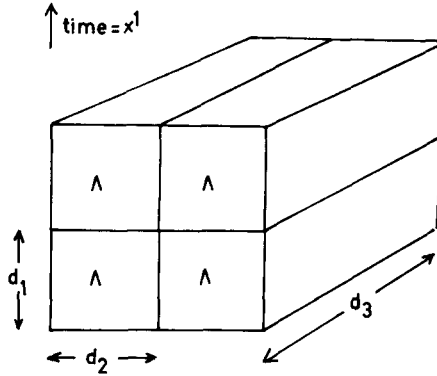
Fig. 2. How to avoid free boundary conditions for the purpose of Monte Carlo computations (drawing
for 3-dimensions, and G = SU(2), n = 2). For explanation see text.

phase, and the confining/screening phase of Fradkin and Shenker are two distinct phases which correspond to different ranges of coupling constants in the Z_2 Higgs model – see sect. 5 (fig. 4).

For an SU(2) model with two Higgs triplets a Higgs phase is known to exist [11, 21], whereas for an SU(2) model with a Higgs doublet there is no indication for its existence [12]. The essential difference between these two models is that the Higgs fields transform non-trivially under the center of the gauge group only in the second model. However, it is conceivable that in sufficiently complicated models with Fermi fields in a fundamental representation of G a Higgs phase could exist, because composite Higgs scalars may form that transform trivially under the center of the gauge group. It would be interesting to study what happens if these Higgs scalars carry flavor (compare refs. [8, 13]).

Finally, we comment on the boundary conditions. Free boundary conditions for the matter fields are a nuisance if one wants to do Monte Carlo computations, because a 4-dimensional lattice of practical size for such computations is mostly boundary.

One way to circumvent this problem is as follows. Take $n_1 n_2$ lattices Λ of size $d_1 \times d_2 \times d_3 \times d_4$ and pack them together to form a lattice Λ_{tot} of size $n_1 d_1 \times n_2 d_2 \times d_3 \times d_4$ as indicated in fig. 2. n_i are chosen as integer multiples of N for an SU(N) or Z_N gauge theory – for instance $n_1 = n_2 = 2$ if G = SU(2). One imposes either periodic or twisted boundary conditions for the gauge fields on each of the lattices Λ, and periodic boundary conditions for the matter fields on Λ_{tot}. (For Fermi field,; it is more natural to take antiperiodic b.c. in the time direction instead of periodic ones [14].)

Another possibility is to use the "boundary conditions" of Lang and Nicolai [15].

2. The models

We will consider two models with gauge group $G = SU(2)$, one with quarks (fermions) and the other with a Higgs doublet in the fundamental representation of $SU(2)$. In addition we study the gauge-invariant Ising model. This is a Higgs model with gauge group Z_2. We set the lattice spacing a equal to one.

The (random) variables of the models are string bit variables $U(b) \in G$ attached to links b of the lattice, and matter fields $\psi(x)$ attached to the sites x. $U(b) \to U(b)^{-1}$ under reversal of the direction of the link b. The action will be a sum of gauge and matter part as usual

$$L(U, \psi) = L_G(U) + L_M(U, \psi). \tag{2.1}$$

Let $\partial_\perp \Lambda$ be the part of the boundary of Λ which bounds Λ in the transverse directions 1 and 2 ($\partial_\perp \Lambda = \partial \Lambda -$ end plates $x_3 = 0$, d_3 in fig. 1). It is convenient to consider partition functions for fixed boundary values $U \equiv U|_{\partial \Lambda}$ of the gauge fields first:

$$Z(\Lambda, U) = \int \mathscr{D} U \mathscr{D} \psi\, e^{L(U, \psi)}, \tag{2.2}$$

with

$$\mathscr{D} U = \prod_{b \in \Lambda - \partial_\perp \Lambda} dU(b). \tag{2.3}$$

$dU(\ )$ is the normalized Haar measure on the group G. $\mathscr{D} \psi$ will be of the following form:

$$\mathscr{D} \psi = \prod_{x \in \Lambda} d\rho_x(\psi). \tag{2.4}$$

The partition functions for twisted and untwisted boundary conditions (as described in sect. 1) are obtained by integrating also over the boundary values U subject to the constraints imposed by the boundary conditions:

$$Z(\Lambda, \text{twisted b.c.}) = \int_{U \text{ periodic}} \prod_{b \in \partial_\perp \Lambda/2} dU(b) Z(\Lambda, U_\gamma), \tag{2.5}$$

$$Z(\Lambda, \text{untwisted b.c.}) = \int_{U \text{ periodic}} \prod_{b \in \partial_\perp \Lambda/2} dU(b) Z(\Lambda, U). \tag{2.6}$$

U_γ is obtained from U by the singular gauge transformation of sect. 1 (see fig. 1). It changes the parallel transporter (see below) $U(C)$ into $-U(C)$ for every path C in $\partial_\perp \Lambda$ that winds once (mod 2) around Λ.

For a path C that consists of a sequence $b_1 \cdots b_n$ of oriented links one defines the parallel transporter

$$U(C) = U(b_n) \cdots U(b_1). $$

In particular, the boundary $\dot{p} \equiv \partial p$ of a plaquette is a sum of four oriented links $b_1 \cdots b_4$, and $U(\dot{p}) = U(b_4) \cdots U(b_1)$. The gauge part of the action will be chosen in the standard Wilson–Wegner form

$$L_G(U) = \sum_p \mathscr{L}_p(U) , \qquad \text{with } \mathscr{L}_p(U) = \begin{cases} \tfrac{1}{2}\beta \text{ tr } U(\dot{p}) + \text{const.} , & \text{for } G = SU(2) , \\ \beta U(\dot{p}) + \text{const.} , & \text{for } G = Z_2 = \{\pm 1\} . \end{cases}$$

$$(2.1')$$

The sum includes only one of the two plaquettes on opposite sides of $\partial_\perp \Lambda$. The matter part of the action will be of the form

$$L_M(U, \psi) = \sum_b \mathscr{L}_b(U, \psi) . \tag{2.1''}$$

Summation is over all links b in Λ, and for $b = (x, y)$ $\mathscr{L}_b$ will only depend on the string bit variables attached to the link b, and on the values of the matter fields at x and y.

Model 1 (SU(2) quark model). We introduce one species of anticommuting quark field ψ. $\psi(x)$ is a Dirac spinor and transforms according to the 2-dimensional fundamental representation of SU(2). As is well known, in the euclidean formulation of such a Fermi field, ψ and $\bar{\psi}$ are treated as independent. We use the Wilson form for the matter part of the action [6]:

$$\mathscr{L}_b(U, \psi) = -K\{\bar{\psi}(x)(1 + \gamma_\mu)U(b)\psi(y) + \bar{\psi}(y)(1 - \gamma_\mu)U(b)^*\psi(x)\} + \text{const.}$$

$$(2.7)$$

Finally,

$$d\rho_x(\psi) = e^{\bar{\psi}(x)\psi(x)} \, d\psi(x) \, d\bar{\psi}(x) .$$

Integration over anticommuting variables ψ, $\bar{\psi}$ is defined in the standard way [16]. Color and spinor indices have been suppressed.

Model 2 (SU(2) Higgs model). This model involves a doublet of complex scalar fields

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} , \qquad |\varphi_1(x)|^2 + |\varphi_2(x)|^2 = 1 . \tag{2.8}$$

The matter part of the action takes its most convenient form when expressed in terms of the SU(2) matrices

$$\psi(x) = \begin{pmatrix} \varphi_1(x) & -\bar{\varphi}_2(x) \\ \varphi_2(x) & \bar{\varphi}_1(x) \end{pmatrix} . \tag{2.9}$$

In this notation, the matter part of the action is given by eq. (2.1'') with

$$\mathscr{L}_b(U, \psi) = \tfrac{1}{4}K\{\text{tr } \psi(x)^* U(b)\psi(y) + \text{c.c.}\} + \text{const.} , \qquad \text{for } b = (x, y) , \quad (2.10)$$

$$\mathscr{D}\psi = \prod_{x \in \Lambda} d\psi(x) , \qquad d\psi = \text{normalized Haar measure on SU(2)} .$$

Under a gauge transformation $V_1(x)$

$$\psi(x) \rightarrow V_1(x)\psi(x) , \qquad (2.11a)$$

$$U(x, y) \rightarrow V_1(x)U(x, y)V_1(y)^{-1} . \qquad (2.11b)$$

From eq. (2.10) one sees [17 (especially sect. 2)] that the model has *in addition* another global SU(2) symmetry. We call it isospin. Isospin rotations act according to

$$\psi(x) \rightarrow \psi(x)V_2^{-1} , \qquad (2.12a)$$

$$U(x, y) \rightarrow U(x, y) , \qquad (2.12b)$$

with $V_2 \in SU(2)$ independent of x. If $V_2 = -1$ then it commutes with all SU(2) matrices. Its action on ψ, U therefore agrees with the action of a global gauge transformation $V_1(x) \equiv -1$ in the center Z_2 of the gauge group. The combined symmetry is therefore

$$(SU(2)_{local} \times SU(2)_{global})/Z_{2\,global}. \qquad (2.13)$$

In the confining phase, all physical states are supposed to have integral isospin (see the discussion in ref. [17]). We are not able to prove that the perimeter law decay of ν as given in table 1 is a sufficient condition for that.

Model 3 (Gauge-invariant Ising model). This is one of Wegner's generalizations of the Ising model [18]. Its string bit variables $U(b)$ and matter fields both take values ± 1. Otherwise the model is the same as model 2. Integration with the normalized Haar measure on $G = Z_2$ amounts to summation:

$$\int dU(\cdots) = \tfrac{1}{2} \sum_{U=\pm 1} (\cdots) , \qquad \int d\psi(\cdots) = \tfrac{1}{2} \sum_{\psi=\pm 1} (\cdots) . \qquad (2.14)$$

The matter part of the action is given by eq. (2.1″) with

$$\mathcal{L}_b(U, \psi) = K\psi(x)U(b)\psi(y) + \text{const.}, \qquad \text{for } b = (x, y) , \qquad (2.15)$$

and

$$d\rho_x(\psi) = d\psi(x) . \qquad (2.16)$$

3. Expansions for the free energy of polymer systems

Expansions of classical statistical mechanics can be obtained in two steps. In the first step one transforms the system which one wishes to consider into a polymer system. In the second step one writes down expansions for the free energy, etc. of this polymer system. In this section we will be concerned with the second step.

We regard a lattice Λ as a cell complex made of 0-cells (sites x), 1-cells (links b), 2-cells (plaquettes p), 3-cells (cubes c), ... To define a polymer system (in the

sense of Gruber and Kunz [19]) certain subsets of the cell complex Λ are declared to be polymers. To every polymer P an activity $\Phi(P)$ is assigned. We use the symbols $+$, $\sum$ for union of disjoint sets. For any $X \subseteq \Lambda$ a partition function is defined by

$$Z_X = 1 + \sum_{\Sigma P \subseteq X} \prod_P \Phi(P). \tag{3.1}$$

Summation is over non-empty collections of disjoint polymers that are contained in X. It is required that $Z(X) > 0$ for all $X \subseteq \Lambda$, but the activities $\Phi(P)$ need not necessarily be positive.

For the purpose of this investigation it is expedient to allow for some generalization. In the case of the low-temperature expansions, two polymers P_1 and P_2 will be called disjoint only if they have vanishing intersection *and* $P_1 \cup P_2$ is not a polymer. In the case of the high-temperature expansions, polymers P with different activities are admitted which can occupy the same set of cells in Λ (=its "support"). Two such polymers will be called disjoint if no cell of Λ is occupied by both. The results of Gruber and Kunz all generalize to these cases [19].

In our applications, the activities $\Phi(P)$ can depend on the boundary conditions when P meets the boundary of Λ. We write $\Phi(P, U)$ or $\Phi(P, \sim b.c.)$ when we want to indicate this dependence. Our polymer systems will be defined in such a way that the partition function is reproduced, viz. $Z(\Lambda, U) = Z_\Lambda$ as defined in eq. (3.1), etc. If the activities $\Phi(P)$ are small enough, then the free energy of the polymer system admits a convergent cluster expansion. A linked cluster $Q = (P_1^{n_N}, \ldots, P_N^{n_N})$ is a non-empty collection of not necessarily distinct polymers. It may contain polymer P_i $n_i \geq 1$ times. It has to be connected in the following sense. Consider the abstract graph $\mathscr{V}_Q$ whose vertices are the polymers in Q, and whose links are the pairs of polymers in Q which are not disjoint. $\mathscr{V}_Q$ has to be connected. The cluster expansion reads (for $X = \Lambda$)

$$\ln Z_\Lambda = \sum_Q a(Q) \prod_{P \in Q} \Phi(P). \tag{3.2}$$

Summation is over all linked clusters Q which consist of polymers in Λ. $a(Q)$ are combinatorial factors which can be computed if the graph $\mathscr{V}_Q$ is given. ($a(Q) = 1$ if Q consists of a single polymer, and -1 if it consists of two distinct polymers.) Sufficient conditions for the convergence of expansion (3.2) are well known [19]. (They are derived by use of Kirkwood Salsburg equations [19, 20].) For an application of this formalism to the computation of the string tension by high-temperature expansions see Münster [7].

4. High-temperature expansions

We consider first the situation where both coupling constants β and K in our models are small. Following standard procedure [20, 21], we write

$$e^{\mathscr{L}_p(U)} = 1 + f_p(U) = 1 + O(\beta), \tag{4.1}$$

$$e^{\mathscr{L}_b(U,\psi)} = 1 + g_b(U, \psi) = 1 + \mathrm{O}(K) \,. \tag{4.2}$$

f_p depends on the gauge field configuration U only through $U(\dot{\mathrm{p}})$, and similarly $g_b(U, \psi)$ depends only on $U(\mathrm{b})$ and on $\psi(x)$, $\psi(y)$ if $\mathrm{b} = (x, y)$ (and on $\bar{\psi}(x)$, $\bar{\psi}(y)$ in the quark model).

We choose the additive constant in the gauge part $\mathscr{L}_p$ of the lagrangian in such a way that the character expansion of f_p contains no constant term. It follows that

$$\int \mathrm{d}U(\mathrm{b})f_p(U) = 0 \,, \qquad \text{for } \mathrm{b} \in \partial\mathrm{p} \,. \tag{4.3}$$

Similarly, we choose the additive constant in the matter part of the lagrangian in such a way that

$$\int \mathrm{d}\rho_x(\psi)g_b(U, \psi) = 0 \,, \qquad \text{for } x \in \partial\mathrm{b} \,. \tag{4.4}$$

To see that this is possible one considers $g_b(U, V\psi)$ as a function of $V(x) \in G$ and expands in irreducible representations of G again. In the case of the quark model one also exploits the presence of the projection operators $\frac{1}{2}(1 \pm \gamma_\mu)$. The details are omitted since high-temperature expansions for theories with matter fields have been treated in the literature before [22, 15].

Consider now the partition functions defined in sect. 2 [eqs. (2.2)–(2.6)]. We wish to exhibit them as partition functions of polymer systems. To achieve this one inserts eqs. (4.1) and (4.2) into their definition and expands in products of f's and g's of the form

$$\prod_{\mathrm{b}\in\mathscr{B}} g_b \prod_{\mathrm{p}\in\mathscr{P}} f_p \,. \tag{4.5}$$

With every term in this sum we associate a graph $\mathscr{F} = \mathscr{F}(\mathscr{B}, \mathscr{P})$ whose vertices are sites and links of Λ. For the term (4.5) the set $v(\mathscr{F})$ of vertices of $\mathscr{F}$ consists of all sites $x \in \Lambda$ which are end points of links $\mathrm{b} \in \mathscr{B}$, of the links $\mathrm{b} \in \mathscr{B}$ themselves, and of the links $\mathrm{b} \in \Lambda$ which are in the boundary of a plaquette $\mathrm{p} \in \mathscr{P}$. Two vertices of the graph $\mathscr{F}$ are linked if one of the following conditions is met:

(i) Both vertices are sites of Λ and they are endpoints of the same links $\mathrm{b} \in \mathscr{B}$.

(ii) The two vertices are a link $\mathrm{b} \in \mathscr{B}$ and a site x which is an endpoint of b.

(iii) The two vertices are both links, and there exists a plaquette $\mathrm{p} \in \mathscr{P}$ which contains both of them in its boundary.

(iv) The two vertices are both links in $\partial_\perp \Lambda$ and are translations of each other in the 1- or 2-direction by d_1 or d_2.

The graphs $\mathscr{F}$ decompose into connected graphs $\mathscr{G}$ and the integrals over expression (4.5) factorize into a product of integrals of the same form, for each connected piece $\mathscr{G}$ of $\mathscr{F}$.

A *polymer* $\mathrm{P} = (\mathscr{B}, \mathscr{P})$ is given by a set $\mathscr{B}$ of links in Λ and a set $\mathscr{P}$ of plaquettes in Λ such that the graph $\mathscr{G} = \mathscr{G}(\mathscr{B}, \mathscr{P})$ that is associated with it in the manner

described above is connected. The support of P will consist of the sites and links in Λ which are contained in $v(\mathcal{G})$. Two polymers P_1 and P_2 are *disjoint* if their supports have no site or link of Λ in common. An *activity* of polymers P is defined by

$$\Phi(P, \boldsymbol{U}) = \int \prod_{x \in v(\mathcal{G})} d\rho_x(\psi) \prod_{b \in v(\mathcal{G})}' d U(b) \prod_{b \in \mathcal{B}} g_b(U, \psi) \prod_{p \in \mathcal{P}} f_p(U) . \tag{4.6}$$

The prime on the second product indicates that integrations over variables $U(b)$ attached to links $b \in \partial_\perp \Lambda$ are to be omitted. Instead, Φ depends on these variables when $v(\mathcal{G})$ contains links in $\partial_\perp \Lambda$. We write $\Phi = \Phi(P, \boldsymbol{U})$ to indicate this dependence on boundary values of the gauge field. By construction

$$Z(\Lambda, \boldsymbol{U}) = 1 + \sum_{\sum P \subseteq \Lambda} \prod_{P} \Phi(P, \boldsymbol{U}) \tag{4.7}$$

(sum over all non-empty collections of disjoint polymers). Finally, we can now integrate over boundary values $\boldsymbol{U}$ of the gauge field. Define

$$\Phi(P, \text{twisted b.c.}) = \int_{\boldsymbol{U} \text{ periodic}} \prod_{b \in \partial_\perp \Lambda/2} d \boldsymbol{U}(b) \Phi(\Lambda, \boldsymbol{U}_\gamma) , \tag{4.8a}$$

$$\Phi(P, \text{untwisted b.c.}) = \int_{\boldsymbol{U} \text{ periodic}} \prod_{b \in \partial_\perp \Lambda/2} d \boldsymbol{U}(b) \Phi(\Lambda, \boldsymbol{U}) , \tag{4.8b}$$

similarly to eqs. (2.5), (2.6). Since dependence on a particular independent integration variable $U(b)$ appears in at most one of the factors $\Phi(P, \boldsymbol{U})$ in eq. (4.8) it follows that

$$Z(\Lambda, \text{b.c.}) = 1 + \sum_{\sum P \subseteq \Lambda} \prod_{P} \Phi(P, \text{b.c.}) , \tag{4.9}$$

where b.c. stands for the twisted or untwisted boundary conditions described in sect. 1. In this way we have exhibited the $Z(\Lambda, \text{b.c.})$ as partition functions of polymer systems.

Now we can apply eq. (3.2). This gives for the semiperiodic boundary conditions b.c. described in sect. 1

$$\nu(\Lambda, \text{b.c.}) = -\tfrac{1}{2} \sum_{Q} a(Q) \left[\prod_{P \in Q} \Phi(P, \text{twisted b.c.}) - \prod_{P \in Q} \Phi(P, \text{untwisted b.c.}) \right] . \tag{4.10}$$

Summation is over linked clusters Q of polymers P as described in sect. 3. For a polymer $P = (\mathcal{B}, \mathcal{P})$, the activity

$$\Phi(P,) = O(K^{|\mathcal{B}|} \beta^{|\mathcal{P}|}) \tag{4.11}$$

by definition and eqs. (4.1) and (4.2). $|\mathcal{B}|$ = number of links in $\mathcal{B}$, etc.

We are interested in small K and β. Our problem is then to find the leading term in the expansion (4.10). For fixed Λ the answer turns out of depend on the relative size of K and β, but for large enough cross section $d_1 d_2$ of Λ (depending on the relative size of K and β) this is no longer the case as we shall see.

Evidently $\Phi(P, \text{twisted b.c.}) = \Phi(P, \text{untwisted b.c.})$ unless $\Phi(P, U) \neq \Phi(P, U_\gamma)$ for some U. But $\Phi(P, U) \neq \Phi(P, U_\gamma)$ is only possible if the support of P contains a link $b \in \partial_\perp \Lambda$ for which $U(b) \neq U_\gamma(b)$. Since $\Phi(P, U)$ is invariant under any ordinary gauge transformation of the boundary values U on $\partial_\perp \Lambda$ (this is seen by extending it to a gauge transformation in Λ which acts trivially on $\Lambda - \partial_\perp \Lambda$), $\Phi(P, U) \neq \Phi(P, U_\gamma)$ must hold for any choice of the singular gauge transformation. Let $[C_1]$ be the (homotopy) class of all closed paths on $\partial_\perp \Lambda$ which wind once around Λ (see fig. 1). The singular gauge transformation can be carried out in such a way that $U_\gamma(b) = U(b)$ except on an arbitrary set T of links which is a closed path (in 3 dimensions) or surface (in 4 dimensions) on the dual of $\partial_\perp \Lambda$ and which meets every path $C \in [C_1]$ once (mod 2) (for $G = SU(2)$ or Z_2; compare fig. 1. Any two such choices differ by an ordinary gauge transformation in Z_2. It follows that a cluster Q can only make a non-zero contribution to the expansion (4.10) if it contains a polymer P with the following properties.

(1) $\Phi(P, U)$ should not be identically zero for all boundary values U.

(2) The support of P should contain all the (unoriented) links b in some closed path $C \in [C_1]$ on $\partial_\perp \Lambda$.

Next we note that the activity $\Phi(P, \cdot)$ of a polymer $P = (\mathcal{B}, \mathcal{P})$ is zero if P has "free ends". A site x in the support of P is a free end if there is only one $b \in \mathcal{B}$ which has x as an endpoint. A link in the support of P is a free end if it is not in $\partial_\perp \Lambda$ or $\mathcal{B}$ and is contained only in the boundary of a single plaquette $p \in \mathcal{P}$. Vanishing of the activities in the presence of free ends follows from eqs. (4.3) and (4.4).

If a polymer P satisfies condition (2), its support must contain at least $|C| = 2(d_1 + d_2)$ links, $|C| = $ minimal length of a path $C \in [C_1]$ on $\partial_\perp \Lambda$. Therefore $|\mathcal{B}| + 4|\mathcal{P}| \geq |C|$ and, because of eqs. (4.1), (4.2),

$$|\Phi(P,)| \leq O([K^4 + \beta]^{(d_1 + d_2)/2}) .$$

Therefore, the leading contribution to the sum (4.10) is at most of the same order in β and K. It follows that $\nu(\Lambda, \text{b.c.})$ has an exponential decay at least as fast as indicated in the 1st row, 2nd column of table 1, if K and β are small.

It is instructive to determine the leading term more explicitly. A typical polymer which satisfies conditions (1) and (2) above is shown in fig. 3a. It consists of a decorated surface Ξ made of plaquettes $p \in \mathcal{P}$ whose intersection with $\partial_\perp \Lambda$ is a closed curve $C \in [C_1]$, with some holes in it that are bordered by links $b \in \mathcal{B}$ (compare ref. [22]). Two limiting cases are shown in figs. 3b, c. The contribution from a cluster Q that consists of a single polymer P of the form shown in figs. 3b, c can easily be evaluated [22]. In the 4-dimensional SU(2) Higgs model one obtains, for
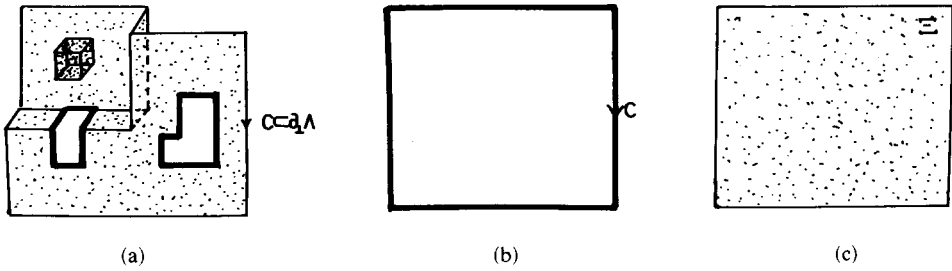
Fig. 3. (a) A typical polymer $P = (\mathscr{B}, \mathscr{P})$ that satisfies conditions (1) and (2) above, when $C \in [C_1]$. The dotted areas are composed of plaquettes $p \in \mathscr{P}$ and the heavy lines are made of links $b \in \mathscr{B}$. (b) A special case: $\mathscr{P}$ is empty, and $\mathscr{B}$ is exhausted by the links in a closed path $C \in [C_1]$ of minimal length. (c) Another special case: a minimal surface with boundary $C \in [C_1]$ that is filled with plaquettes $p \in \mathscr{P}$. $\mathscr{B}$ is empty.

instance ($a = 1$)

$$\nu(\Lambda, \text{b.c.}) = d_3 d_4 \{ e^{-2m_0(d_1 + d_2)} + e^{-\alpha_0 d_1 d_2} + \cdots \},$$

and similarly in the other two models. $\alpha_0 = -\ln(\frac{1}{4}\beta)$ is the string tension to leading order of the high-temperature expansion in the pure gauge theory and $m_0 = -\ln(\frac{1}{4}K)$. For large d_1 and d_2 the first term dominates.

5. Low-temperature expansions for the gauge-invariant Ising model

We will now consider the gauge-invariant Ising model (model 3) for large β and small K. For the sake of definiteness we shall consider the case of a 3-dimensional lattice. The 4-dimensional case can be handled in the same way.

Concerning the phase diagram of this model, rigorous results of Marra and Miracle Solé are available [23] which establish analyticity of the free energy in two regions I and II of the β, K diagram. These regions are bounded by the dashed lines in fig. 3. Monte Carlo data are also available [24]. They confirm that the two
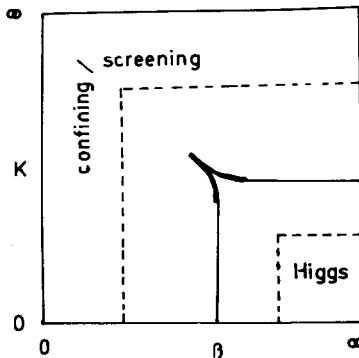

Fig. 4. Phase diagram of the gauge invariant Ising model in 3 dimensions.

regions are separated by a phase transition line. In our terminology, region II is the Higgs phase, as we shall see, whereas region I is the confining/screening phase that was also investigated by Fradkin and Shenker [10]. It is customary to speak of a Higgs mechanism when both β and K are large. We have already commented on that at the end of sect. 1. Our high-temperature result of sect. 4 covers only a small part of region I. It would be of interest to show that it remains true throughout region I.

In this section we are interested in region II. It is convenient to introduce

$$Z_{\pm}(\Lambda) = Z(\Lambda, \text{untwisted b.c.}) \pm Z(\Lambda, \text{twisted b.c.}) . \qquad (5.1)$$

In computing the vortex free energy, the limit $d_3 \to \infty$ is taken first. It turns out that in this limit the second term in (5.1) becomes very small compared to the first. Therefore, in this limit

$$Z(\Lambda, \text{twisted b.c.})/Z(\Lambda, \text{untwisted b.c.}) = \tfrac{1}{2} \ln \frac{Z_{+}(\Lambda)}{Z_{-}(\Lambda)} . \qquad (5.2)$$

We compute the expression on the right-hand side. To get the vortex free energy one must take the logarithm of the result, at the end.

To compute $\ln Z_{\pm}$ we reinterpret the model as a polymer system with activities $\Phi_{\pm}(\mathrm{P})$ in such a way that [compare eq. (3.1)]

$$Z_{\pm}(\Lambda) = 1 + \sum_{\Sigma} \prod_{\mathrm{P} \subseteq \Lambda} \Phi_{\pm}(\mathrm{P}) . \qquad (5.3)$$

Summation is over non-empty sets of disjoint polymers. We must find a suitable definition of polymer, of disjointness, and determine the activities $\Phi_{\pm}(\mathrm{P})$ in such a way that eq. (5.3) holds. We follow Marra and Miracle Solé and generalize their procedure in such a way that the dependence on boundary conditions, in which we are interested, can be determined.

Let $\mathrm{C} = \partial \Xi$ be a closed path which is boundary of a surface Ξ in Λ. Since we are now dealing with an abelian theory we have

$$U(\mathrm{C}) = \prod_{p \in \Xi} U(\dot{\mathrm{p}}) , \qquad \text{if } \mathrm{C} = \partial \Xi . \qquad (5.4)$$

Let Ξ be the intersection of any plane $x^3 = \text{const.}$ with Λ. Then

$$U(\mathrm{C}) = 1 , \qquad \text{for the untwisted b.c.} , \qquad (5.5a)$$

$$U(\mathrm{C}) = -1 , \qquad \text{for the twisted b.c.} . \qquad (5.5b)$$

as described in sect. 1.

Given a gauge field configuration U on Λ, let S consist of those plaquettes p where $U(\dot{\mathrm{p}}) = -1$. S is *coclosed* $\equiv$ closed on the dual lattice. This means that every cube c in Λ contains an even number of plaquettes $\mathrm{p} \in \mathrm{S}$ in its boundary. It follows

from eqs. (5.4) and (5.5) that the number $|S \cap \Xi|$ of plaquettes in $S \cap \Xi$ satisfies

$$|S \cap \Xi| = \begin{cases} \text{even,} & \text{for untwisted b.c.,} & (5.6a) \\ \text{odd,} & \text{for twisted b.c..} & (5.6b) \end{cases}$$

Conversely, if the *"field strengths"* $U(\dot{\mathrm{p}})$ satisfy periodic boundary conditions in directions 1 and 2 and also eq. (5.6a) or (5.6b) then $U(\dot{\mathrm{p}})$ is field strength of a gauge field $U(\mathrm{b})$ which satisfies the untwisted or the twisted b.c., respectively. Therefore, the partition functions can be written as follows (for suitable choice of the additive constants in the action)

$$Z_{\pm}(\Lambda) = \sum_{S} (\pm)^{|S \cap \Xi|} \mathrm{e}^{-2\beta|S|} \int \prod_{x} \mathrm{d}\psi(x) \int_{U(\dot{\mathrm{p}})=-1 \text{ iff } \mathrm{p} \in S} \prod \mathrm{d}U(\mathrm{b})$$

$$\times \prod_{(x,y)} \{1 + \psi(x)U(x, y)\psi(y) \tanh K\} . \tag{5.7}$$

Let Λ_{per} be obtained from Λ by identifying plaquettes in opposite sides of $\partial_{\perp}\Lambda$. The sum over S in (5.7) extends over all coclosed sets of plaquettes p in Λ_{per}. The sum over gauge fields $U(\mathrm{b})$ is restricted as indicated. Ξ is intersection of an arbitrary plane $x^3 = \mathrm{const.}$ with Λ.

The product $\prod\{ \}$ is expanded into a sum of terms that are labelled by sets $\mathscr{B}$ of links. Now the ψ summations are carried out. This annihilates all the terms with $\mathscr{B}$ not closed in Λ ($\mathscr{B}$ is closed if every site $x \in \Lambda$ has an even number if links $\mathrm{b} \in \mathscr{B}$ incident on it). Since Λ is topologically trivial, $\mathscr{B} = \partial\Sigma$ for some sum of plaquettes Σ if $\partial\mathscr{B} = 0$. One defines the winding number of $\mathscr{B}$ relative to S by

$$n(S, \mathscr{B}) = (-1)^{|S \cap \Sigma|} . \tag{5.8}$$

Because of eq. (5.4), the result of the ψ summation can be expressed as

$$Z_{\pm}(\Lambda) = \sum_{\substack{S \\ \partial^* S = 0 \text{ in } \Lambda_{\mathrm{per}}}} (\pm)^{|S \cap \Xi|} \mathrm{e}^{-2\beta|S|} \sum_{\substack{\mathscr{B} \\ \partial\mathscr{B} = 0 \text{ in } \Lambda}} (\tanh K)^{|\mathscr{B}|} n(S, \mathscr{B}) . \tag{5.9}$$

The final step is to factorize the terms appearing in these partition functions. Consider sets $\mathrm{P} = \{s_1, \ldots, s_n; c_1 \cdots c_m\}$, $n + m \geqslant 1$, where s_i are coclosed sets of plaquettes in Λ_{per}, and c_j are closed sets of links in Λ. With every such P we associate an abstract graph as follows. The vertices of the graph are the sets s_i and c_j. Two vertices s_i and s_k are linked with each other if there exists a cube c whose boundary contains a plaquette in s_i and a plaquette in s_j. Two vertices c_j and c_k are linked if there is a point x such that a link in c_j and a link in c_k are incident on it. Finally a vertex s_i is linked to a vertex c_j if the relative winding number $n(s_i, c_j) = -1$. P is called a polymer if the graph associated with it is connected. Two polymers P_1 and P_2 are *disjoint* if the graph associated with $\mathrm{P}_1 \cup \mathrm{P}_2$ is not connected (i.e. $\mathrm{P}_1 \cup \mathrm{P}_2$ is not a polymer). We write $\sum \mathrm{P}$ for the union of
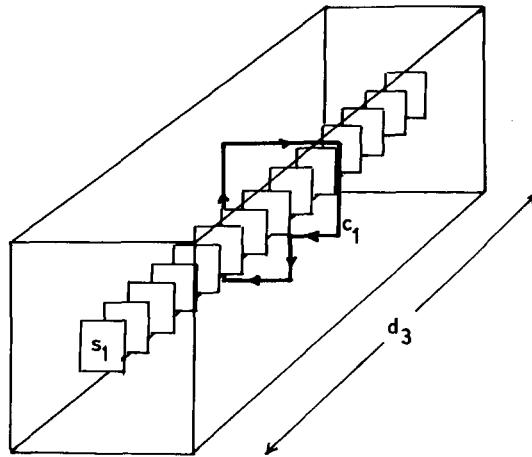
Fig. 5. A polymer (s_1, c_1) whose vortex s_1 winds once through Λ.

mutually disjoint polymers. The activities of the polymers shall be given by

$$\Phi_{\pm}(P) = (\pm)^{\Sigma |s_i \cap \Xi|} e^{-2\beta \Sigma |s_i|} (\tanh K)^{\Sigma |c_j|} \prod_{i=1}^{n} \prod_{j=1}^{m} n(s_i, c_j) \qquad (5.10)$$

for $P = (s_1 \cdots s_n; c_1 \cdots c_m)$. s_i will be called vortices and c_j will be called loops. The factor $(-)^{|s_i \cap \Xi|}$ counts how many times the vortex s_i winds through $\Lambda \pmod 2$. As an example, a polymer (s_1, c_1) whose single vortex winds once through Λ is shown in fig. 5. With these definitions, the partition functions (5.9) take the form (5.3). The first term 1 in (5.3) is the contribution from the term in (5.9) with $\mathscr{B}$ and S both empty.

Having exhibited Z_+ and Z_- as partition functions of polymer systems with activities Φ_+ and Φ_- we can now apply expansion (3.2) for $\ln Z_{\pm}$. With eq. (5.2) we obtain, in the limit of large d_3

$$Z(\Lambda, \text{twisted b.c.})/Z(\Lambda, \text{untwisted b.c.}) = \tfrac{1}{2} \ln Z_+(\Lambda) - \tfrac{1}{2} \ln Z_-(\Lambda)$$

$$= \tfrac{1}{2} \sum_Q a(Q) \left[\prod_{P \in Q} \Phi_+(P) - \prod_{P \in Q} \Phi_-(P) \right]. \qquad (5.11)$$

Summation is over linked clusters $Q = (P_1^{n_1}, \ldots, P_k^{n_k})$ of polymers. They are collections of polymers P_i such that the graph that is associated with $\cup P_i$ is connected.

$-\ln Z_+$ is the free energy of a system with semiperiodic boundary conditions for field strengths $U(\dot{p})$. Absolute convergence of its low-temperature series follows from the results of Marra and Miracle Solé [23] for low enough β^{-1} and K. The series for $\ln Z_-$ differs only in the sign of the individual terms and is therefore also

absolutely convergent. In conclusion, eq. (5.11) is a convergent expansion for low enough β^{-1} and K.

Next we will identify the leading term. The activities Φ_+ and Φ_- differ only in sign. All those terms in which $\prod \Phi_-$ and $\prod \Phi_+$ have the same sign drop out in eq. (5.11). We get opposite signs if the total winding number $(-)^{\sum_{s_i \in Q} |s_i \cap \Xi|}$ of all the vortices in all the polymers in Q is -1. Therefore, we need at least one vortex which winds through Λ in such a way that it meets every plane Ξ: $x_3 = \text{const}$. It must therefore have at least d_3 plaquettes. Every plaquette contributes a depression factor $e^{-2\beta}$. The leading term is thus given by a cluster Q which consists of a single polymer $P = (s_1;)$ which consists only of a single vortex s_1 that winds through the lattice. The situation is as shown in fig. 5 except that the loop c_1 is absent. For such Q one has $a(Q) = 1$. Thus

$$Z(\Lambda, \text{twisted b.c.})/Z(\Lambda, \text{untwisted b.c.}) = d_1 d_2 \, e^{-2d_3\beta} + \cdots . \qquad (5.12)$$

To obtain ν/d_3 we must take the logarithm and divide by $-2d_3$. The contribution of the leading term to ν/d_3 is therefore equal to β, in the limit $d_3 \to \infty$, independent of d_1 and d_2.

Finally, we will discuss correction terms. They have one "principal" vortex s_1 which winds through Λ and may differ from the minimal one shown in fig. 5 by some deformations. It may have some decorations attached to it. The whole cluster Q may consist of one or several polymers, one of them contains s_1. (Contributions with several vortices that wind through Λ become negligible when $d_3 \to \infty$.) The leading corrections come from a kink in the vortex s_1, which gives it an extra plaquette, or from the presence of a loop c_1 of four links that winds around s_1. In the second case, the vortex and the loop may each form a polymer, or they may be in one polymer. The kink or loop may be positioned anywhere on the vortex s_1. Therefore, one gets a correction to the leading term that is proportional to d_3. There are other corrections which involve 2, 3, ... such loops and kinks. They give contributions proportional d_3^2, d_3^3, ... The point is that all these contributions (and more complicated ones) sum up to an exponential. The argument is exactly the same as in Münsters computation of the string tension [7] and the details will therefore not be repeated here. (He had to consider decorations and deformations of the minimal surface with prescribed boundary = Wilson loop.) The idea is to rewrite expression (5.11) as

$$Z(\Lambda, \text{twisted b.c.})/Z(\Lambda, \text{untwisted b.c.}) = \sum_{\substack{Q \\ \text{winding no.}-1}} \prod_P \Phi_+(P) ,$$

to regard this expression as partition function of still another polymer system, and to use eq. (3.2) again to compute its logarithm. The result is that all the correction terms to ν/d_3 go to zero when β^{-1}, $K \to 0$, uniformly in d_1 and d_2. The leading correction terms give, for large d_3 and $d_3 \gg d_1, d_2 \geq 1$,

$$\nu(\Lambda, \text{b.c.}) = d_3[\beta - 2 \, e^{-2\beta} + (\tanh K)^4 + \cdots] . \qquad (5.13)$$

Comparing with table 1 we see that this is in agreement with the expected behavior for a Higgs phase.

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