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THE 2-DIMENSIONAL O(4) SYMMETRIC HEISENBERG FERROMAGNET IN TERMS OF ROTATION-INVARIANT VARIABLES*

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After the introduction of rotation-invariant auxiliary variables, the integration over all rotation-variant variables (spins) in the O(4) symmetric two-dimensional Heisenberg ferromagnet can be performed. The resulting new hamiltonian involves a sum over closed loops. It is complex and invariant under U(1) gauge transformations. Rühl's boson representation is used to derive the result.

1. Introduction

Duality transformations have proved to be a useful tool in the investigation of ferromagnets and gauge theories with abelian symmetry group. One of their main ingredients consists of the introduction of suitable auxiliary variables in order to perform the integration over the rotation and gauge-variant field variables. These rotation or gauge-invariant auxiliary variables then play the rôle of random variables in a new system of statistical mechanics. The aim of this paper is to answer the question whether an analogous formulation can be found for non-abelian theories – if inevitable even with a not necessarily positive measure instead of a Gibbs measure. (In the abelian case one obtains positive measure only for "ferromagnetic" systems.)

As a simple example of a theory with non-abelian global symmetry, the O(4) symmetric Heisenberg ferromagnet in two dimensions is studied. Its partition function is reformulated in terms of SU(2) variables and expanded into characters. Then, following Rühl's investigation of SU(N) invariant lattice field theories [1, 2], the integration over the field variables is carried out using the Bargmann space realization of group representations of SU(2) [3]. One advantage of this formalism is that no vector coupling coefficients arise and the summation over the irreducible unitary representations of SU(2) may be performed explicitly. This causes the rotation-invariant U(1) content of the initial SU(2) variables to reappear, U(1) being the maximal torus of SU(2). Thus, only the rotation-variant variables are eliminated.

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The group integration implies the introduction of new rotation variant spins. They are $\mathbb{C}^2$ variables with quartic interaction. With the help of rotation-invariant complex variables this is brought into quadratic form so that a gaussian integration can be carried out.

The expansion of the resulting determinant leads to a system of closed loops. The new hamiltonian is complex and invariant under U(1) gauge transformations. It is even possible to formulate the theory as a local gauge theory where the gauge group is a semi direct product of U(1) and the Weyl group of SU(2). This gauge invariance reminds one of the equivalence between the O(3) symmetric Heisenberg ferromagnet and the CP^1 lattice model, as the latter exhibits a local U(1) symmetry not visible in the Heisenberg model [4]. Perhaps this hidden U(1) invariance is a common feature of O(N) symmetric Heisenberg ferromagnets?

As intended, the new system of closed loops is formulated entirely in rotation-invariant variables. The elimination of gauge freedom has proved crucial in the investigation of the 3-dimensional U(1) gauge theory [5] where mass generation turned out to be a perturbative effect in the dually transformed system. So one may hope that this work will be a first step towards a useful duality transformation for theories with non-abelian symmetry group.

In contrast to other methods of introducing rotation-invariant variables (see e.g. [6]), our method is also applicable to non-abelian lattice gauge theories, as will be shown for SU(2) gauge theories in a forthcoming paper.

2. The model

The euclidean action $L(s)$ of the O(4) symmetric Heisenberg ferromagnet in two dimensions is a function of spins $s_x \in \mathrm{S}^3$ which are attached to the sites x of a two-dimensional quadratic (or hexagonal) lattice $\Lambda \subset \mathbb{Z}^2$:

$$L(s) = \beta \sum_{\mathrm{b}=\langle xy \rangle} s_x \cdot s_y \,, \tag{2.1}$$

where b is a link between nearest neighbor vertices $x, y \in \Lambda$. The partition function of the system is given by

$$Z = \int \prod_{x\in\Lambda} \frac{\mathrm{d}^4 s_x}{\pi^2} \delta(s_x^2 - 1)\, \mathrm{e}^{L(s)} \,. \tag{2.2}$$

Periodic boundary conditions are assumed.

The links are oriented in alternating order, as shown in fig. 1. Consequently, the sites of the lattice fall into a set Λ_i of starting points of links and a set Λ_f of end points. Then all links are labelled by $\mathrm{b} = \langle xy \rangle$ with $x \in \Lambda_\mathrm{i}$, $y \in \Lambda_\mathrm{f}$.

Making use of the isomorphism between S^3 and SU(2):

$$s \in \mathrm{S}^3 \leftrightarrow u = \begin{pmatrix} s_1 + is_2 & s_3 + is_4 \\ -s_3 + is_4 & s_1 - is_2 \end{pmatrix} \in \mathrm{SU}(2) \,,$$

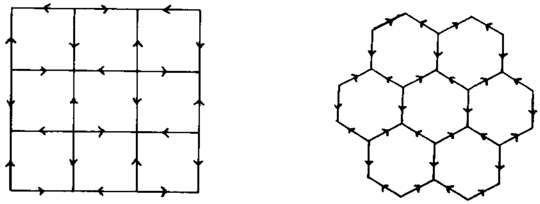

Fig. 1.

the action can be rewritten as

$$L(u)=\sum_{\mathrm{b}} L(u_{\mathrm{b}})\,, \tag{2.3}$$

with

$$\begin{aligned} L(u_{\mathrm{b}}) &= \tfrac{1}{2}\beta \operatorname{tr} u_{\mathrm{b}}\,, \\ u_{\mathrm{b}} &= u_x u_y^{-1}\,, \qquad \text{if } \mathrm{b}=\langle xy\rangle\,. \end{aligned} \tag{2.4}$$

The partition function now reads

$$Z=\int \prod_x \mathrm{d}u_x\, \mathrm{e}^{L(u)}\,, \tag{2.5}$$

where $\mathrm{d}u$ is the Haar measure on SU(2).

3. Integration of the field variables

The action being a class function, $\mathrm{e}^{L(u)}$ can be expanded into characters of irreducible unitary representations of SU(2) which are labelled by half integers $j=0, \frac{1}{2}, 1, \ldots$:

$$\mathrm{e}^{(\beta/2)\operatorname{tr} u}=\sum_j c_j \chi_j(u)\,. \tag{3.1}$$

The expansion coefficients are given by

$$c_j=(2j+1)\frac{2}{\beta} I_{2j+1}(\beta)\,, \tag{3.2}$$

with the modified Bessel functions I_n.

Expanding the characters one gets

$$Z=\int \prod_x \mathrm{d}u_x \prod_{\mathrm{b}} \sum_{j_{\mathrm{b}}} c_{j_{\mathrm{b}}} \sum_{m_{\mathrm{b}}, m'_{\mathrm{b}}} D^{j_{\mathrm{b}}}_{m_{\mathrm{b}} m'_{\mathrm{b}}}(u_x) D^{j_{\mathrm{b}}}_{m'_{\mathrm{b}} m_{\mathrm{b}}}(u_y^{-1})\,. \tag{3.3}$$

The integration over the group variables will be done in the Bargmann space formalism used by Rühl (for details see [1, 3]). It amounts to introducing a Hilbert space of complex analytic functions which is a reducible representation space for SU(2) that contains every unitary irreducible representation of SU(2) once.

The representation matrices are of the form

$$D^j_{mm'}(u) = (v^j_m, T_u v^j_{m'})$$

$$= \int \mathrm{d}\mu(z)\,\mathrm{d}\mu(z')\bar{v}^j_m(z)K(u;z,z')v^j_{m'}(z')\,, \tag{3.4}$$

where

$$\mathrm{d}\mu(z) = \frac{1}{\pi^2}\prod_{i=1,2}\mathrm{d}x_i\,\mathrm{d}y_i\,\mathrm{e}^{-x_i^2-y_i^2} \equiv \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{e}^{-z^+z}$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2\,, \qquad z_i = x_i + iy_i\,,$$

$$K(u;z,z') = \mathrm{e}^{z'^+(u^\mathrm{T}z)}$$

$$v^j_m(z) = \frac{z_1^{j+m}z_2^{j-m}}{[(j+m)!(j-m)!]^{1/2}}\,.$$

The variables associated with m are denoted by ζ, the others by z.

The partition function is now

$$Z = \int \prod_x \mathrm{d}u_x \prod_{\mathrm{b}=\langle xy\rangle}\left\{\int \mathrm{d}\mu(\zeta_{\mathrm{b},x})\,\mathrm{d}\mu(z_{\mathrm{b},x})\,\mathrm{d}\mu(\zeta_{\mathrm{b},y})\,\mathrm{d}\mu(z_{\mathrm{b},y})\right.$$

$$\times \sum_{j_\mathrm{b}} c_{j_\mathrm{b}} \sum_{m_\mathrm{b},m'_\mathrm{b}} \bar{v}^{j_\mathrm{b}}_{m_\mathrm{b}}(\zeta_{\mathrm{b},x})K(u_x;\zeta_{\mathrm{b},x},z_{\mathrm{b},x})v^{j_\mathrm{b}}_{m'_\mathrm{b}}(z_{\mathrm{b},x})$$

$$\left.\times\,\bar{v}^{j_\mathrm{b}}_{m'_\mathrm{b}}(z_{\mathrm{b},y})K(u_y^{-1};z_{\mathrm{b},y},\zeta_{\mathrm{b},y})v^{j_\mathrm{b}}_{m_\mathrm{b}}(\zeta_{\mathrm{b},y})\right\}. \tag{3.5}$$

By means of the formula [1]

$$\sum_m \bar{v}^j_m(z)v^j_m(z') = \frac{(z^+z')^{2j}}{(2j)!} \equiv Q^j(z,z') \tag{3.6}$$

the summations over m, m' can be performed:

$$Z = \int \prod_x \mathrm{d}u_x\,\mathrm{D}\mu(\zeta,z)\prod_{\mathrm{b}=\langle xy\rangle}\left\{\sum_{j_\mathrm{b}} c_{j_\mathrm{b}}Q^{j_\mathrm{b}}(\zeta_{\mathrm{b},x},\zeta_{\mathrm{b},y})Q^{j_\mathrm{b}}(z_{\mathrm{b},y},z_{\mathrm{b},x})\right.$$

$$\left.\times K(u_x;\zeta_{\mathrm{b},x},z_{\mathrm{b},x})K(u_y^{-1};z_{\mathrm{b},y},\zeta_{\mathrm{b},y}\right)\,, \tag{3.7}$$

with the abbreviation

$$\mathrm{D}\mu(\zeta,z) = \prod_\mathrm{b}\mathrm{d}\mu(\zeta_{\mathrm{b},x})\,\mathrm{d}\mu(z_{\mathrm{b},x})\,\mathrm{d}\mu(\zeta_{\mathrm{b},y})\,\mathrm{d}\mu(z_{\mathrm{b},y})\,.$$

The integrals over the group elements u are evaluated with the help of the formula

$$\int \mathrm{d}u\,\exp\left(\sum_i z_i^+(u^\mathrm{T}z'_i)\right) = \frac{i}{2\pi}\oint \mathrm{d}v\,\exp\left(-\frac{1}{v}+v\sum_{(ij)}(z'_i\varepsilon z'_j)(z_i^+\varepsilon^{-1}z_j^+)\right). \tag{3.8}$$

Summation is over unordered pairs (ij), i.e. (12) and (21) are not counted separately.

The result is

$$Z=\int\prod_x\left[\frac{i}{2\pi}\mathrm{d}v_x\,\mathrm{e}^{-1/v_x}\right]\mathrm{D}\mu(\zeta,z)$$
$$\times\prod_{\mathrm{b}}\sum_{j_{\mathrm{b}}}c_{j_{\mathrm{b}}}Q^{j_{\mathrm{b}}}(\zeta_{\mathrm{b},x},\zeta_{\mathrm{b},y})Q^{j_{\mathrm{b}}}(z_{\mathrm{b},y},z_{\mathrm{b},x})$$
$$\times\prod_{x\in\Lambda_{\mathrm{i}}}\exp\left[v_x\sum_{(\mathrm{b},\mathrm{b}')\wedge x}(\zeta_{\mathrm{b},x}\varepsilon\zeta_{\mathrm{b}',x})(z^+_{b,x}\varepsilon^{-1}z^+_{\mathrm{b}',x})\right]$$
$$\times\prod_{y\in\Lambda_{\mathrm{f}}}\exp\left[v_y\sum_{(\mathrm{b},\mathrm{b}')\wedge y}(z_{\mathrm{b},y}\varepsilon z_{\mathrm{b}',y})(\zeta^+_{\mathrm{b},y}\varepsilon^{-1}\zeta^+_{\mathrm{b}',y})\right] \tag{3.9}$$

$(\mathrm{b},\mathrm{b}')\wedge x$ denotes an unordered pair of links touching at the site x.

4. Formulation of the partition function in terms of rotation-invariant variables

By introducing rotation-invariant auxiliary variables, the quartic terms in the exponent can be brought into quadratic form:

$$\mathrm{e}^{v(\zeta_i\varepsilon\zeta_j)(z_i^+\varepsilon^{-1}z_j^+)}=\frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}\eta\,\mathrm{d}\bar{\eta}\,\mathrm{e}^{-\eta\bar{\eta}}\,\mathrm{e}^{\zeta_i\varepsilon\zeta_j\eta+vz_i^+\varepsilon^{-1}z_j^+\bar{\eta}}\,. \tag{4.1}$$

We associate a complex variable

$$\eta_{\mathrm{bb}'}=-\eta_{\mathrm{b}'\mathrm{b}} \tag{4.2}$$

with each pair of distinct links (b,b') that touch at a site.

The partition function is now

$$Z=\int\mathrm{D}v\,\mathrm{D}\mu(\eta)\,\mathrm{D}\mu(\zeta,z)$$
$$\times\prod_{\mathrm{b}}\sum_{j_{\mathrm{b}}}c_{j_{\mathrm{b}}}Q^{j_{\mathrm{b}}}(\zeta_{\mathrm{b},x},\zeta_{\mathrm{b},y})Q^{j_{\mathrm{b}}}(z_{\mathrm{b},y},z_{\mathrm{b},x})$$
$$\times\prod_{x\in\Lambda_{\mathrm{i}}}\exp\left[\tfrac{1}{2}\sum_{(\mathrm{b},\mathrm{b}')\wedge x}(\zeta_{\mathrm{b},x}\varepsilon\zeta_{\mathrm{b}',x}\eta_{\mathrm{bb}'}+v_xz^+_{\mathrm{b},x}\varepsilon^{-1}z^+_{\mathrm{b}',x}\bar{\eta}_{\mathrm{bb}'})\right]$$
$$\times\prod_{y\in\Lambda_{\mathrm{f}}}\exp\left[\tfrac{1}{2}\sum_{(\mathrm{b},\mathrm{b}')\wedge y}(\zeta^+_{\mathrm{b},y}\varepsilon^{-1}\zeta^+_{\mathrm{b}',y}\bar{\eta}_{\mathrm{bb}'}+v_yz_{\mathrm{b},y}\varepsilon z_{\mathrm{b}',y}\eta_{\mathrm{bb}'})\right], \tag{4.3}$$

where

$$\mathrm{D}v=\prod_x\frac{i}{2\pi}\mathrm{d}v_x\,\mathrm{e}^{-1/v_x},$$

$$\mathrm{D}\mu(\eta)=\prod_{(\mathrm{b},\mathrm{b}')}\frac{1}{\pi}\mathrm{d}\eta_{\mathrm{bb}'}\,\mathrm{d}\bar{\eta}_{\mathrm{bb}'}\,\mathrm{e}^{-\eta_{\mathrm{bb}'}\bar{\eta}_{\mathrm{bb}'}}\,.$$

The product $\prod_{(\mathrm{b},\mathrm{b}')}$ runs over unordered pairs, whereas the sum $\sum_{(\mathrm{b},\mathrm{b}')}$ in the exponent is now over ordered pairs of links. This is compensated by the factor $\frac{1}{2}$.

The projector Q^j can be represented by a complex contour integral [1]

$$Q^j(z, z') = \frac{1}{2\pi i} \oint \frac{\mathrm{d}\tau}{\tau^{2j+1}} \, \mathrm{e}^{\tau z + z'} . \tag{4.4}$$

Thus

$$\begin{aligned} Q^j(z, z') Q^j(\zeta, \zeta') = \frac{1}{(2\pi i)^2} \oint \mathrm{d}\tau \, \mathrm{d}\tau' \, \mathrm{e}^{\tau z + z' + \tau' \zeta + \zeta'} \\ \times [(\tau\tau')^{-2j-1} - (\tau\tau')^{2j+1}] . \end{aligned} \tag{4.5}$$

The term proportional to $(\tau\tau')^{2j+1}$ does not contribute to the integral but is inserted to make the summation over j feasible. The sum to evaluate is

$$B(\tau) \equiv \sum_j c_j (\tau^{-2j-1} - \tau^{2j+1}) .$$

If $|\tau|$ is chosen equal to unity, $\tau \equiv \mathrm{e}^{i\varphi/2}$, then

$$\begin{aligned} B(\tau) &= -2i \sum_j c_j \sin (2j+1)\tfrac{1}{2}\varphi = -i \sin \tfrac{1}{2}\varphi \sum_j c_j \chi_j(u) \\ &= -2i \sin \tfrac{1}{2}\varphi \, \mathrm{e}^{L(u)} , \end{aligned}$$

u being a rotation by the angle of φ:

$$u = \begin{pmatrix} \mathrm{e}^{i\varphi/2} & 0 \\ 0 & \mathrm{e}^{-i\varphi/2} \end{pmatrix} .$$

Analytic continuation leads to

$$B(\tau) = \left(\frac{1}{\tau} - \tau\right) \mathrm{e}^{(\beta/2)(\tau + 1/\tau)} . \tag{4.6}$$

Essentially, the link variable $\tilde{\tau}_\mathrm{b} \equiv \tau_\mathrm{b}\tau'_\mathrm{b}$ corresponds to the invariant rotation angle of the SU(2) element u_b. This means that some U(1) variables, U(1) being the maximal torus of SU(2), survive, whereas the rotation-variant parts of the field variables are integrated out.

The partition function now reads

$$\begin{aligned} Z = \int & \mathrm{D}v \, \mathrm{D}\mu(\eta) \, \mathrm{D}\mu(\zeta, z) \\ & \times \prod_\mathrm{b} \oint \frac{\mathrm{d}\tau_\mathrm{b} \, \mathrm{d}\tau'_\mathrm{b}}{(2\pi i)^2} B(\tau_\mathrm{b}\tau'_\mathrm{b}) \, \mathrm{e}^{\tau_\mathrm{b} \overset{+}{z}_{\mathrm{b},y} z_{\mathrm{b},x} + \tau'_\mathrm{b} \overset{+}{\zeta}_{\mathrm{b},x} \zeta_{\mathrm{b},y}} \\ & \times \prod_{x \in \Lambda_\mathrm{i}} \exp\left[\tfrac{1}{2} \sum_{(\mathrm{b},\mathrm{b}') \wedge x} (\zeta_{\mathrm{b},x} \varepsilon \zeta_{\mathrm{b}',x} \eta_{\mathrm{bb}'} + v_x \overset{+}{z}_{\mathrm{b},x} \varepsilon^{-1} \overset{+}{z}_{\mathrm{b}',x} \bar{\eta}_{\mathrm{bb}'})\right] \\ & \times \prod_{y \in \Lambda_\mathrm{f}} \exp\left[\tfrac{1}{2} \sum_{(\mathrm{b},\mathrm{b}') \wedge y} (\overset{+}{\zeta}_{\mathrm{b},y} \varepsilon^{-1} \overset{+}{\zeta}_{\mathrm{b}',y} \bar{\eta}_{\mathrm{bb}'} + v_y z_{\mathrm{b},y} \varepsilon z_{\mathrm{b}',y} \eta_{\mathrm{bb}'})\right] . \end{aligned} \tag{4.7}$$

The formula of the reproducing kernel for the Bargmann space [1, 3],

$$\int \mathrm{d}\mu(z')\, \mathrm{e}^{\bar{z}\cdot z'} f(z') = f(z)\,, \tag{4.8}$$

allows one to perform the integrations over $z_{\mathrm{b},x}$ and $\zeta_{\mathrm{b},y}$ with $x \in \Lambda_{\mathrm{i}}$, $y \in \Lambda_{\mathrm{f}}$. This amounts to the substitution

$$z_{\mathrm{b},x} \to \bar{\tau}_{\mathrm{b}} z_{\mathrm{b},y}\,, \qquad \zeta_{\mathrm{b},y} \to \bar{\tau}'_{\mathrm{b}} \zeta_{\mathrm{b},x}\,.$$

Afterwards, each link carries just one z and one ζ.

With the notation

$$\mathrm{D}'\mu(\zeta, z) = \prod_{\mathrm{b}} \mathrm{d}\mu(\zeta_{\mathrm{b}})\, \mathrm{d}\mu(z_{\mathrm{b}})\,,$$

$$z_{\mathrm{b}} \equiv z_{\mathrm{b},y}\,, \qquad \zeta_{\mathrm{b}} \equiv \zeta_{\mathrm{b},x}\,,$$

the result is

$$\begin{aligned} Z = \int & \mathrm{D}v\, \mathrm{D}\mu(\eta)\, \mathrm{D}'\mu(\zeta, z) \oint \prod_{\mathrm{b}} \frac{\mathrm{d}\tau_{\mathrm{b}}\, \mathrm{d}\tau'_{\mathrm{b}}}{(2\pi i)^2} B(\tau_{\mathrm{b}}\tau'_{\mathrm{b}}) \\ & \times \prod_{x \in \Lambda_{\mathrm{i}}} \exp\left[\tfrac{1}{2} \sum_{(\mathrm{b},\mathrm{b}')\wedge x} (\zeta_{\mathrm{b}} \varepsilon \zeta_{\mathrm{b}'} \eta_{\mathrm{bb}'} + v_x \tau_{\mathrm{b}} \tau_{\mathrm{b}'} z^{+}_{\mathrm{b}} \varepsilon^{-1} z^{+}_{\mathrm{b}'} \bar{\eta}_{\mathrm{bb}'})\right] \\ & \times \prod_{y \in \Lambda_{\mathrm{f}}} \exp\left[\tfrac{1}{2} \sum_{(\mathrm{b},\mathrm{b}')\wedge y} (\tau'_{\mathrm{b}} \tau'_{\mathrm{b}'} \zeta^{+}_{\mathrm{b}} \varepsilon^{-1} \zeta^{+}_{\mathrm{b}'} \bar{\eta}_{\mathrm{bb}'} + v_y z_{\mathrm{b}} \varepsilon z_{\mathrm{b}'} \eta_{\mathrm{bb}'})\right]. \end{aligned} \tag{4.9}$$

We define antisymmetric matrices λ, λ' and κ, κ', the elements of which are labelled by the links of the lattice.

$$\lambda_{\mathrm{bb}'} = \begin{cases} \eta_{\mathrm{bb}'}\,, & \text{if } (\mathrm{b}, \mathrm{b}') \wedge x \text{ for any } x \in \Lambda_{\mathrm{i}}\,, \\ 0\,, & \text{otherwise}\,; \end{cases}$$

$$\kappa_{\mathrm{bb}'} = \begin{cases} \tau'_{\mathrm{b}} \tau'_{\mathrm{b}'} \bar{\eta}_{\mathrm{bb}'}\,, & \text{if } (\mathrm{b}, \mathrm{b}') \wedge y \text{ for any } y \in \Lambda_{\mathrm{f}}\,, \\ 0\,, & \text{otherwise}\,; \end{cases}$$

$$\lambda'_{\mathrm{bb}'} = \begin{cases} v_y \eta_{\mathrm{bb}'}\,, & \text{if } (\mathrm{b}, \mathrm{b}') \wedge y \text{ for any } y \in \Lambda_{\mathrm{f}}\,, \\ 0\,, & \text{otherwise}\,; \end{cases}$$

$$\kappa'_{\mathrm{bb}'} = \begin{cases} v_x \tau_{\mathrm{b}} \tau_{\mathrm{b}'} \bar{\eta}_{\mathrm{bb}'}\,, & \text{if } (\mathrm{b}, \mathrm{b}') \wedge x \text{ for any } x \in \Lambda_{\mathrm{i}}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Application of the formula [1, 7]

$$\int \prod_k \mathrm{d}\mu(z_k) \exp \tfrac{1}{2} \sum_{ij} (z_i \varepsilon z_j \lambda_{ij} + z^{+}_j \varepsilon^{-1} z^{+}_j \kappa_{ij}) = \det(1 - \lambda\kappa)^{-1}\,, \tag{4.10}$$

valid for antisymmetric matrices λ and κ, leads to

$$Z=\int \mathrm{D}v\, \mathrm{D}\mu(\eta) \oint \prod_{\mathrm{b}} \frac{\mathrm{d}\tau_{\mathrm{b}}\, \mathrm{d}\tau'_{\mathrm{b}}}{(2\pi i)^2} B(\tau_{\mathrm{b}}\tau'_{\mathrm{b}}) \det(1-\lambda\kappa)^{-1} \det(1-\lambda'\kappa')^{-1}\,. \tag{4.11}$$

At this stage, the partition function involves only rotation invariant variables.

5. Loop expansion

The formula

$$\det(1-\lambda\kappa)^{-1}=\mathrm{e}^{-\operatorname{tr}\ln(1-\lambda\kappa)} \tag{5.1}$$

can be used to derive a loop expansion. The circular path of the line integrals over τ variables should be chosen such that no cut of the logarithm is crossed. Since $\kappa^{(\prime)}\to 0$ when $\tau, \tau'\to 0$, this is possible, but it requires that the η-integrations are performed after the τ-integrations only. Keeping this restriction in mind, we may expand

$$\det(1-\lambda\kappa)^{-1}=\exp \sum_{n\geq 1} \frac{1}{n} \operatorname{tr}(\lambda\kappa)^n\,. \tag{5.2}$$

The contributions to $(\lambda\kappa)^n$ can be represented by graphs composed of $2n$ double links which belong alternatingly to points of Λ_{i} and Λ_{f}. Therefore, the graph of lowest, i.e. second, order is a plaquette.

Consider an oriented closed path C of order n_{C}, consisting of double links $(\mathrm{b}_1, \mathrm{b}_2)$, $(\mathrm{b}_2, \mathrm{b}_3) \cdots (\mathrm{b}_{2n}, \mathrm{b}_1)$. The algebraic expression corresponding to C involves

$$\eta(\mathrm{C})\equiv \eta_{\mathrm{b}_1\mathrm{b}_2}(x_1)\bar{\eta}_{\mathrm{b}_2\mathrm{b}_3}(y_2)\cdots \bar{\eta}_{\mathrm{b}_{2n}\mathrm{b}_1}(y_{2n})\,, \tag{5.3}$$

$$\tau'(\mathrm{C})\equiv \tau'_{\mathrm{b}_1}\tau'_{\mathrm{b}_2}\cdots \tau'_{\mathrm{b}_{2n}}\,. \tag{5.4}$$

The same path C appears in $(\lambda'\kappa')^n$, but τ'_{b} is replaced by τ_{b}, $\eta_{\mathrm{bb'}}(x)$ by $\bar{\eta}_{\mathrm{bb'}}(x)$ and each site carries a factor v_x.

The determinants are thus replaced by the loop expansion

$$\det(1-\lambda\kappa)^{-1}\det(1-\lambda'\kappa')^{-1}=\exp\sum_{\mathrm{C}}\frac{1}{n_{\mathrm{C}}}\left[\tau'(\mathrm{C})\eta(\mathrm{C})+\tau(\mathrm{C})\bar{\eta}(\mathrm{C})\prod_{x\in\mathrm{C}} v_x\right]. \tag{5.5}$$

The sum extends over all oriented closed paths C that visit sites of Λ_{i} and Λ_{f} alternatingly, i.e. no spikes like $\perp$ are possible. Paths which contain the same double links but start at different sites are not identified. After the variable transformation

$$\eta_{\mathrm{bb'}}(x)\to\begin{cases}\left(\dfrac{\tau_{\mathrm{b}}v_x}{\tau'_{\mathrm{b}}}\right)^{1/2}\eta_{\mathrm{bb'}}(x)\,, & x\in\Lambda_{\mathrm{i}}\,,\\[2ex] \left(\dfrac{\tau_{\mathrm{b}}v_x}{\tau'_{\mathrm{b}}}\right)^{-1/2}\eta_{\mathrm{bb'}}(x)\,, & x\in\Lambda_{\mathrm{f}}\,,\end{cases}$$

setting $|\tau_b v_x/\tau'_b| = 1$, and integration over τ', we finally arrive at

$$Z = \int \mathrm{D}v\, \mathrm{D}\mu(\eta) \oint \prod_b \frac{d\tilde{\tau}_b}{2\pi i} B(\tilde{\tau}_b) \exp \sum_C \frac{1}{n_C} \tau(C) v(C)[\eta(C) + \bar{\eta}(C)], \tag{5.6}$$

where

$$\tau(C) \equiv \prod_{b \in C} \tilde{\tau}_b^{1/2}, \qquad v(C) \equiv \prod_{x \in C} v_x^{1/2},$$

and $\bar{\eta}(C)$ is the complex conjugate of $\eta(C)$.

We are thus led to a system of closed loops and a hamiltonian which is complex and, surprisingly, invariant under U(1) gauge transformations:

$$\eta_{bb'}(x) \to \eta_{bb'}(x) g_b(x) g_{b'}(x).$$

Inserting

$$B(\tau) = \sum_j c_j \tau^{-2j-1}$$

into the partition function (5.6) we may integrate over all variables, thus reproducing the standard high-temperature expansion. In the usual derivation of this expansion one has to deal with Clebsch–Gordan series, and Clebsch–Gordan coefficients are involved in the computation of complicated graphs. They do not appear here. Instead, our method amounts to counting all paths which may be built out of a given set of links.

6. Formulation of the Heisenberg ferromagnet as a local gauge theory

One could think of regarding the phase of $\eta_{bb'}$ as a parallel transporter of a lattice gauge theory on a lattice whose sites are our links (see fig. 2). However, this is not possible because the η satisfy the antisymmetry condition (4.2) instead of $\eta_{bb'}(x) = \bar{\eta}_{b'b}(x)$. Therefore the parallel transporter would not go over into its inverse under reversal of the direction of the link.

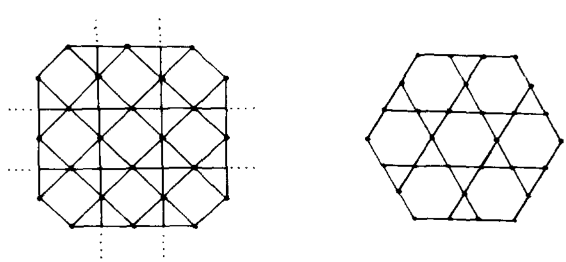

Fig. 2.

However, it is possible to transform our system into a local gauge theory with a nonabelian but solvable gauge group $Z_2 \otimes U(1) = T^*$. Separating the η phase

$$\eta \equiv r\vartheta, \qquad \vartheta \equiv e^{i\varphi},$$
$$D\mu(\eta) = D\mu(r)\, D\vartheta,$$

with

$$D\mu(r) \equiv \prod_{(bb')} \tfrac{1}{2} r_{bb'}\, dr_{bb'}\, e^{-r^2_{bb'}},$$

$$D\vartheta \equiv \prod_{(bb')} \frac{1}{2\pi} d\varphi_{bb'},$$

the partition function reads

$$Z = \int Dv\, D\tau\, D\mu(r)\, D\vartheta \exp\left(\sum_C \frac{1}{n_C} \tau(C) v(C) r(C)[\vartheta(C) + \text{c.c.}]\right). \qquad (6.1)$$

If we define 2×2 matrices

$$t_{bb'} = \begin{pmatrix} \vartheta_{bb'} & 0 \\ 0 & \bar{\vartheta}_{bb'} \end{pmatrix},$$

which are elements of the maximal torus $T = U(1)$ of SU(2) we may write

$$\vartheta(C) + \text{c.c.} = \operatorname{tr} t(C), \qquad t(C) \equiv t_{b_1 b_2}(x_1) t^*_{b_2 b_3}(y_2) \cdots t^*_{b_{2n} b_1}(y_{2n}). \qquad (6.2)$$

The Weyl group $W = T^*/T$ of SU(2), where T^* is the normalizer of T, consists of two elements

$$W = \{1, \hat{\tau}\},$$

where

$$1 \equiv T, \qquad \hat{\tau} \equiv T\tau, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

τ corresponds to complex conjugation:

$$\tau t \tau^{-1} = t^*, \qquad \tau^{-1} = -\tau.$$

We may thus write

$$t(C) = (-1)^{n_C} t_{b_1 b_2}(x_1) \tau t_{b_2 b_3}(y_2) \tau \cdots \tau t_{b_{2n} b_1}(y_{2n}) \tau. \qquad (6.3)$$

We regard the variables $t\tau \in T^*$ as the new variables of our system. They show the desired behavior under U(1) gauge transformations:

$$(t\tau)_{bb'} \to g_b t_{bb'} g_{b'} \tau = g_b (t\tau)_{bb'} g_{b'}^{-1}, \qquad g \in T, \qquad (6.4)$$

$$(t\tau)_{bb'} = (t\tau)^{-1}_{b'b}. \qquad (6.5)$$

We introduce T* variables

$$u_{\mathrm{bb'}} \equiv t_{\mathrm{bb'}}\sigma_{\mathrm{bb'}}\,, \qquad t \in T\,, \qquad \sigma \in \{1, \tau\}\,. \tag{6.6}$$

In contrast to $\eta_{\mathrm{bb'}}$ they satisfy

$$u_{\mathrm{bb'}} = u_{\mathrm{b'b}}^{-1}\,. \tag{6.7}$$

The partition function in terms of these new variables is

$$Z = \int \mathrm{D}'u\, \mathrm{D}v\, \mathrm{D}\mu(r)\, \mathrm{D}\tau \exp\left(\sum_{\mathrm{C}} \frac{(-1)^{n_{\mathrm{C}}}}{n_{\mathrm{C}}} \tau(\mathrm{C}) v(\mathrm{C}) r(\mathrm{C}) \operatorname{tr} u(\mathrm{C})\right), \tag{6.8}$$

where

$$\mathrm{D}'u \equiv \prod_{(\mathrm{bb'})} \mathrm{d}u_{\mathrm{bb'}} \delta_\tau(u_{\mathrm{bb'}})\,.$$

$\mathrm{d}u$ is a Haar measure on T*, viz.

$$\int \mathrm{d}u\, f(u) = \int \mathrm{d}t \sum_{\sigma=1,\tau} f(t\sigma)\,,$$

and

$$\delta_\sigma(u) = \begin{cases} 1, & \text{if } \sigma' = \sigma\,, \\ 0, & \text{if } \sigma' \neq \sigma\,, \end{cases} \qquad u = t\sigma'\,.$$

The exponential is invariant under T* transformations so that each configuration which is obtained from the original one by a W transformation

$$u_{\mathrm{bb'}} \to \sigma_{\mathrm{b}} u_{\mathrm{bb'}} \sigma_{\mathrm{b'}}^{-1}\,, \qquad \sigma \in \{1, \tau\}\,,$$

gives the same contribution.

The allowed configurations are determined by the gauge-invariant constraints

$$\delta_\tau(u_{b_1b_2} u_{\mathrm{b_2b_3}} u_{\mathrm{b_3b_1}}) = 1\,, \qquad \text{for each triangle}\,;$$

$$\delta_1(u_{\mathrm{b_1b_2}} u_{\mathrm{b_2b_3}} u_{\mathrm{b_3b_4}} u_{\mathrm{b_4b_1}}) = 1\,, \qquad \text{for each quadrangle}\,,$$

if we start on a quadratic lattice.

Thus we arrive at the partition function of a lattice gauge theory with local T* invariance:

$$Z = \int \mathrm{D}u\, \mathrm{D}v\, \mathrm{D}\mu(r)\, \mathrm{D}\tau \exp\left(\sum_{\mathrm{C}} \frac{(-1)^{n_{\mathrm{C}}}}{n_{\mathrm{C}}} \tau(\mathrm{C}) v(\mathrm{C}) r(\mathrm{C}) \operatorname{tr} u(\mathrm{C})\right), \tag{6.9}$$

with

$$\mathrm{D}u = 2^{-2N} \prod_{(\mathrm{bb'})} \mathrm{d}u_{\mathrm{bb'}} \prod_{\triangle} \delta_\tau(uuu) \prod_{\diamond} \delta_1(uuuu)\,,$$

where N is the number of sites of the original lattice.

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