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THE RENORMALIZATION OF FF

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A fairly complete study of the renormalization of $F_a^{\mu\nu}F_{\mu\nu a}$ is presented in the gauge in which it is simplest: the background field gauge. This gauge allows one to go to second-order perturbation theory without evaluating a single Feynman integral. The use of the equations of motion, the gauge invariance of a classically gauge-invariant renormalized operator and the renormalization group invariance are studied. Its two-loop anomalous dimension is given and its relation to the trace anomaly obtained.

1. Introduction

Since only colourless states have been observed in nature, it is clear that the successes of quantum gauge field theory in explaining the physical reality will be correlated to the understanding one develops of gauge-invariant operators in field theory. Unfortunately gauge-invariant operators are either composite or non-local (i.e. path dependent). Within the first approach, where one deals with composite local operators, $FF \equiv F_a^{\mu\nu}(x)F_{\mu\nu a}(x)$ is the gauge-invariant scalar operator of lowest dimension in pure QCD. Here $F_a^{\mu\nu}(x)$ is the field strength defined as $F_a^{\mu\nu}(x) = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + gf_{abc}A_b^\mu A_c^\nu$, where A_a^ν is the gluon field of colour index a and g is the coupling constant. It will appear in its normal product form as the first non-trivial operator in the short-distance expansion of the product of gauge-invariant operators. The normal product makes its vacuum expectation value irrelevant for perturbation theory, but this is not so for non-perturbative physics, where one expects its physical vacuum expectation value to be non-zero. This is on the basis of a non-perturbative approach to low-energy physics, which has achieved remarkable successes in the past years [1, 2]. However, and in order to be of direct physical relevance, FF, once renormalized, has to appear in an expression such that it is renormalization group invariant, i.e. independent of the renormalization scale μ. Its physical vacuum expectation value will then be, due to its gauge and renormalization group invariance, the scale of non-perturbative physics.

The renormalization of composite operators has been studied in the BPH scheme by Zimmermann [3] and in dimensional schemes by Collins [4] and Breitenlohner and Maison [5]. The renormalization of gauge-invariant operators like FF has special problems related to mixing. In usual covariant gauges FF mixes not only

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with other gauge-invariant operators of the same dimension but also with non-gauge-invariant operators. This is because continuum quantization requires the breaking of the gauge symmetry. Nevertheless, the problem is understood in spite of its complexity [6, 7]. There is, however, as we see it, a more elegant approach based on the background field method [8]. The action is then a functional of the classical background field A, and retains its gauge invariance for the background field, since quantization in the background field gauge only requires the breaking of the gauge symmetry for the quantum gauge field Q. As a consequence of this gauge symmetry many problems related to renormalization in gauge theories are simplified. Thus, the calculation of the Callan–Symanzik β-function [9] only requires the computation of the background field self-energy and no vertex has to be considered [10]. More relevant to us, it has been shown by Kluberg-Stern and Zuber that under renormalization a gauge-invariant operator which does not vanish by virtue of the equations of motion only mixes with other gauge invariant operators [11]. This is both conceptually appealing and computationally very helpful, as we will see in this work, where we will use always the background field gauge.

The introduction of matter fields complicates the renormalization of FF. This is because one has to add three new local scalar gauge-invariant operators, $\bar{\psi}\psi$ of dimension three and $i\,\partial^\mu(\bar{\psi}\gamma_\mu\psi)$ and $\bar{\psi}(i\not{D}-m)\psi$ of dimension four, $D_\mu \equiv \partial_\mu - ig\frac{1}{2}\lambda^a A^a_\mu$ being the covariant derivative. They are, however, not on the same footing as FF; the first because it only mixes with FF if the quarks are massive, the second and third because they vanish if one uses the equations of motion. These we will see, are important features for the renormalization of these four operators.

Now, turning to the renormalization group invariant expression of FF there are, to our knowledge, two sources of information. For pure QCD and from a lowest order computation one knows that αFF, where $\alpha = g^2/4\pi$, is independent of the renormalization scale [6]. On the other hand, FF appears in the trace anomaly [9, 12]. The exact expression for the trace of the bare energy-momentum tensor on the mass shell and at non-zero momentum is [13, 14]

$$\theta^\mu{}_\mu = (1+\gamma_m)\sum_i m_i[\bar{\psi}_i\underline{\psi_i}] + \tfrac{1}{4}\beta[\underline{FF}]\,, \tag{1.1}$$

where $i = 1, \dots, N_f$ is the flavour index. The Callan–Symanzik β-function and mass anomalous dimension γ_m are given by

$$\alpha(\mu)\beta(\alpha(\mu)) = \mu\frac{d\alpha(\mu)}{d\mu}\,, \qquad m_i(\mu)\gamma_m(\alpha(\mu)) = -\mu\frac{dm_i(\mu)}{d\mu}\,, \tag{1.2}$$

α and m_i being the renormalized coupling constant and mass, respectively, and brackets mean dimensionally renormalized normal order products, the lines under them indicate that the equations of motion have been used. Eqs. (1.2) are already written for the minimal subtraction renormalization scheme [15] for which both the β and γ_m functions are gauge parameter and quark mass independent [16].

Since $\theta^{\mu}{}_{\mu}$ in eq. (1.1) is written for dimensionally regularized bare fields, the r.h.s. of eq. (1.1) can not depend on μ and is thus renormalization group invariant. For massless quarks this means that $\beta(\alpha)[\underline{FF}]$ is μ-independent. If one then supposes that this is also so for massive quarks, because one can always work in a mass-independent renormalization scheme, one reaches the conclusion that $(1+\gamma_m)\sum_i m_i[\underline{\bar{\psi}_i\psi_i}]$ is also μ-independent [1]. This is, however, not so. Taking, e.g., the MS scheme [15], one knows that to all orders in perturbation theory $m_i[\bar{\psi}_i\psi_i]$ is renormalization group invariant, i.e.

$$m_i[\bar{\psi}_i\psi_i] = m_{i_0}(\bar{\psi}_{i_0}\psi_{i_0}) , \tag{1.3}$$

where the r.h.s. is written in terms of bare masses and bare quark fields. This is so because a zero momentum insertion of $(\bar{\psi}_{i_0}\psi_{i_0})$,

$$(\bar{\psi}_{i_0}\psi_{i_0})\check{} \equiv \int d^n x (\bar{\psi}_{i_0}\psi_{i_0})_n(x) ,$$

where $n = 4+2\varepsilon$ is the number of dimensions, is equivalent to $i\,\partial/\partial m_{i_0}$ and the renormalization does not depend on the momentum of the insertion. Recall that in n dimensions the dimensions of the bare quark field $\psi_{i_{0_n}}$ are $\frac{3}{2}+\varepsilon$ so that eq. (1.3) gets its meaning from

$$m_i[\bar{\psi}_i\psi_i]\check{} = m_{i_0}(\bar{\psi}_{i_0}\psi_{i_0}) , \tag{1.4}$$

with $[\bar{\psi}_i\psi_i]\check{} \equiv \int d^4x[\bar{\psi}_i\psi_i](x)$ after erasing the condition of zero momentum. As a consequence of eq. (1.3), $\beta(a)[\underline{FF}]$ will in general not be μ-independent. Instead, the correct renormalization group invariant expression for FF is

$$\gamma_m \sum_i m_i[\bar{\psi}_i\psi_i] + \tfrac{1}{4}\beta[\underline{FF}] . \tag{1.5}$$

The aim of this work is to study fairly exhaustively the renormalization of FF up to second order in perturbation theory. We will work with zero momentum insertions, but comment on the only new aspect which non-zero momentum insertions introduce. We will in general consider matrix elements where the bare equations of motion cannot be used, but make their usefulness clear in the search of renormalization group invariance. The identification of non-renormalization (i.e. the renormalization constant is $Z=1$) with renormalization group invariance to lowest order, but their incompatibility at higher orders, will be studied with care. This is related to the fact that the bare coupling constant in n dimensions, α_{0_n}, is not dimensionless. Its dimensions may be made explicitly introducing a mass scale μ, so that $\alpha_{0_n} = (\mu^2)^{\,\varepsilon}\alpha_0$. Notice that then α_0, which is dimensionless, is μ-dependent. Thus, if renormalization group invariance requires expressions inhomogeneous in the renormalized coupling constant α, they cannot be formally equal to their unrenormalized analogues, since an inhomogeneous expression in α_0 is necessarily μ-dependent by virtue of the μ-dependence of α_0 which does not factorize now as an infinitesimal factor. These subtleties will be made clear by the explicit two-loop computation.

As a byproduct, the renormalization group invariance of the trace anomaly will be proven in perturbation theory. This will imply that the trace anomaly gets renormalized in higher than the first orders of perturbation theory. This is in agreement with the fact that the non-renormalization of the dilatation anomaly, which is the trace anomaly inserted in a matrix element of current operators, can only be proven non-perturbatively [13]. The origin of this problem lies in the appearance of the renormalization group functions in the trace anomaly. The reason for this can be traced back to the fact that this anomaly is caused by the unavoidable breaking of scale invariance by regularization and thus renormalization [9]. In this it differs from the triangle anomaly [17], where, since no renormalization group functions appear, one can prove its renormalization group invariance by proving its non-renormalization order by order in perturbation theory. This difference can be understood because there are regularization procedures which maintain chiral invariance in all diagrams except the lowest order axial triangle diagram, whereas at all orders scale invariance is necessarily broken [18]. But we will also give non-renormalized expressions at higher orders, which then, of course, are necessarily μ-dependent!

We will work in the background field gauge; it is the only one in which *FF* is multiplicatively renormalizable in pure QCD. From the computational point of view we will show that it is so advantageous that all our two-loop computations will not require a single Feynman integral calculation.

We hope that the precise meaning of the renormalized operator *FF* will be made clear in the course of this work as well as what has happened to its classical gauge invariance in the process of renormalization.

In sect. 2 we study the renormalization of *FF* without fermion fields. It is multiplicatively renormalizable and we obtain its anomalous dimension up to two loops. In sect. 3 all the complications due to the inclusion of massless quark fields will be studied. Sect. 4 includes quark masses and makes the connection to the trace anomaly. We will draw some conclusions in sect. 5. The appendix collects some useful formulae from renormalization theory which are frequently used in omputations.

2. Renormalization of *FF* in pure QCD

We will give by an explicit two-loop calculation in the background field gauge the renormalized expression for *FF*, check from it that $\beta(\alpha)[FF]$ is renormalization group invariant but that it gets renormalized and give an expression which does not get renormalized but is not renormalization group invariant. We will follow the background field gauge Feynman rules as given in ref. [10], to which we will refer throughout this section. We will work in the Landau background field gauge

$a=0$, because then one does not have to consider renormalization of the gauge parameter and because of other advantages which will become evident immediately. Notice that as in the Feynman rules for some vertices terms proportional to $1/a$ appear, these have to be kept until they are multiplied by terms proportional to a; those which are not do not contribute anyhow [10].

Since we do not consider quark fields in this section, FF is the only scalar gauge-invariant local operator of dimension four and one does not have to worry about mixing, as in the background field gauge FF only mixes with other gauge-invariant operators [11]. Thus, FF is multiplicatively renormalizable:

$$[FF]=Z_{FF}(F_0F_0)\,, \tag{2.1}$$

Z_{FF} being the renormalization constant. Recall that in the Landau background field gauge one only has to consider background field and coupling constant renormalization, and these are related by eq. (A.2).

In order to compute Z_{FF} it is enough to consider the Fourier transform of the Green function $\langle A_a^\mu(x)(FF)_0(0)A_b^\nu(y)\rangle$, which we will write as

$$\langle A_a^\mu(FF)_0A_b^\nu\rangle=Z_{FF}^{-1}Z_\alpha\langle A_a^\mu[FF]A_b^\nu\rangle\,, \tag{2.2}$$

where we have written renormalized background fields so that one does not have to consider external field renormalization. Thus we use the notation [see eq. (A.1)]

$$(FF)_0=Z_\alpha(F_0F_0) \tag{2.3}$$

for the composite operators written in terms of renormalized fields.

The Feynman rules for the insertion of $-\frac{1}{4}i(FF)_0$ of zero momentum are given by

$$-i\delta_{ab}(p^2g_{\mu\nu}-p_\mu p_\nu) \tag{2.4}$$

for an insertion on a gluon propagator of momentum p, and by the ordinary three- and four-gluon vertices for quantum fields for insertions on three- and four-gluon vertices independently of whether background fields flow into these or not. At the one loop-level the diagrams which contribute to eq. (2.2) are the first two shown in fig. 1, where we recall that the external fields are background fields A.

From eq. (2.4) it is clear that the insertion of $(FF)_0$ into a gluon propagator transforms it into the same propagator in the Landau gauge $a=0$ but with opposite sign. Recalling the above given comment on the gluon vertex insertion, one obtains the first equality of fig. 1. Notice that now the second diagram has a quantum external field Q and that proper account has been taken of the weight factors. Now recalling that the difference in the Feynman rules for A and Q fields are the terms in $1/a$, one gets the second equality of fig. 1, where in the last diagram only the term proportional to a is left for the upper propagator and since we work in the Landau gauge and the external fields are quantum fields this will not contribute. Finally, as the $1/a$ terms always go with momenta corresponding to quantum fields and which carry the Lorentz index of precisely the same field, they just pick out

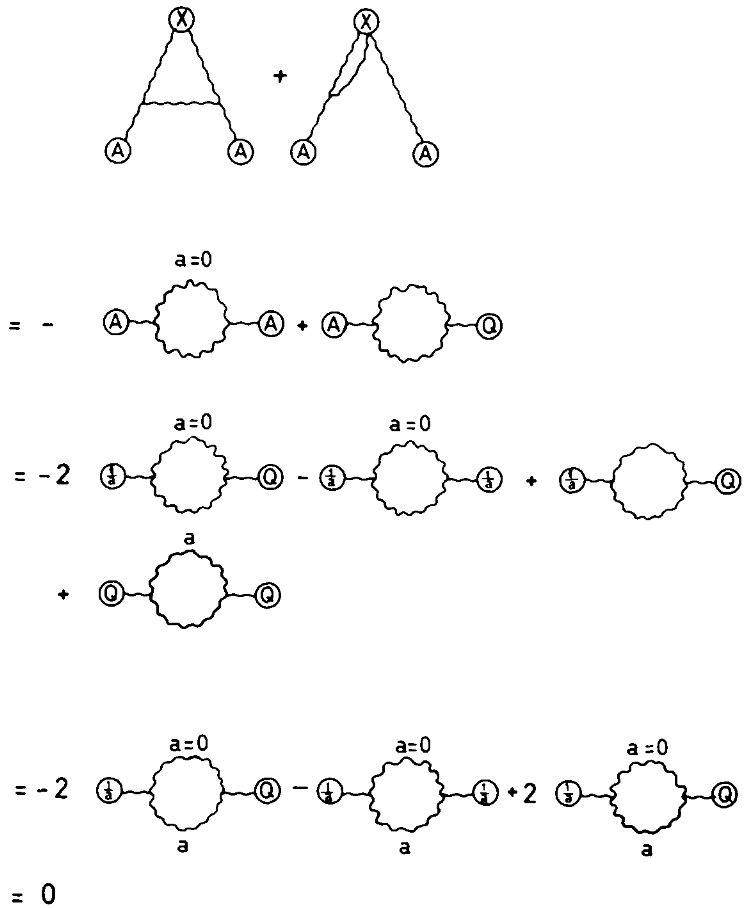

Fig. 1. Diagrammatic proof of $Z_{FF}^{(2)} = Z_{\alpha}^{(2)}$ in pure QCD.

the a-term when contracted with the corresponding propagator according to

$$\frac{1}{a} p^{\nu} \frac{g_{\mu\nu} - (1-a)p_{\nu}p_{\mu}/p^2}{p^2} = \frac{p_{\mu}}{p^2}. \tag{2.5}$$

This then implies the following equality of fig. 1, where in the last diagram we have used the fact that we work in the Landau gauge. Only the second diagram is left over, but this is proportional to $1/a$ and thus vanishes.

This concludes our one-loop proof. There is no renormalization of $(FF)_0$ at this level,

$$Z_{FF}^{(2)} = Z_{\alpha}^{(2)}, \tag{2.6}$$

and therefore

$$\alpha^{(2)}[FF]^{(2)} = \alpha_0(F_0F_0)\,. \tag{2.7}$$

The l.h.s. is μ-independent; it is enough to recall that $\mu(\mathrm{d}/\mathrm{d}\mu)\alpha_0 = 2\varepsilon\alpha_0$ and that (F_0F_0) is μ-independent. Furthermore, there is no renormalization of $\alpha_0(F_0F_0)$ at this order, since the renormalized expression is formally equal to the unrenormalized one.

It is an amusing exercise to do the same calculation to next order. Again one only needs to draw diagrams, 32 topologically different ones to start with, 12 of them involving ghosts, and check carefully all the weight factors. Following rules similar to the ones used in fig. 1 one finds that they are reduced to the 12 background field self-energy two-loop diagrams, 7 out of them involving ghosts, and with just the right weight factors but with opposite sign. As $Z_A\alpha_0 = \alpha$ implies that the two-loop poles do not get any contribution from renormalization of the one-loop self-energy, we find immediately from eqs. (A.4) and (A.8) that the result is

$$-i\delta_{ab}(p^2 g_{\mu\nu} - p_\mu p_\nu)\left(\frac{\alpha}{\pi}\right)^2 \frac{\beta_2}{4\varepsilon}\,, \tag{2.8}$$

so that we find from eq. (2.2)

$$Z_{FF}^{(4)} = Z_\alpha^{(4)}\left(1 - \left(\frac{\alpha}{\pi}\right)^2 \frac{\beta_2}{4\varepsilon}\right), \tag{2.9}$$

which is the main result of this section. It shows that the non-renormalization of $\alpha_0(F_0F_0)$ is no longer true at two loops. What happens with $\beta(\alpha)[FF]$? Let us consider its first two terms:

$$\left[\beta_1 \frac{\alpha^{(4)}}{\pi} + \beta_2\left(\frac{\alpha^{(2)}}{\pi}\right)^2\right][FF]^{(4)} = \left[\beta_1 \frac{\alpha_0}{\pi}\left(1 - \left(\frac{\alpha}{\pi}\right)^2 \frac{\beta_2}{4\varepsilon}\right) + \beta_2 \frac{\alpha_0}{\pi}\frac{\alpha^{(2)}}{\pi}\right](F_0F_0)\,, \tag{2.10}$$

where we have used $2\varepsilon[FF]^{(4)} = 0$, when $\varepsilon \to 0$. Using eq. (A.4) the r.h.s. can be written as

$$\left[\beta_1 \frac{\alpha_0}{\pi} + \beta_1\beta_2\left(\frac{\alpha_0}{\pi}\right)^3 \frac{1}{4\varepsilon} + \beta_2\left(\frac{\alpha_0}{\pi}\right)^2\right](F_0F_0)\,, \tag{2.11}$$

which is not formally equal to the renormalized expression. Thus, also $\beta(\alpha)[FF]$ gets renormalized. However, it is μ-independent. To see this, apply $\mu\,\mathrm{d}/\mathrm{d}\mu$ to eq. (2.11). The result from eq. (A.4) is

$$2\varepsilon\beta_2 \frac{\alpha_0}{\pi}\frac{\alpha}{\pi}(F_0F_0) = \mathrm{O}(\alpha^4)\,, \tag{2.12}$$

where eq. (2.9) has been used. It vanishes, therefore, at the level we are working for $\varepsilon \to 0$. This concludes our proof: $\beta(\alpha)[FF]$ is renormalization group invariant but gets renormalized in perturbation theory.

From eq. (2.9) one can compute the anomalous dimension of $[FF]$ up to two loops. It is given by

$$\gamma_{FF}=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_{FF} \tag{2.13}$$

which leads to

$$\gamma_{FF}^{(4)}=-\frac{\alpha}{\pi}\beta_1-2\left(\frac{\alpha}{\pi}\right)^2\beta_2\,. \tag{2.14}$$

Notice finally that from eq. (2.9) one can find a combination which does not get renormalized at the two-loop level:

$$\left[2\beta_1\frac{\alpha^{(4)}}{\pi}+\beta_2\left(\frac{\alpha^{(2)}}{\pi}\right)^2\right][FF]^{(4)}=\left[2\beta_1\frac{\alpha_0}{\pi}+\beta_2\left(\frac{\alpha_0}{\pi}\right)^2\right](F_0F_0)\,. \tag{2.15}$$

It is, however, μ-dependent. Indeed $\mu\,\mathrm{d}/\mathrm{d}\mu$ gives

$$2\epsilon\beta_2\left(\frac{\alpha_0}{\pi}\right)^2(F_0F_0)=-\beta_1\beta_2\left(\frac{\alpha}{\pi}\right)^3[FF]\,. \tag{2.16}$$

This is an uncommon situation. Usually non-renormalization implies renormalization group invariance. Here this is not so because of the inhomogeneous character in α of the functions we are considering and because of the fact that the dimensionless bare coupling constant is μ-dependent.

Notice that $[FF]$ only depends on gauge-invariant quantities in pure QCD as (F_0F_0) is gauge invariant and the renormalization constant depends only on the β-function coefficients which in the MS scheme we are using are gauge-parameter independent. This is, however, a result one only obtains in the background field gauge.

3. Introduction of massless fields

There is now a new set of gauge-invariant scalar operators of dimension 4 and which therefore mix with FF: $i\bar{\psi}_j \not{D}\psi_j$, j being the flavour index of the quark field. At non-zero momentum there is still another operator of the same type $i\partial^\mu(\bar{\psi}_j\gamma_\mu\psi_j)$; however, we will not consider it in order to keep the computations manageable, so that our results from now on will be valid for zero momentum operators.

We will consider the effects of the operators $i\bar{\psi}_j\not{D}\psi_j$ on the one- and two-loop renormalization of FF in the background field gauge. One expects from the general renormalization theory of gauge-invariant operators in the background field gauge that these operators enter into the renormalized FF but that their renormalized expressions contain non-gauge-invariant operators because they vanish when the equations of motion are used [11]. Thus nothing will be said about the renormalization of these operators.

For one flavour and with the zero momentum notation $O_1 = -\frac{1}{4}iF\tilde{F}$ and $O_2 = -\bar{\psi}\not{D}\tilde{\psi}$ we will have to compute

$$O_1 = Z_{11}O_{1_0}^0 + Z_{12}O_{2_0}^0 \tag{3.1}$$

where we use the notation $O_{i_0}^0$ for bare operators written in terms of bare fields and

$$O_{1_0} = Z_\alpha O_{1_0}^0 , \qquad O_{2_0} = Z_F^{-1} O_{2_0}^0 , \tag{3.2}$$

for the bare operators written in terms of renormalized fields [see eq. (A.1)]. The renormalization constants Z_{11} and Z_{12} will be obtained from the divergent parts of the insertion of the operators $O_{1_0}^0$ and $O_{2_0}^0$ at zero momentum into the two background field and the two quark field Green functions. Then, and for renormalized fields,

$$\begin{aligned} \langle A_a^\mu O_1 A_b^\nu\rangle &= Z_\alpha^{-1} Z_{11}\langle A_a^\mu O_{1_0} A_b^\nu\rangle + Z_{12}\langle A_a^\mu O_{2_0}^0 A_b^\nu\rangle , \\ \langle \psi O_1 \bar{\psi}\rangle &= Z_{11}\langle \psi O_{1_0}^0 \bar{\psi}\rangle + Z_F Z_{12}\langle \psi O_{2_0}\bar{\psi}\rangle . \end{aligned} \tag{3.3}$$

The Feynman rules for an insertion of O_{2_0} of zero momentum on a quark propagator of momentum p are given by

$$i\not{p} \tag{3.4}$$

and for an insertion on a quark–quark–gluon vertex, independently on whether the gluon is a background or quantum gauge field by

$$ig\tfrac{1}{2}\lambda^a\gamma_\mu . \tag{3.5}$$

To lowest order the diagrams which contribute to $\langle A_a^\mu O_{1_0} A_b^\nu\rangle$ are the same ones as considered in sect. 2. The diagrams which contribute to $\langle A_a^\mu O_{2_0}^0 A_b^\nu\rangle$ are shown in fig. 2. There are two of each type and from eqs. (3.4) and (3.5) they cancel. The diagram which contributes to $\langle \psi O_{1_0}^0 \bar{\psi}\rangle$ is shown in fig. 3, which is equivalent to the quark self-energy diagram with changed sign in the Landau gauge, which is again zero. Finally the diagrams which contribute to $\langle \psi O_{2_0} \bar{\psi}\rangle$ are shown in fig. 4 and they again do not contribute because we have chosen to work in the Landau gauge. Since also in this gauge $Z_F^{(2)} = 1$ [see eq. (A.5)], the solution of eq. (3.4) is

$$Z_{11}^{(2)} = Z_\alpha^{(2)} , \qquad Z_{12}^{(2)} = 0 . \tag{3.6}$$

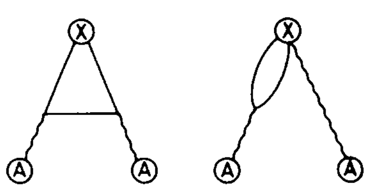

Fig. 2. Diagrams contributing to $\langle A(\bar{\psi}\not{D}\psi)A\rangle^{(2)}$.

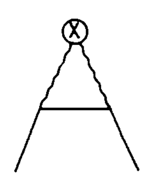

Fig. 3. Diagrams contributing to $\langle \psi(FF)\bar{\psi}\rangle^{(2)}$.

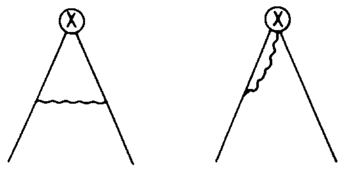

Fig. 4. Diagrams contributing to $\langle\psi(\bar{\psi}\not{D}\psi)\bar{\psi}\rangle^{(2)}$.

The renormalized operator O_1 is defined by the condition that Z_{12} starts directly with the poles in ε, and not with 1, or equivalently by

$$\langle\psi O_1\bar{\psi}\rangle^{(0)}=0\,. \tag{3.7}$$

Eq. (3.6) shows that at this level FF does not mix with $i\bar{\psi}\not{D}\psi$ and that eq. (2.7) still holds at least at zero momentum in presence of massless quark fields

$$\alpha^{(2)}[FF]^{(2)^{\sim}}=\alpha_0(F_0F_0)^{\sim}\,. \tag{3.8}$$

However, a different situation is encountered at the two-loop level as we will now sketch.

The diagrams which contribute to $\langle A^{\mu}_{a}O_{1_0}A^{\nu}_{b}\rangle$ are the same ones as in the quarkless case plus 7 new ones involving quark loops. The result is the same as in the previous section, i.e. (2.8), only that now the β_2 coefficient includes quarks. There are 11 topologically different diagrams which contribute to $\langle\psi O^0_{1_0}\bar{\psi}\rangle$. Their divergent contribution can be easily evaluated in the Landau gauge, because it is then precisely twice the divergent part of the two-loop quark self-energy, $-i\Sigma_0^{(4)}(\not{p})$, with changed sign. From eq. (A.9) this is in the massless case $2i\not{p}\Sigma_{0_2}^{(4)}(p^2)$. Then the result follows immediately from eqs. (A.10) and (A.4) and is, recalling that $\gamma_{F1}(a=0)=0$,

$$i\not{p}\,\frac{\gamma_{F2}(a=0)}{2}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}\,. \tag{3.9}$$

We do not need the other two matrix elements, because Z_{12} does not start with 1. The solution of the system of eq. (3.3) then is

$$\begin{aligned}Z_{11}^{(4)}&=Z_\alpha^{(4)}\left(1-\left(\frac{\alpha}{\pi}\right)^2\frac{\beta_2}{4\varepsilon}\right),\\ Z_{12}^{(4)}&=-\frac{\gamma_{F2}(a=0)}{2}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}\,.\end{aligned} \tag{3.10}$$

This is the main result of this section. The two-loop renormalized FF is now given by the expression

$$[FF]^{(4)^{\sim}}=Z_\alpha^{(4)}\left(1-\left(\frac{\alpha}{\pi}\right)^2\frac{\beta_2}{4\varepsilon}\right)(F_0F_0)^{\sim}+2\gamma_{F2}(a=0)\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}\,(\bar{\psi}_0 i\not{D}_0\psi_0)^{\sim}\,. \tag{3.11}$$

The generalization to several flavours is straightforward, one just substitutes $(\bar{\psi}_0 i\not{D}_0\psi_0)$ by $\sum_i(\bar{\psi}_{0_i} i\not{D}_0\psi_{0_i})$. Eq. (3.11) shows that FF is no longer multiplicatively renormalizable but mixes with $i\bar{\psi}\not{D}\psi$. Only when the equations of motion are used (on the mass shell), where this operator is zero, as well as $i\partial^\mu(\bar{\psi}\gamma_\mu\psi)$, do we recover the result of sect. 2,

$$[F\underline{F}]^{(4)} = Z_\alpha^{(4)}\left(1-\left(\frac{\alpha}{\pi}\right)^2\frac{\beta_2}{2\varepsilon}\right)(F_0F_0)\,, \tag{3.12}$$

and FF is still multiplicatively renormalizable.

It is clear from eq. (3.11) that in presence of massless quark fields $\beta(\alpha)[FF]^{\check{}}$ is no longer renormalization group invariant, whereas it is still true for $\beta(\alpha)[F\underline{F}]$.

Notice the appearance of a gauge-parameter dependent anomalous dimension in eq. (3.11). This is not surprising, since even in the background field gauge one expects that somewhere something gauge dependent has to show up, since quantization always requires the breaking of gauge invariance. On the mass shell, however, as expected on physical grounds, this gauge dependence disappears.

4. Massive quark fields

There is now a third set of operators which enters the game, $m_i\bar{\psi}_i\psi_i$. For one flavour one then has to consider the three operators $O_1 \equiv -\frac{1}{4}iFF^{\check{}}$, $O_2 \equiv -\bar{\psi}(\not{D}+im)\psi^{\check{}}$ and $O_3 \equiv im\bar{\psi}\psi^{\check{}}$, where the second one vanishes when the equations of motion are used and the third one is multiplicatively renormalizable and unrenormalized to all orders in perturbation theory, eq. (1.3). For the renormalization of FF we will have to compute

$$O_1 = Z_{11}O^0_{1_0} + Z_{12}O^0_{2_0} + Z_{13}O^0_{3_0}\,, \tag{4.1}$$

which, for $m=0$ has to reproduce the results of sect. 3. Since the renormalization constants are mass independent we already know $Z_{11}^{(4)}$ and $Z_{12}^{(4)}$ from the previous section. It only remains to compute $Z_{13}^{(4)}$.

To do so we will consider

$$\langle\psi O_1\bar{\psi}\rangle = Z_{11}\langle\psi O^0_{1_0}\bar{\psi}\rangle + Z_F Z_{12}\langle\psi O_{2_0}\bar{\psi}\rangle + Z_F Z_{13}\langle\psi O_{3_0}\bar{\psi}\rangle\,, \tag{4.2}$$

with the notation

$$\begin{aligned} O_{2_0} &\equiv -(\bar{\psi}(\not{D}_0+im_0)\psi)^{\check{}}_0 \equiv Z_F^{-1}O^0_{2_0}\,,\\ O_{3_0} &\equiv im_0(\bar{\psi}\psi)^{\check{}}_0 \equiv Z_F^{-1}O^0_{3_0}\,. \end{aligned} \tag{4.3}$$

The Feynman rule for an insertion of O_{2_0} of zero momentum into a quark propagator of momentum p is

$$i(\not{p}-m_0)\,, \tag{4.4}$$

and of O_{3_0} into a quark propagator is

$$im_0 \,. \tag{4.5}$$

Let us start at the one-loop level. The calculation of $\langle\psi O^0_{1_0}\bar{\psi}\rangle$ corresponds to the diagram of fig. 3, but it is now not zero because of the non-vanishing quark mass. Indeed it is given by $i\Sigma_0^{(2)}(\not{p}, a=0)$ which from eqs. (A.11) and (A.4) has the divergent part

$$-im_0\frac{\gamma_{m1}}{2}\frac{\alpha_0}{\pi}\frac{1}{\varepsilon}\,. \tag{4.6}$$

As Z_{12} does not start with 1 we do not need to calculate $\langle\psi O_{2_0}\bar{\psi}\rangle$. Finally the calculation of $\langle\psi O_{3_0}\bar{\psi}\rangle$ is given by fig. 5 and recalling that an insertion of im_0 is equivalent to $-m_0\partial/\partial m_0$, the result is $im_0(\partial/\partial m_0)\,\Sigma_0^{(2)}(\not{p}, a=0)$, the divergent part of which is given from eq. (A.9) by the divergent part of $im_0\,\bar{\Sigma}_{0_1}^{(2)}(p^2)$ which, as before, is again given by eq. (4.6). Putting this together one finds

$$Z_{13}^{(2)}=\frac{\gamma_{m1}}{2}\frac{\alpha}{\pi}\frac{1}{\varepsilon}\,, \tag{4.7}$$

where, as in sect. 3, the renormalized operator O_1 is defined by the condition that Z_{13} starts directly with the poles in ε, or equivalently by eq. (3.7).

From eq. (4.1) our result is up to one-loop level

$$[FF]^{(2)\breve{}}=\left(1-\frac{\alpha}{\pi}\frac{\beta_1}{2\varepsilon}\right)(F_0F_0)\breve{}-2\gamma_{m1}\frac{\alpha}{\pi}\frac{1}{\varepsilon}\,m_0(\bar{\psi}_0\psi_0)\breve{}\,, \tag{4.8}$$

and FF is no longer multiplicatively renormalizable. We can easily build a renormalization group invariant expression from eq. (4.8). Consider

$$\tfrac{1}{4}\beta_1\frac{\alpha^{(2)}}{\pi}[FF]^{(2)\breve{}}=\tfrac{1}{4}\beta_1\frac{\alpha_0}{\pi}(F_0F_0)\breve{}-\tfrac{1}{2}\beta_1\gamma_{m1}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}\,m_0(\bar{\psi}_0\psi_0)\breve{}\,, \tag{4.9}$$

and from eq. (1.3)

$$\gamma_{m1}\frac{\alpha^{(2)}}{\pi}m^{(2)}[\bar{\psi}\psi]^{(2)\breve{}}=\gamma_{m1}\frac{\alpha_0}{\pi}m_0(\bar{\psi}_0\psi_0)\breve{}+\tfrac{1}{2}\beta_1\gamma_{m1}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}\,m_0(\bar{\psi}_0\psi_0)\breve{}\,, \tag{4.10}$$

and summing both expressions

$$\tfrac{1}{4}\beta_1\frac{\alpha^{(2)}}{\pi}[FF]^{(2)\breve{}}+\gamma_{m1}\frac{\alpha^{(2)}}{\pi}m^{(2)}[\bar{\psi}\psi]^{(2)\breve{}}=\tfrac{1}{4}\beta_1\frac{\alpha_0}{\pi}(F_0F_0)\breve{}+\gamma_{m1}\frac{\alpha_0}{\pi}m_0(\bar{\psi}\psi_0)\breve{}\,. \tag{4.11}$$

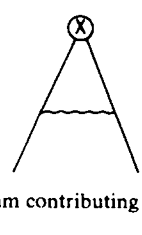

Fig. 5. Diagram contributing to $\langle\psi(m\bar{\psi}\psi)\bar{\psi}\rangle^{(2)}$.

This proves both the renormalization group invariance and the non-renormalization of this expression, which is precisely the lowest order trace anomaly, eq. (1.5).

In order to perform the two-loop computation only $\langle\psi O^0_{1_0}\bar{\psi}\rangle$ has to be computed up to two loops. There are as before 11 topologically different diagrams which contribute at this level. Their result is exactly the two-loop part of $2i\Sigma_0^{(4)}(\not{p}, a=0)$. Its divergent part can be traced back from eqs. (A.9)–(A.12) and (A.4). It is, being careful in subtracting the two-loop contribution which comes from the coupling constant renormalization of the one-loop contribution,

$$2i\left[m_0\left(-\left(\frac{\alpha}{\pi}\right)^2\frac{\gamma_{m2}}{4\varepsilon}+\left(\frac{\alpha}{\pi}\right)^2\frac{\gamma_{m1}(\gamma_{m1}-\beta_1)}{8\varepsilon^2}\right)+(\not{p}-m_0)\left(\frac{\alpha}{\pi}\right)^2\frac{\gamma_{F2}(a=0)}{4\varepsilon}\right]. \quad (4.12)$$

With this and putting together eqs. (3.10) and (4.4)–(4.8) we find from a careful analysis of eq. (4.2) that

$$Z_{13}^{(4)}=\frac{\gamma_{m1}}{2}\frac{\alpha}{\pi}\frac{1}{\varepsilon}+\frac{\gamma_{m2}}{2}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}-\left(\frac{\alpha}{\pi}\right)^2\frac{\beta_1\gamma_{m1}}{4\varepsilon^2}. \quad (4.13)$$

This then leads to our final result:

$$\begin{aligned}[FF]^{(4)\check{}}=Z_\alpha^{(4)}\left(1-\left(\frac{\alpha}{\pi}\right)^2\frac{\beta_2}{4\varepsilon}\right)(F_0F_0)^{\check{}}+2\gamma_{F2}(a=0)\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\varepsilon}(\bar{\psi}_0(i\not{D}_0-m_0)\psi_0)^{\check{}}\\ -\left(2\frac{\alpha}{\pi}\frac{\gamma_{m1}}{\varepsilon}+2\left(\frac{\alpha}{\pi}\right)^2\frac{\gamma_{m2}}{\varepsilon}-\left(\frac{\alpha}{\pi}\right)^2\frac{\beta_1\gamma_{m1}}{\varepsilon^2}\right)m_0(\bar{\psi}_0\psi_0)^{\check{}}.\end{aligned} \quad (4.14)$$

The generalization to several flavours is again immediate; one only has to sum over them.

An analysis like that done in eqs. (2.10)–(2.12), but more laborious, allows one to prove that the expression

$$\frac{1}{4}\left[\beta_1\frac{\alpha^{(4)}}{\pi}+\beta_2\left(\frac{\alpha^{(2)}}{\pi}\right)^2\right][\underline{FF}]^{(4)}+\left[\gamma_{m1}\frac{\alpha^{(4)}}{\pi}+\gamma_{m2}\left(\frac{\alpha^{(2)}}{\pi}\right)^2\right]m^{(4)}[\underline{\bar{\psi}\psi}]^{(4)} \quad (4.15)$$

is renormalization group invariant. It is precisely the first two terms of the trace anomaly, eq. (1.5).

On the contrary, one can easily prove the equality

$$\begin{aligned}&\frac{1}{4}\left[2\beta_1\frac{\alpha^{(4)}}{\pi}+\beta_2\left(\frac{\alpha^{(2)}}{\pi}\right)^2\right][\underline{FF}]^{(4)}+\left[2\gamma_{m1}\frac{\alpha^{(4)}}{\pi}+\gamma_{m2}\left(\frac{\alpha^{(2)}}{\pi}\right)^2\right]m^{(4)}[\underline{\bar{\psi}\psi}]^{(4)}\\ &=\frac{1}{4}\left[2\beta_1\frac{\alpha_0}{\pi}+\beta_2\left(\frac{\alpha_0}{\pi}\right)^2\right](F_0F_0)+\left[2\gamma_{m1}\frac{\alpha_0}{\pi}+\gamma_{m2}\left(\frac{\alpha_0}{\pi}\right)^2\right]m_0(\bar{\psi}_0\psi_0),\end{aligned} \quad (4.16)$$

which shows the non-renormalization of this expression, which, however, is not renormalization group invariant. In fact, eq. (4.16) gives a multiplicative renormalized expression involving $[\underline{FF}]$ with renormalization constant 1.

5. Conclusions

We have studied the renormalization of *FF* in the background field gauge up to second-order perturbation theory. The diagrammatic technique we have used allows one to go easily to third order; however, no new qualitative features come in and from the quantitative point of view both γ_{m3} and γ_{F3} are unknown.

Let us shortly recall our main results together with some facts known from the general theory of the renormalization of gauge-invariant operators in the background field gauge [11].

Mixing: In pure QCD *FF* is multiplicatively renormalizable. It mixes with $i\bar{\psi}\not{D}\psi$, $i\partial^\mu(\bar{\psi}\gamma_\mu\psi)$ and $m\bar{\psi}\psi$ if quark fields are included. Only the mass operator is left if the equation of motion can be used.

Gauge invariance: Although all these operators are gauge invariant, those which vanish when the equations of motion are used appear with gauge-parameter dependent anomalous dimensions. In order to obtain physically meaningful matrix elements the equations of motion have to be used.

Renormalization group invariance: The renormalization group invariant expression built from the renormalization of *FF* is precisely the trace anomaly, and requires the use of the equations of motion.

Non-renormalization: We have found expressions of a structure similar to the one of the trace anomaly which do not get renormalized but are, precisely because of this, μ-dependent.

Anomalous dimension: The anomalous dimension of *FF* we have found for pure QCD is also correct for the complete theory. This is due to two facts: that $m\bar{\psi}\psi$ is multiplicatively renormalizable and that *FF* does not enter into the renormalization of gauge-invariant operators which vanish when the equations of motion are used. Diagonalization then leads to our result.

Background field technique: These results have been found in the Landau background field gauge without doing a single Feynman integral computation. In any other gauge one would have had to calculate a very large amount of two-loop integrals and, depending on the gauge, include several more operators.

Phenomenology: The leading non-perturbative effects will be parametrized by the physical vacuum expectation values of the renormalization group and gauge-invariant operators

$$\langle m_i[\bar{\psi}_i\psi_i]\rangle\,, \qquad \left\langle \sum_i m_i\gamma_m[\bar{\psi}_i\psi_i]+\tfrac{1}{4}\beta[FF]\right\rangle. \tag{5.1}$$

PCAC allows estimates for the first of these [19]; however, one does not know a safe value for the second because all the analyses have been performed for $\langle\beta[FF]\rangle$ in the belief that it is renormalization group invariant. This difference might be important specially in studies of quark masses including non-perturbative corrections [20]. We will come back to these problems and others related to the operators product expansion elsewhere.

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Appendix

The renormalization of the background field, the coupling constant, the quark field and the quark mass is given by

$$A^{\mu}_{0_a} = Z_A^{1/2} A^{\mu}_a\,, \qquad \alpha_0 = Z_\alpha \alpha\,, \qquad \psi_{0_i} = Z_F^{1/2}\psi_i\,, \qquad m_{0_i} = Z_m m_i\,, \tag{A.1}$$

in any flavour independent renormalization scheme. The subscript 0 denotes bare quantities. We will follow the MS scheme [15] and work in the background field gauge, where

$$Z_A = Z_\alpha^{-1}\,. \tag{A.2}$$

The anomalous dimensions are given by

$$\begin{aligned} \gamma_X &= \mu \frac{d}{d\mu} \ln Z_X = \sum_{j=1} \left(\frac{\alpha}{\pi}\right)^j \gamma_{X_j}\,, \\ \beta &= 2\varepsilon - \mu \frac{d}{d\mu} \ln Z_\alpha = 2\varepsilon + \sum_{j=1} \left(\frac{\alpha}{\pi}\right)^j \beta_j \end{aligned} \tag{A.3}$$

$X = A, F, m$ and $2\varepsilon = n-4$, n being the number of dimensions. Then the following expressions follow:

$$\begin{aligned} Z_A &= 1 + \frac{\alpha}{\pi}\frac{\beta_1}{2\varepsilon} + \left(\frac{\alpha}{\pi}\right)^2 \frac{\beta_2}{4\varepsilon}\,, \\ Z_\alpha &= 1 - \frac{\alpha}{\pi}\frac{\beta_1}{2\varepsilon} + \left(\frac{\alpha}{\pi}\right)^2 \frac{\beta_1^2}{4\varepsilon^2} - \left(\frac{\alpha}{\pi}\right)^2 \frac{\beta_2}{4\varepsilon}\,, \\ Z_F &= 1 + \frac{\alpha}{\pi}\frac{\gamma_{F1}}{2\varepsilon} + \left(\frac{\alpha}{\pi}\right)^2 \frac{\gamma_{F1}(\gamma_{F1}-\beta_1)}{8\varepsilon^2} + \left(\frac{\alpha}{\pi}\right)^2 \frac{\gamma_{F2}}{4\varepsilon}\,, \\ Z_m &= 1 + \frac{\alpha}{\pi}\frac{\gamma_{m1}}{2\varepsilon} + \left(\frac{\alpha}{\pi}\right)^2 \frac{\gamma_{m1}(\gamma_{m1}-\beta_1)}{8\varepsilon^2} + \left(\frac{\alpha}{\pi}\right)^2 \frac{\gamma_{m2}}{4\varepsilon}\,. \end{aligned} \tag{A.4}$$

The coefficients are known to one-loop order:

$$\begin{aligned} \beta_1 &= -\tfrac{11}{6} C_2(\mathrm{G}) + \tfrac{1}{3} N_f\,, \\ \gamma_{m1} &= \tfrac{3}{2}\, C_2(\mathrm{R})\,, \\ \gamma_{F1} &= \frac{a}{2}\, C_2(\mathrm{R})\,, \end{aligned} \tag{A.5}$$

and to two-loop order [21, 22, 23]:

$$\beta_2 = -\tfrac{17}{12} C_2^2(\mathrm{G}) + \tfrac{5}{12} C_2(\mathrm{G})N_f + \tfrac{1}{4}C_2(\mathrm{R})N_f ,$$

$$\gamma_{m2} = \tfrac{3}{16}C_2^2(\mathrm{R}) + \tfrac{97}{48}C_2(\mathrm{R})C_2(\mathrm{G}) - \tfrac{5}{24}C_2(\mathrm{R})N_f , \qquad (\mathrm{A}.6)$$

$$\gamma_{F2} = (\tfrac{25}{32} + \tfrac{1}{4}a + \tfrac{1}{32}a^2)C_2(\mathrm{R})C_2(\mathrm{G}) - \tfrac{1}{8}C_2(\mathrm{R})N_f - \tfrac{3}{16}C_2^2(\mathrm{R}) ,$$

where $C_2(\mathrm{G}) = N$, $C_2(\mathrm{R}) = (N^2-1)/2N$ for colour SU(N), N_f is the number of flavours and a is the gauge parameter. The coefficient β_3 has also been computed recently, but we do not need it [24].

If the background field self-energy tensor is written as

$$i\Pi_{ab}^{\mu\nu}(p) = i(p^\mu p^\nu - p^2 g^{\mu\nu})\delta_{ab}\Pi(p^2) , \qquad (\mathrm{A}.7)$$

the renormalization of the self-energy is given by

$$1 + \Pi(p^2) = Z_A(1 + \Pi_0(p^2)) . \qquad (\mathrm{A}.8)$$

If the quark self-energy matrix is written as

$$-i\Sigma(\not p) = -i[m\bar\Sigma_1(p^2) + (\not p - m)\Sigma_2(p^2)] , \qquad (\mathrm{A}.9)$$

the renormalization proceeds according to

$$1 - \Sigma_2(p^2) = Z_F(1 - \Sigma_{0_2}(p^2)) , \qquad (\mathrm{A}.10)$$

and

$$1 - \Sigma_2(p^2) + \bar\Sigma_1(p^2) = Z_4(1 - \Sigma_{0_2}(p^2) + \bar\Sigma_{0_1}(p^2)) , \qquad (\mathrm{A}.11)$$

with

$$Z_4 = Z_m Z_F . \qquad (\mathrm{A}.12)$$

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