Nuclear Physics B196 (1982) 62–82

LOCAL OBSERVABLES IN NON-ABELIAN GAUGE THEORIES

H.S. SHARATCHANDRA[1]

Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

Received 18 September 1981

Labelling of the physical states of a non-abelian gauge theory on a lattice in terms of local observables is considered. The labelling is in terms of local color electric field observables and (separately) local color magnetic field observables. Matter fields are also included. Non-local observables required when space is multiply connected, are specified. The non-abelian version of Stokes' theorem is considered. Relevance to the continuum theory is discussed in detail.

1. Introduction

Gauge theories form the underlying framework for both strong and electroweak interactions. Therefore it is imperative that we understand their structure as well as possible. When considering the observables, they possess a special status among all field theory models. The dynamics is formulated in terms of the potentials but these are not the observables of the system. Only gauge-invariant quantities are observable. It has been shown [1] both in the continuum and the lattice theories that the set of all Wilson loops forms a sufficient set of observables of the system. But obviously this is not a minimal set. Moreover, large Wilson loops are not local observables and it is not evident that a set of local measurements is enough to determine the state of the system. The question then arises as to what is the minimal set of (local) measurements to be made to determine the state of the system. We consider such questions in this paper.

We begin with the problem of labelling the physical subspace of the Hilbert space by gauge-invariant local observables. Our considerations are mostly in lattice gauge theories in the hamiltonian formalism [2] because in this way our results become precise and we are able to evade a variety of technical difficulties of working with the Hilbert space of the continuum theory. (However, we will give a detailed discussion of the relevance of our results to the continuum theory.) In sect. 2 we consider labelling of the physical states using color electric field observables. For this we first label the extended Hilbert space by a color electric field and then obtain the physical states by implementing the non-abelian Gauss' law. (For such attempts in continuum theory see ref. [3].) Since the color components of the

[1] Supported by Alexander von Humboldt Stiftung.

electric field form the generators of a Lie group, the problem reduces to a Kronecker product decomposition. The labelling [eq. (2.22)] finally involves the $(N-1)$ color invariants [for the SU(N) group] for each space component of the electric field and, in addition, the invariants involving certain covariant derivatives of the electric field. (For $N>2$, it is also necessary to specify some symmetry properties under the interchange of the electric field operators.) We also consider the labelling problem when matter fields are present, matter being described by the color spin, apart from the spin labels and other global quantum numbers. In case there are fields with different global quantum numbers at the same point, the labelling involves the net color spin of various combinations of such fields.

In sect. 3 we consider labelling in terms of color magnetic field observables. We obtain the generalization of the result in the abelian case where it is sufficient to specify two components of the magnetic field, say B_x and B_y (in 3-space dimensions), plus some boundary data on B_z. The analogue for the non-abelian theory involves invariants of both B_x and B_y and their covariant derivatives along the z-axis. (Some boundary data involving B_z are also needed.) We show that in the generic case it is sufficient to specify Wilson loops involving at most three adjacent plaquettes. By the generic case we mean those states in which the color magnetic fields at neighboring points do not have a special relationship to each other, e.g. the covariant derivative is not zero. (For a precise definition see the text.) Non-generic states are expected to belong to a set of measure zero in the Hilbert space [4] and are perhaps not very relevant. Nevertheless, they illustrate the point that sometimes non-local observables are necessary in order to specify the state completely. This is by now well known [4, 5]. We also consider the labelling when matter fields are present, the matter field now being described by the configuration space. In the generic case this involves observables like $\varphi^* D_{i_1} D_{i_2} \ldots D_{i_n} \varphi$.

In sect. 3 it is presumed that the space is simply connected. In sect. 4 we consider multiply connected regions, relevant for a Bohm–Aharanov experiment. In the abelian theory it is well known that a complete specification of the state includes the net flux through the hole. We obtain such non-local observables required in the non-abelian theory. This involves a set of N^2 Wilson loops going around the hole.

In sect. 5 we consider the non-abelian Stokes' theorem. Versions of such a theorem have been proposed in literature before [6]. However, we take a different interpretation of what is meant by Stokes' theorem. Given gauge-invariant *local* data entirely on some surface spanning a loop, can we reconstruct all observables involving the loop? If yes, what is the kind of data to be specified on the surface? In contrast, earlier versions of the theorem consider non-local data on the surface. They compare the magnetic fields at different points on the surface by a parallel transport along some arbitrary path. We obtain one kind of local data to be specified. Whereas in the abelian case just the magnetic field on the surface suffices, the non-abelian case requires certain invariants involving certain covariant derivatives of the magnetic field along the surface.

All previous sections are concerned with gauge theory on a lattice. One may expect most of the results to carry over to the continuum theory basically unchanged. But crucial simplifications can occur for the continuum theory. Such issues are discussed in sect. 6.

In appendix A we prove a result used in the text. For the sake of completeness, we mention the minimal complete local gauge-invariant data for the classical (lattice) theory in appendix B.

2. Labelling the physical states by local color electric field observables

Consider a $(d+1)$-dimensional lattice gauge theory in the hamiltonian formalism [2]. We have a d-dimensional hypercubic lattice labelled by a d-tuple of ordered integers, $n=\{n_i\}$, $i=1-d$. The coordinate variables are a matrix $U(ni)\in \mathrm{SU}(N)$ for each link (ni), where i stands for the positive unit vector along ith direction. The conjugate variables are $E(ni)\in \mathscr{S}\mathrm{U}(N)$ defined at each site n of the lattice. Here $\mathscr{S}\mathrm{U}(N)$ denotes the Lie algebra of $\mathrm{SU}(N)$. The basic equal-time commutation relation is

$$[E(ni)^{\alpha}, (U(ni))_{IJ}] = (T^{\alpha}U(ni))_{IJ}, \tag{2.1}$$

other commutators being zero. (Unless otherwise specified all our variables will refer to the same time henceforth.) In eq. (2.1), α labels the adjoint representation and I the fundamental representation. T^{α} are the generators of the fundamental representation of $\mathrm{SU}(N)$:

$$[T^{\alpha}, T^{\beta}] = if_{\alpha\beta\gamma}T^{\gamma}. \tag{2.2}$$

Thus E generates an infinitesimal group transformation of U on the left. For self-consistency (e.g. the validity of Jacobi identities among E^{α}, E^{β} and U) we require

$$[E(ni)^{\alpha}, E(ni)^{\beta}] = -if_{\alpha\beta\gamma}E(ni)^{\gamma}, \tag{2.3}$$

so that E^{α} obey the commutation relations of $-T^{\alpha}$. Note that E^{α} do not commute among themselves in contrast with the continuum theory.

Given the set E, U we can construct [2] a generator $\mathscr{E}_{ni}$ of infinitesimal right transformations:

$$\mathscr{E}(ni)^{\alpha} = E(ni)^{\beta}(U^{(1)}(ni))_{\beta\alpha}, \tag{2.4}$$

$$[\mathscr{E}(ni)^{\alpha}, (U(ni))_{IJ}] = (U(ni)T^{\alpha})_{IJ}, \tag{2.5}$$

where $U^{(1)}$ is the adjoint representation for U:

$$(U^{(1)})_{\alpha\beta} = \mathrm{tr}\,(T^{\alpha}UT^{\beta}U^{\dagger}), \qquad U^{(1)\dagger} = U^{(1)}, \tag{2.6}$$

$\mathscr{E}^{\alpha}$ and E^{α} commute with each other. $\mathscr{E}^{\alpha}$ as defined above is not hermitian. The satisfactory modification is

$$\mathscr{E}(ni) = U(ni)^{\dagger}E(ni)U(ni) - T^{2}, \tag{2.7}$$

where

$$E = T^{\alpha} E^{\alpha}, \qquad \mathscr{E} = T^{\alpha} \mathscr{E}^{\alpha}. \tag{2.8}$$

Subtraction of T^2 in eq. (2.7) is necessary to have a common spectrum for E and $\mathscr{E}$.

The hamiltonian for our system is

$$H = \tfrac{1}{2} g^2 \sum_{ni} \operatorname{tr} (E(ni)^2) + \frac{1}{4g^2} \sum_{nij} \operatorname{tr} (U(ni) U(n+i, j) \times U(n+j, i)^{\dagger} U(nj)^{\dagger} + \text{h.c.}). \tag{2.9}$$

This hamiltonian and the commutation relations are invariant under the local gauge transformations

$$\begin{aligned} U(ni) &\to V(n) U(ni) V(n+i)^{\dagger}, \\ E(ni) &\to V(n) E(ni) V(n)^{\dagger}. \end{aligned} \tag{2.10}$$

Note that $E(ni)$ transforms as though it is located at the site n. This is required (e.g.) by the validity of the commutation relations (2.1) among the transformed variables. From (2.10) it follows that

$$\mathscr{E}(ni) \to V(n+i) \mathscr{E}(ni) V(n+i)^{\dagger}, \tag{2.11}$$

so that $\mathscr{E}(ni)$ behaves like an object located at $(n+i)$. The transformations (2.10) and (2.11) are generated by

$$G = \exp \left[i \sum_{n\alpha} \theta_n^{\alpha} \sum_i (E(ni)^{\alpha} - \mathscr{E}(n-i, i)^{\alpha}) \right]. \tag{2.12}$$

Since $\mathscr{E}(n-i, i)^{\alpha}$ is $E(n-i, i)^{\alpha}$ shifted to the site n by a parallel transport along i, the difference in the exponent in (2.12) is the discrete version of the covariant derivative $(D_i E(ni))^{\alpha}$. We will use this compact notation henceforth.

The physical states of the lattice theory have to satisfy the (non-abelian) Gauss' law:

$$\left(\sum_i D_i E(ni) \right)^{\alpha} | \rangle = 0, \tag{2.13}$$

i.e. the physical states must be in the singlet representation of $\sum_i D_i E(ni)$ at each lattice site. To construct such states we will first define an extended Hilbert space. Since E^{α}'s do not commute among themselves, they cannot be simultaneously diagonalized. However, since $-E^{\alpha}$'s obey the commutation relations of the SU(N) Lie algebra, the states can be labelled by the representation spaces of the algebra. A convenient labelling is as follows:

(a) $N-1$ invariants of SU(N) label the irreducible representations of the group. These are

$$\sum_{\alpha} E^{\alpha}E^{\alpha},$$

$$\sum_{\alpha\beta\gamma} d_{\alpha\beta\gamma}E^{\alpha}E^{\beta}E^{\gamma},$$

$$\sum_{\alpha\beta\gamma\delta\epsilon} d_{\alpha\beta\gamma}d_{\gamma\delta\epsilon}E^{\alpha}E^{\beta}E^{\delta}E^{\epsilon},\ldots,$$

and are obviously gauge-invariant local variables. These variables characterize the Young tableau uniquely and just count the number of boxes in the $N-1$ rows of the tableau.

(b) To uniquely label the states of a given irreducible representation construct any chain of subgroups

$$\mathrm{U}(1)\times\mathrm{SU}(N-1)/\mathrm{Z}(n-1)\subset\mathrm{SU}(N),$$

$$\mathrm{U}(1)\times\mathrm{SU}(N-2)/\mathrm{Z}(n-2)\subset\mathrm{SU}(N-1)/\mathrm{Z}(N-1),$$

$$\ldots$$

$$\mathrm{U}(1)\subset\mathrm{SU}(2)/\mathrm{Z}(2).$$

The $N-1$ invariants of $\mathrm{U}(1)\times\mathrm{SU}(N-1)/\mathrm{Z}(N-1)$, the $N-2$ invariants of $\mathrm{U}(1)\times\mathrm{SU}(N-2)/\mathrm{Z}(N-2)$, etc. and finally the invariant of U(1) of this chain label all the states. These labels are not gauge invariant as they depend on the choice of the chain.

We denote the labels of type (a) by $E(ni)^k$ and of type (b) by $E(ni)^A$.

The singlet state $|0\rangle$ of every $E(ni)$,

$$E(ni)^{\alpha}|0\rangle = 0, \tag{2.14}$$

is obviously gauge invariant. It is the strong coupling vacuum. All states of the extended Hilbert space are obtained by applying products of $U(ni)_{IJ}$ and $U(ni)^{\dagger}_{KL}$ on $|0\rangle$. Our choice of labelling corresponds to characterizing the initial set of indices I. But this is not sufficient: the final set of indices must also be characterized, so that we need the labels of $\mathscr{E}(ni)$ too. However, once the initial set of indices in

$$c_{I_1I_2\ldots}U(ni)_{I_1J_1}U(ni)_{I_2J_2}\ldots$$

has a certain symmetry and antisymmetry property, so does the final set of indices. Therefore, the labels of the invariants $E(ni)^k$ and $\mathscr{E}(ni)^k$ are identical. Thus a unique minimal labelling of the extended Hilbert space is

$$|\{E(ni)^k(=\mathscr{E}(ni)^k),\ E(ni)^A,\ \mathscr{E}(ni)^A\}\rangle. \tag{2.15}$$

We will now obtain a labelling for the physical subspace. Using the Kronecker product decomposition, we relabel the extended Hilbert space in terms of ($E(ni)$ –

$\mathscr{E}(n-i, i)$). The result is

$$|\{E(ni)^k, (D_iE(ni))^K, (D_iE(ni))^A\rangle \tag{2.16}$$

(no sum over i). $(D_iE(ni))^K$ label the irreducible representations obtained from the direct product of the representations of $E(ni)$ and $\mathscr{E}(n-i, i)$. They are not just the $(N-1)$ invariants of the SU(N) group, except for the case $N=2$. When $N \neq 2$, the same representation may appear more than once in a direct product decomposition. To distinguish between such representations we cannot use invariant numbers but rather we have to specify symmetry properties under interchanges of the labels. A familiar example is provided by the SU(3) case:

$$\underline{8} \times 8 = 1 + \underline{8} + \underline{8} + \underline{10} + 10^* + \underline{27}\,. \tag{2.17}$$

The representation $\underline{8}$ occurs twice on the right-hand side. If we denote the two representations on the left-hand side by $|A\rangle$ and $|B\rangle$, the two 8's on the right-hand side correspond to

$$d_{\alpha\beta\gamma}|A_\beta\rangle|B_\gamma\rangle\,, \qquad f_{\alpha\beta\gamma}|A_\beta\rangle|B_\gamma\rangle\,, \tag{2.18}$$

and can be distinguished only by symmetry/antisymmetry under $A \leftrightarrow B$. We will not worry about how such distinctions are to be made for an arbitrary group but tacitly assume that the label K includes them.

For $d=2$, we again make the Kronecker product decomposition:

$$\left|\left\{E(ni)^k, (D_iE(ni))^K, \left(\sum_i D_iE(ni)\right)^K, \left(\sum_i D_iE(ni)\right)^A\right\}\right\rangle\,. \tag{2.19}$$

The physical subspace is a singlet of $(\sum_i D_iE(ni))$ and hence easily isolated. The condition requires that $D_1E(n1)$ and $D_2E(n2)$ be in conjugate representations. Therefore it is sufficient to specify the subset k of the labels K [i.e. the SU(N) invariants] for just one of them. However, the labels distinguishing identical representations of the Kronecker product decomposition have to be specified for both. Henceforth we will denote such labels by λ. Hence the physical states are labelled by

$$|\{E(ni)^k, (D_1E(n1))^k, (D_1E(n1))^\lambda, (D_2E(n2))^\lambda\}\rangle\,. \tag{2.20}$$

We have therefore obtained a labelling by gauge-invariant local observables.

For $d=3$ we have to go one step further. First we combine D_1E_1 and D_2E_2 and then the result with D_3E_3. The extended Hilbert space is

$$\left|\left\{E(n1)^k, (D_iE(ni))^K, (D_1E(n1)+D_2E(n2))^K, \left(\sum_i D_iE(ni)\right)^K, \left(\sum_i D_iE(ni)\right)^A\right\}\right\rangle\,. \tag{2.21}$$

Thus the physical subspace is

$$|\{E(ni)^k (=\mathscr{E}(ni)^k), (D_iE(ni))^K, (D_1E(n1)+D_2E(n2))^\lambda\}\rangle\,. \tag{2.22}$$

We could have followed an alternate chain in decomposing the Kronecker product.

For $d = 2$ we may use

$$E(n1)+E(n2),\ \mathscr{E}(n-1,1)+\mathscr{E}(n-2,2)$$
$$\to D_1E(n1)+D_2E(n2)\,.$$

Then the physical states are labelled by

$$\left|\left\{E(ni)^k, \left(\sum_i E(ni)\right)^K, \left(\sum_i \mathscr{E}(n-i,i)\right)^\lambda\right\}\right\rangle. \tag{2.23}$$

For $d = 3$ we get

$$\left|\left\{E(ni)^k, (E(n1)+E(n2))^K, (\mathscr{E}(n-1,1)+\mathscr{E}(n-2,2))^K,\right.\right.$$
$$\left.\left.\left(\sum_i E(ni)\right)^K, \left(\sum_i \mathscr{E}(n-i,i)\right)^\lambda\right\}\right\rangle. \tag{2.24}$$

Invariance like $E(ni)^k$, $(E(n1)+E(n2))^K$ specify the color flux at a point whereas invariants like $(D_iE(ni))^K$ and $(\mathscr{E}(n-1,1)+\mathscr{E}(n-2,2))^k$ specify the correlation in color electric fields at neighboring points. It is not possible to find a labelling which does not require specifying the correlation in color electric fields at adjacent points. This is in marked contrast to the labelling in the continuum theory (sect. 6). The difference is due to the different commutation relations.

For U(1) lattice gauge theory, E and $\mathscr{E}$ are identical and are by themselves gauge invariant. The physical Hilbert space is labelled by the electric field on each link with the net electric field flowing into each point being zero (Gauss' law), i.e. the label is

$$|E(ni);\ \Delta_iE(ni)=0\rangle\,.$$

Thus for $d = 3$, the label is $|E(n1), E(n2)\rangle$ [plus some boundary data for $E(n3)$].

We remark that our labelling gives an orthonormal basis for the strong coupling expansion. It is possible to rewrite the hamiltonian in this space directly. However, it is complicated and does not seem to serve any purpose. To search for a hamiltonian that is simpler in this space which may have the correct continuum limit is futile.

We will now include matter fields in the above analysis. As an illustrative example we will consider fermions in some representation R of the color group. The form of the hamiltonian is irrelevant. We have fermions $\psi(n)_{a,A}$ and $\psi(n)^\dagger_{a,A}$ at each site n with

$$\{\psi(n)_{a,A},\ \psi(m)^\dagger_{b,B}\} = \delta_{n,m}\delta_{A,B}\delta_{a,b}\,.$$

Here a labels the Dirac components and A the representation space of R. Under local gauge transformations

$$\psi(n)_{a,A} \to \exp\left(i\theta(n)^\alpha T^\alpha_{\mathrm{R}}\right)_{AB}\psi(n)_{a,B}\,, \tag{2.25}$$

where T_R are the generators in the representation R. The state $|0\rangle$ defined by

$$\psi(n)_{a,A}|0\rangle = 0 \tag{2.26}$$

in addition to (2.14) is still gauge invariant. The extended Hilbert space is now constructed by applying $\psi(n)^\dagger$ in addition to $U(ni)$ and $U^\dagger(ni)$ on $|0\rangle$. Thus the matter fields are characterized by the states of the representation R and by the set of Dirac indices at each site. We thus have the additional labels

$$T(n,p)^k, \qquad T(n,p)^A, \tag{2.27}$$

where $p = 1, \ldots, P$ labels the Dirac components (and other global quantum numbers). Again considering the Kronecker product we get an equivalent set

$$\begin{aligned} &T(n,p)^k, \\ &(T(n,1)+T(n,2))^K, \qquad (T(n,3)+T(n,4))^K, \ldots, \\ &\left(\sum_1^4 T(n,p)\right)^K, \qquad \left(\sum_5^8 T(n,p)\right)^K, \ldots, \\ &\left(\sum_1^P T(n,p)\right)^K, \qquad \left(\sum_1^P T(n,p)\right)^A. \end{aligned} \tag{2.28}$$

We will now combine $\sum_p T(n,p)$ with $\sum_i D_i E(ni)$ to get

$$\begin{aligned} \Big| &E(ni)^k, (D_iE(ni))^K, \ldots, \\ &T(n,p)^k, (T(n1)+T(n2))^K, \ldots, \left(\sum_1^P T(n,p)\right)^K, \\ &\left(\sum_i D_iE(ni)+\sum_p T(n,p)\right)^K, \left(\sum_i D_iE(ni)+\sum_p T(n,p)\right)^A \Big\rangle. \end{aligned} \tag{2.29}$$

Gauss' theorem in presence of matter fields is

$$\sum_i D_iE(ni)+\sum_p T(n,p) = 0, \tag{2.30}$$

so that the physical states are labelled by

$$\begin{aligned} \Big| &E(ni)^k, (D_iE(ni))^K, (D_1E(n1)+D_2E(n2))^K, \\ &\left(\sum_i D_iE(ni)\right)^K, T(n,p)^k, (T(n1)+T(n2))^K, \ldots \left(\sum^P T(n,p)\right)^A \Big\rangle. \end{aligned} \tag{2.31}$$

Thus the matter fields are characterized in a gauge-invariant way by their color spin and by the net color spin of various subsets of such fields at a given point with different (Lorentz) spin components (and possibly the indices of global symmetries).

3. Labelling the physical states by local color magnetic field observables

The extended Hilbert space in terms of the link-variables is simply the configuration space $|\{U(ni)\}\rangle$. For the physical states we require

$$\exp\left(i\sum_n \theta(n)\sum_i D_i E(ni)\right)|\ \rangle = |\ \rangle\,. \tag{3.1}$$

Since

$$\exp\left(i\sum_n \theta(n)\sum_i D_i E(ni)\right)|\{U(ni)\}\rangle = |\{V(n)U(ni)V(n+i)^\dagger\}\rangle\,, \tag{3.2}$$

where $V = \exp i\theta\cdot T$, we can form such states simply from the extended Hilbert space by averaging over the gauge transformations at each lattice site:

$$|\ \rangle = \prod_n \int \mathscr{D}V(n)|\{V(n)U(ni)V(n+i)^\dagger\}\rangle\,, \tag{3.3}$$

where $\mathscr{D}V(n)$ is the group-invariant measure. Thus the physical states may be characterized by the equivalence class

$$|\{U(ni)\}\rangle \sim |\{V(n)U(ni)V(n+i)^\dagger\}\rangle\,. \tag{3.4}$$

In order to label this equivalence class in terms of gauge-invariant quantities we first consider a representative member of each equivalence class obtained by a complete gauge fixing. As the boundary plays a crucial role, we choose a finite lattice with free boundary conditions. We first consider the case of a simply connected region, a cube to be specific. For $d = 2$ we may choose $V(n)$'s such that all $U(n2)$ are transformed to 1 as also all $U(n1)$ on one edge [which we will denote by $n_2 = 0$ (fig. 1)]. $V(n)$ at 0 (to be denoted by $n = 0$) is still not chosen. This freedom is equivalent to the equivalence of the gauge-fixed set under a global gauge transformation since we may make a global transformation without altering any of the links chosen to be 1. Thus the physical states are characterized by the equivalence class

$$|\{U(n1)\};\ n_2 \neq 0\rangle \sim |\{\Omega U(n1)\Omega^\dagger\},\ n_2 \neq 0\rangle\,. \tag{3.5}$$

It will prove to be useful to rewrite this class as

$$|\{U(n)\}\rangle \text{ modulo a global transformation}\,, \tag{3.6}$$

where

$$U(n) = U(n1)U(n+e_2, 1)^\dagger \tag{3.7}$$

are the plaquette variables. Now we do not have the restriction $n_2 \neq 0$. Note that we have used an anti-clockwise orientation for our plaquette variables starting from the "smallest" plaquette coordinate (where n_1 is defined to be smaller than n_2 if $n_{1i} - n_{2i} < 0$ for the smallest i for which this quantity is non-zero).

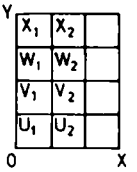

Fig. 1. Gauge fixing in the 2d case.

The class invariants in eq. (3.5) has the form tr $(U(n1)^{r(n)}U(m1)^{r(m)}\ldots)$. Such objects can be readily rewritten in a gauge-invariant form by a suitable insertion of $U(n2)$ variables. Our problem is to label the equivalence classes in eq. (3.5) by class invariants which, when rewritten in a gauge invariant form, are local. We will see below that a lattice with just three plaquettes contains all the features of the general problem. We will therefore first consider 1-, 2- and 3-plaquette cases:

1-plaquette world. The physical Hilbert space may be labelled the equivalence class

$$|U\rangle \sim |\Omega U \Omega^{\dagger}\rangle , \tag{3.8}$$

and hence by the eigenvalues

$$\lambda_i = e^{i\theta_i}, \qquad \prod_i \lambda_i = 1 , \qquad i = 1, \ldots, N. \tag{3.9}$$

These eigenvalues are obtained as the roots of the equation

$$\lambda^N - \{U\}\lambda^{N-1} + \tfrac{1}{2}\{U^2\}\lambda^{N-2} - \cdots + \frac{(-1)^N}{N!}\{U^N\} = 0 , \tag{3.10}$$

where

$$\begin{aligned} \{U\} &= \operatorname{tr} U , \\ \{U^2\} &= (\operatorname{tr} U)^2 - (\operatorname{tr}(U^2)) , \\ &\vdots \\ \{U^{r+1}\} &= \{U\}\{U^r\} - r(\{U\}\operatorname{tr}(U^r) - \operatorname{tr}(U^{r+1})) . \end{aligned} \tag{3.11}$$

Thus the physical states are completely labelled by

$$|\operatorname{tr} U^r ; r = 1, \ldots, (N-1)\rangle \tag{3.12}$$

for SU(N). [For U(N), $r = N$ should also be included.] Any observable of the form tr U^p can be evaluated in terms of tr U^r, $r = 1, \ldots, N-1$, using eq. (3.10) with λ replaced by U. (A better set is tr B^r, $r = 2, \ldots, N$ where $U = \exp iB$. Now we have $(N-1)$ *real* variables.)

2-plaquette world. Now the physical states are labelled by two matrices U, V with

$$|U, V\rangle \sim |\Omega U \Omega^{\dagger}, \Omega V \Omega^{\dagger}\rangle . \tag{3.13}$$

We may choose Ω to diagonalize U. For a generic case (i.e. the eigenvalues of U all distinct) this fixes Ω up to the set of diagonal SU(N) matrices. We may use this freedom to make the first non-zero, non-diagonal element in each row (except the first row) of V real and positive. Denoting such a matrix by V', we have to find the minimal set of class invariants that completely determine the set $\{\Lambda, V'\}$, where Λ is the diagonalized form of U. In special cases, e.g. when a set of eigenvalues of U coincide, there is more arbitrariness in Ω and V can be further reduced to a special form. We will show in appendix A that the set $\mathrm{tr}\, U^r V^s$ with $r, s = 0, \ldots, (N-1)$ and $r+s \neq 0$ is sufficient to fix up the equivalence class uniquely. For special cases this set may overspecify the data: e.g. when $\Lambda = 1$, the equivalence class is simply labelled by the eigenvalues of V and hence by just $(N-1)$ invariants. However, given our set such special configurations can be readily recognized.

3-plaquette world. Now the states are

$$|U, V, W\rangle \sim |\Omega U \Omega^\dagger, \Omega V \Omega^\dagger, \Omega W \Omega^\dagger\rangle . \tag{3.14}$$

In the generic case, Ω is chosen to bring the set $\{U, V\}$ to the standard form $\{\Lambda, V'\}$ as before. Then W has to be completely specified with respect to this set. Since in the generic case the set

$$U^r V^s, \qquad r, s = 1, \ldots, (N-1), \tag{3.15}$$

forms a linearly independent basis in the vector space of $N \times N$ matrices, all we have to do is to specify the inner product of W with respect to this basis. Therefore we need the set

$$\mathrm{tr}\,(U^r V^s W), \qquad r, s = 0, \ldots, (N-1). \tag{3.16}$$

For special configurations this is not sufficient. In the extreme case when $U = V$, the set (3.15) is not linearly independent. (This can be stated in a gauge-invariant language.) In this case our system reduces to the 2-plaquette case and we have to specify the set $\mathrm{tr}\, U^r W^s$, $r, s = 0, \ldots, (N-1)$.

We remark that the variables we have specified in 2- and 3-plaquette cases are not the least constrained one can think of. Thus, e.g., in the 2-plaquette case we have only (N^2-1) real variables to be specified whereas we are using (N^2-1) complex numbers to describe them. Specifying the modulus of our set does not suffice, because there are many discrete solutions even in the generic case. Instead consider (in the 2-plaquette case)

$$\mathrm{tr}\,(U^r), \qquad \mathrm{tr}\,(V^r), \qquad \mathrm{tr}\,(UV)^r, \qquad \mathrm{tr}\,(U^2V)^r, \ldots, \qquad \mathrm{tr}\,(U^{N-1}V)^r, \tag{3.17}$$

where $r = 1, \ldots, (N-1)$. This system can be equivalently described in terms of the traces of the hermitian matrices B_s^n, defined by $U^s V = \exp iB_s$, etc. Though this set appears to be a sufficient set, we have not been able to provide a formal proof.

In the 3-plaquette case, we have used N^2 complex variables whereas again we need just (N^2-1) real variables to fix W in terms of U and V. In this case we

may use the set

$$\mathrm{tr}\,(W^r), \quad \mathrm{tr}\,(VW)^r, \quad \mathrm{tr}\,(V^2W)^r, \ldots, \quad \mathrm{tr}\,(V^{N-1}W)^r, \quad \mathrm{tr}\,(UW)^r, \tag{3.18}$$

which can also be specified in terms of traces of hermitian matrices.

Back to the 2d world. We choose Ω to diagonalize U_1 and to bring V_1 to the standard form V_1^r (fig. 1). Then all invariants involving U_1, V_1 and W_1 are obtained as in the 3-plaquette case from $\mathrm{tr}\,U_1^r V_1^s$ and $\mathrm{tr}\,U_1^r V_1^s W_1$, $r, s = 0, \ldots, (N-1)$. (This is in the generic case.) Since V_1^r and W_1 are now known, we may determine X_1 completely from the set $\mathrm{tr}\,V_1^r W_1^s X_1$. In this way at most three successive plaquettes are involved in the gauge-invariant data that are specified. To completely determine the matrix U_2, we may use the set $\mathrm{tr}\,V_1^r U_1^s U_2$ and for V_2 the set $\mathrm{tr}\,U_1^r U_2^s V_2$. All these traces can be rewritten as local gauge invariant variables by suitable insertion of $U(n2)$.

We now consider special cases. If $U_1 = V_1$, as discussed earlier we need $\mathrm{tr}\,V_1^r W_1^s$. On the other hand, if a successive set of plaquette variables, say $W_1, X_1, Y_1, \ldots, Z_1$ are identical (which can readily be given a gauge invariant meaning), then to determine the next plaquette variable A_1 completely we need the set $\mathrm{tr}\,V_1^r W_1^s A_1$. If W_1 and A_1 are separated by macroscopic distances, this kind of gauge-invariant data is non-local. Thus under special conditions, non-local data become unavoidable [4, 5]. However, such special configurations form a set of measure zero [4]. We remark that with the data for the generic case it is possible to recognize whether special configurations are present and then make further measurements to completely determine the configuration.

3d world. We first specify a complete gauge fixing. On each $n_3 =$ constant plane we choose the gauge as in the 2d case. We still have the freedom of local gauge transformations along the line OZ. We choose these to make all the links of the line 1. We are still left with the gauge freedom at O which is equivalent to a global gauge transformation. Thus the physical states may be labelled by the class

$$|U(n1),\ n_2 \neq 0;\ U(n3),\ n_1, n_2 \neq 0\rangle \text{ modulo a global gauge transformation}, \tag{3.19}$$

or, equivalently, the plaquette variables

$$|U(n)_3,\ U(n)_1;\ U(n)_2,\ n_2 = 0\rangle \text{ modulo a global gauge transformation}. \tag{3.20}$$

In (3.20), $U(n)_1$, for example, refers to a plaquette variable in the 2-3 plane, n referring, as earlier, to the "smallest" plaquette coordinate with the orientation fixed by the right-hand-screw rule.

It is now easy to construct the minimal set of local data needed. On each $n_3 =$ constant or $n_1 =$ constant plane, we use the set of variables as in the 2d case. The U_2 plaquettes in the plane $n_2 = 0$ are again fixed with respect to the line OZ as in the 2d case. Since we do not have the freedom of global transformation in every $n_3 =$ constant plane, we have to fix two plaquettes on each $n_3 =$ constant plane

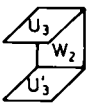

Fig. 2. Plaquettes used to fix U_3.

with respect to the lower planes. For this we use variables like tr $(U_3'^{r}W_2^{s}U_3)$ (see fig. 2).

We remark that our local variables correspond to invariants like tr $(B_3^{r}D_2^{s}B_3^{t})$ and tr $(B_1^{r}D_2^{s}B_1^{t})$. Thus it suffices to specify invariants involving B_1 and B_3 separately (along with covariant derivatives along y-direction). Invariants involving B_2 or combinations of B_i's are then completely determined. This is analogous to the U(1) case in which it suffices to specify only 12 and 23 plaquette variables (and boundary data on 31 plaquettes). 31-plaquette variables then get determined from

$$\prod_{\text{box}} U_{\text{plaquette}} = 1 \tag{3.21}$$

(which is the lattice analogue of $\nabla \cdot \boldsymbol{B} = 0$).

Inclusion of matter fields. We will now consider the minimal data that is to be specified when matter fields are present. In sect. 2 the physical states were labelled by electric field invariants and it was natural to describe the matter fields by their color spins. Since we are now labelling the states by the configuration space variables, it is natural to describe matter also by configuration space. We will therefore consider a scalar field $\varphi(n)_A$ in a representation R of the color group, A being the color label. The extended Hilbert space is now labelled by $|U(ni)_{IJ}, \varphi(n)_A\rangle$. An analogous consideration [7] for fermions requires the use of coherent states and anticommuting c-number fields at each lattice point. We will not treat this case here.

The physical states are again obtained by averaging over the gauge transformations at each site. We will again consider a representative of each such state obtained by a complete gauge fixing. We first concentrate on the generic case. The gauge is chosen with respect to the link variables as earlier. In the pure gauge case with SU(N) as the local gauge invariance, the residual global transformation is fixed only up to a Z($N-1$) global transformation. However, as the link variables are invariant under such a transformation, it is irrelevant. When matter fields transform non-trivially under the centre of the gauge group this is no longer true. To be specific we will consider matter in the fundamental representation of SU(N). (It is easy to extend the results to the general case.) In this case we choose the residual Z($N-1$) global transformation such that

$$0 \leq \theta < 2\pi/N, \tag{3.22}$$

where θ is the phase of the first non-vanishing component of $\varphi(0)$. (In case $\varphi(0) = 0$ we have to choose the matter field at some other site.) We have to now specify

the matter fields completely with respect to the gauge-fixed link variables using local gauge-invariant combinations.

To specify $\varphi(0)$ (up to a phase) we may use local gauge invariants such as $\varphi(0)^\dagger U^{(A)}\varphi(0)$ where $U^{(A)}$'s are along loops beginning and ending at 0. The minimal set of $U^{(i)}$'s needed would appear to be N but this set allows more than one (but a finite number of) solutions. Even in the simplest case of 2-component real φ, there are in general four solutions corresponding to the four points of intersection of the two conical sections involved. In the generic case an additional condition will suffice. However, solving for $\varphi(0)$ is in general difficult. But by specifying N^2 variables corresponding to N^2 matrices $U^{(A)}$ linearly independent in the space of $N\times N$ matrices, it is easy to obtain the $N\times N$ matrix $\varphi(0)_i^*\varphi(0)_j$ by inversion. We also remark that the matrices $U^{(A)}$ should not be simultaneously diagonalizable. This rules out repetitions of the same loop in choosing $U^{(A)}$. It also shows that when the gauge-field configuration is degenerate [5] in a macroscopic region, non-local variables are needed to fix $\varphi(0)$.

For a moment we presume that the overall phase of $\varphi(0)$ is also known. Then the matter field ψ at a neighboring point is easily fixed by using the local observables $\varphi(0)^\dagger U^{(i)}\psi$ where $U^{(i)}$'s are along paths joining the two points involved. Now a set of N variables suffices, with the requirement $\det D\neq 0$ where $D_{ij}=(\varphi(0)^\dagger U^{(i)})_j$. Note that this is a gauge-invariant condition.

In this way matter fields at successive sites may be fixed in terms of the previous ones. Since there is still an ambiguity in the overall phase of $\varphi(0)$, this ambiguity carries over to all the sites. To fix this phase we have to specify a local variable such as the phase of the determinant

$$\det(\varphi(0), \psi(1)_0, \psi(2)_0, \ldots, \psi(N-1)_0), \tag{3.23}$$

where the columns of the determinant are formed of the matter vectors indicated, and $\psi(i)_0$ stands for the parallel transport of $\psi(i)$ to the site 0 along a specified path. This only fixes the phase modulo an element of $Z(N-1)$ but this is sufficient because of eq. (3.20).

4. Minimal data for the Bohm–Aharanov experiment

Till now we considered simply connected regions of space. We will now consider the changes affected when the space in which observations are made is multiply connected. We begin with a rectangular hole cut into a finite rectangular 2d lattice and fix a gauge completely as before (fig. 3). All the y-links as well as the x-links on the x-axis are made 1. In addition all the x-links on the line CD can be chosen to be 1 except for one, say C'C. All other links have to be completely specified up to a global gauge transformation. In the abelian case the 1-plaquette variables $UUUU$ were sufficient for the simply connected regions. This set is not enough to fix the link C'C in the present case. It is necessary to specify a Wilson loop going

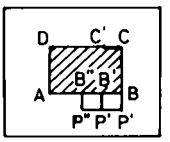

Fig. 3. Link variable on C'C cannot be gauged to one.

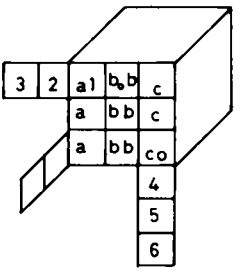

Fig. 4. Link variables that cannot be gauged to one on one surface of a hole are lettered.

around the hole, e.g. the loop ABCDA. Thus a complete specification of the state requires a specification of the magnetic field enclosed in the hole. That this has observable effects is well known [8].

In the non-abelian case, we need many more variables to specify the link C'C. In the generic case a minimal set is tr $(U^r V^s W)$, $0 \leq r, s \leq (N-1)$, where U = loop B'B"P"P'B', V = loop B'P'PBB', W = loop BCDAB. These gauge-invariant data are sufficient for the non-abelian Bohm–Aharanov effect. Specifying at least N^2-1 loop variables going around the hole is unavoidable.

We will now consider the three-dimensional case. If the hole is such that the region is not simply connected, then again it is necessary to specify Wilson loops going around the hole exactly as in the two-dimensional case. On the other hand, if the space is simply connected, this is not necessary. Some local boundary data on the hole is sufficient. To be specific we consider a box-like hole (fig. 4). With the gauge fixing, we have to specify all the links marked in fig. 4 on one face of the hole. The z-links marked (a) are specified using U_x plaquettes as usual. The z-links marked (b) are specified using at most three U_y plaquettes at a time. E.g. the z-link b_0 is specified via the y-plaquettes labelled 1, 2 and 3. The x-links marked (c) are specified again with the y-plaquettes, but now arranged along the z-direction. For the link c_0 for instance, we use the plaquettes 4, 5 and 6. These boundary data correspond to invariants of $D_1^r B_y$ and of $D_2^r B_y$, $r = 0, 1, 2, \ldots$, on a face of the hole.

5. Non-abelian Stokes' theorem

In the abelian theory, Stokes' theorem relates the Wilson loop variable to the magnetic field on some surface bounding the loop:

$$\exp\left(i \oint_C A_\mu \, dx^\mu\right) = \exp\left(i \int_S F_{\mu\nu} \, d\sigma^{\mu\nu}\right). \tag{5.1}$$

The analogue for the non-abelian theory is to specify some minimal local data on a surface which enables us to compute the Wilson loop variable for the contour

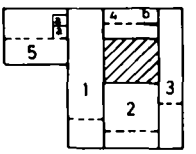

Fig. 5. The dotted lines are the base lines chosen.

bounding it. With the experience gained in sect. 3, it is easy to obtain the data to be specified. For this we first give a complete gauge fixing for an arbitrary non-selfintersecting open surface. We begin with the $d = 2$ case.

Consider a connected section of an x-line (i.e. a line parallel to the x-axis) intersecting the surface. We will call this base line 1. Consider connected strips of plaquettes along y-lines intersecting the base line and lying on the surface. These form a (connected) region to be called region 1 (fig. 5). Consider some other x-line intersecting part of the surface exterior to region 1. A connected part of this line lying on the surface but exterior to region 2 will be called base line 2. Region 2 is defined with respect to this base line as before. We continue this procedure until the regions cover the entire surface. Our choice of gauge corresponds to choosing all the y-links to be unit matrices as also all the x-links on each base line (fig. 5). In case the surface is multiply connected, one of the link variables (e.g. b) on some of the base lines cannot be gauged to one. Finally there is the freedom of global gauge transformation.

Starting with region 1 the free links are successively fixed with respect to the base line using Wilson loops involving at most three plaquettes at a time (generic case) as in sect. 3. In each successive region, two of the links, e.g. links (a) in fig. 5, have to be completely fixed using some of the plaquettes of the regions already covered. But for this, other link variables are fixed as for region 1. In case of multiply connected regions, the link variables on the base line which is not gauged to one has to be fixed by specifying N^2 Wilson loops around the corresponding hole.

Once such gauge-invariant data on the surface are specified, it is easy to reconstruct the link variables with a complete gauge fixing and thereby the Wilson loop for the contour bounding the loop.

In the case of $d = 3$, we consider sections of the surface in various $z =$ constant planes. In general we encounter disconnected regions and lines in each plane. In each plane the gauge is chosen as for $d = 2$. There is a freedom of global gauge transformation for each disconnected region in each plane. This completes the gauge fixing.

The data are specified for each disconnected region in each $z =$ constant plane as for $d = 2$. The plaquettes connecting two adjacent planes are fixed as follows: two plaquettes with the smallest (x, y) coordinates are fixed with respect to the z-plane below. Then all the plaquettes connected to these are successively fixed using three neighboring plaquettes at a time. This is repeated for each disjoint set

of plaquettes perpendicular to the z-plane. Moreover, it is necessary to fix two plaquettes from each disjoint section in each z-plane with respect to the lower sections.

Thus given any surface bounding a given contour and enough gauge-invariant data on such a surface, Wilson loops of the form tr c^n (where c^n corresponds to n traverses of the contour) can be calculated. We get the same values for this observable, whatever surface we choose. We may therefore use these variables to define non-abelian magnetic fluxes [9] through an open surface as the N phases $\theta_i[c]$ (with $0 \leq \theta_i < 2\pi$) of the eigenvalues of c. We may use these fluxes to provide a gauge-invariant criterion for the $(N-1)$ varieties of monopoles in a SU(N) gauge theory recently proposed by 't Hooft [10]. To know the monopole types at a given point, we surround the point by a small sphere and parametrize the sphere by mapping it onto a unit square S whose boundary is identified with a point p on the sphere (our construction works only for the continuum theory):

$$S = \{(s, t),\ 0 \leq s, t \leq 1\}. \tag{5.2}$$

We may now construct a sequence [6] of loops $c(s)$, $0 \leq s \leq 1$ corresponding to the points $s = \text{const.}$ on S which begin and end at P and which sweep out the sphere as s changes from 0 to 1. This gives us N functions $\{\theta_i(s)\}$ where $\theta_i(s)$ are the fluxes associated with the contour $c[s]$. $\theta_i(s)$ are arranged to be differentiable functions of s by suitable ordering of the eigenvalues and by suitable additions of multiples of 2π. Then for any smooth gauge field configuration

$$\theta_i(1) = 2\pi n_i, \qquad i = 0, \pm 1, \pm 2, \ldots, \tag{5.3}$$

and n_i determines the monopole number of type i.

6. Comments and conclusions

Our aim in this paper was to understand certain questions of principle in connection with non-abelian gauge theories: labelling the physical states in terms of local observables both of color electric field and color magnetic field; the number and type of non-local variables to be specified when space is multiply connected; the local gauge-invariant data to be specified on a surface to be able to reconstruct observables connected with the contour bounding the surface etc.

Though our labelling in terms of the color electric field observables was the minimal possible, the parameters involved are not all independent of each other. Thus given the Casimir invariants of $E(n, i)$ and $\mathscr{E}(n-i, i)$ for some (n, i), those of $D_i E(n, i)$ are not altogether independent. Neither is the hamiltonian simple in terms of these variables. The same is true for the case of magnetic field observables. In this sense, labelling by the gauge-dependent potentials has unquestionable advantage. Nevertheless, our efforts have the advantage of telling what is the

minimal set of measurements needed to completely specify the physical states of the system.

Our labelling in terms of the color electric field provides a set of products of Wilson loop operators which operating on the strong coupling vacuum generate a complete set of orthonormal states.

All our considerations were in gauge theory on a lattice. We may now raise the question of the relevance of our results to the continuum theory. The continuum theory may have simplifications not to be seen in the lattice theory. In fact in the continuum theory the color components of the electric field commute among each other (at equal times) whereas this is not true in the lattice theory. Therefore the states of the extended Hilbert space can be labelled simultaneously by the color components: $|\{E(x)_i^\alpha\}\rangle$. (Our analysis is heuristic in what follows. We will not worry about the care to be taken in defining the Hilbert space.) As usual the physical states are obtained by averaging over the gauge transformations:

$$|\ \rangle = \int \mathcal{D}\theta(x) \exp\left(i \int \theta(x)^\alpha (D_i E_i(x))^\alpha\right) |\{E(x)_i^\alpha\}\rangle\,.$$

Thus the physical states are simply characterized by the equivalence class

$$\{E(x)_i;\ E(x)_i \sim V(x) E(x)_i V(x)^\dagger\}\,,$$

and from sect. 3 it follows that the minimal set of class invariants characterizing this class is tr $(E(x)_1^r E(x)_2^s)$ and tr $(E(x)_1^r E(x)_2^s E(x)_3)$, $r, s = 0, \ldots, (N-1)$, $r+s \neq 0$ (in the generic case). Thus the labelling now is altogether different from what it is in the lattice theory even in the content. It is not necessary to specify correlations of the color field at the neighboring points. It is impossible to find a corresponding labelling in the lattice theory. On the other hand the continuum theory must have a labelling similar to that of the lattice theory.

The above labelling in the continuum theory, in terms of the electric field may raise the hope that a similar labelling may exist in terms of the magnetic field in the continuum theory. There would then be $E \leftrightarrow B$ symmetry at the level of the physical Hilbert space. However, it is well known [10] that the color magnetic field by itself (without covariant derivatives) does not uniquely characterize the potential up to a gauge transformation so that the possibility is excluded.

Many of our results for the magnetic case are directly known in the continuum theory, e.g. ref. [5]. The new results in the present paper are the specification of the minimal number of derivatives required, the boundary data to be employed, data expressed in terms of gauge-invariant quantities, inclusion of matter fields, etc.

We have also found that for the non-abelian case it is necessary to specify not only the magnetic field (invariants) but also certain of its covariant derivatives (along the surface) to be able to compute the Wilson loop for the boundary of the surface. For the continuum theory, there is a more natural procedure for specifying

the data on the surface. The surface can be mapped onto a unit square and the data specified as in the 2d case.

I have benefitted from a crucial discussion with Professor G. Mack. I thank Dr. P. Weisz and Professor H. Joos for carefully reading the manuscript and for helpful suggestions.

Appendix A

Consider the equivalence class

$$\{U, V; (U, V) \sim (\Omega U \Omega^{\dagger}, \Omega V \Omega^{\dagger}),\ U, V, \Omega \in \mathrm{SU}(N)\}\,. \tag{A.1}$$

Then a minimal set of class invariants characterizing this class completely is

$$\mathrm{tr}\ U^r V^s, \qquad 0 \leq r, s \leq N-1\,, \qquad r+s \neq 0\,. \tag{A.2}$$

To see this we choose a representative matrix of each class as follows: Ω [eq. (A.1)] is chosen to diagonalize U. In the generic case this fixes Ω up to a diagonal SU(N) matrix. (Non-generic cases considered later.) Let A be a matrix diagonalizing V:

$$V = AMA^{\dagger}, \qquad A \in \mathrm{U}(N)\,. \tag{A.3}$$

If A is a solution, so is $A\Omega'$, where Ω' is a diagonal U(N) matrix. We now fix Ω and Ω' completely by requiring

$$A_{1i},\ A_{i1} \qquad (i = 1, \ldots, N) = \text{real positive}\,.$$

If some of the elements are zero (a non-generic case) we choose the first non-zero element in the corresponding row or column to be real positive. We have

$$\mathrm{tr}\ U^r V^s = \sum_{i,j} \lambda_i^r \mu_j^s |A_{ij}|^2 \tag{A.4}$$

where λ_i and μ_j $(i = 1, \ldots, N)$ are the eigenvalues of U, V to be determined from (A.2) for $r = 0$, $s = 0$ respectively. We have the additional identity

$$N = \sum_{i,j} |A_{ij}|^2\,. \tag{A.5}$$

We have (the Van-der-Monde determinant)

$$|\det D_{(ij)}{}^{(rs)}| = \prod_{\substack{i,j=1 \\ (i \neq j)}}^{N} (\lambda_i - \lambda_j)^N (\mu_i - \mu_j)^N\,, \tag{A.6}$$

where

$$D_{(ij)}{}^{(rs)} = \lambda_i^r \mu_j^s\,, \tag{A.7}$$

so that in the generic case we may invert (A.4) and (A.5) to get $|A_{ij}|$.

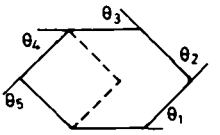

Fig. 6. There are two choices for θ_4 (and θ_5).

We see that we have N orthonormal vectors $A_i^{(j)} = A_{ij}$ with $A^{(1)}{}_i$ and $A^{(i)}{}_1$ real positive and with $|A^{(j)}{}_i|$ specified. We will show how $A^{(j)}{}_i$ may be completely recovered. The $\frac{1}{2}N(N-1)$ (complex) orthogonality equations are all independent. Each (complex) equation imposes two constraints. In fact each condition requires that the vectors $A^{(j)}{}_i A^{(k)}{}_i$ (j and $k \neq j$ fixed, $i = 1, \ldots, N$) in the complex plane form a closed N-polygon (fig. 6). Moreover, the base of the polygon (corresponding to $i = 1$) is specified to be along the real axis. Therefore if we fix $\theta_1, \ldots, \theta_{N-2}$ there are at most two solutions for θ_{N-1} and θ_N (fig. 6).

This way we may undertake to express the phases θ_{ij} (where $A_{ij} = |A_{ij}| \exp i\theta_{ij}$) successively in terms of other phases (up to the discrete ambiguity). For example, the orthogonality of $A^{(1)}$ and $A^{(j)}$ ($j > 1$) is used to express θ_{2j} and θ_{3j} ($j \neq 1$) in terms of the other phases. Next the orthogonality of $A^{(j)}$ ($j > 2$) and $A^{(2)}$ is used to eliminate θ_{4j} and θ_{5j} ($j = 1, \ldots, N$). This procedure finally leads to solutions for θ_{ij} making use of just $\frac{1}{2}(N-1)^2$ (for N = odd) orthogonality conditions for any choice of the discrete ambiguities. We have to now use the remaining $\frac{1}{2}(N-1)$ conditions to eliminate this discrete ambiguity. We have not found a general proof that such ambiguities are uniquely resolved. But in all concrete examples this has been found to be the case. We have also not found any invariances (in the generic case) in the orthogonality equations which allow more than one solution (apart from the complex conjugate solutions which are resolved because the eigenvalues of V are known).

We will now consider non-generic cases. If a set of eigenvalues of U coincide, then the diagonalizing matrix Ω has more arbitrariness. Thus if the last m eigenvalues coincide, we may use SU(N) matrices of the form ($k_1 \neq k_2 \neq \cdots \neq k_{N-m}$)

$$\left(\begin{array}{ccc|c} k_1 & & & \\ & k_2 & & \\ & & \ddots & 0 \\ & & k_{N-m} & \\ \hline & 0 & & m \times m \end{array}\right)$$

to further bring A to a special form. To be specific, we require the corresponding $m \times m$ submatrix of A to have the upper-triangular form with the diagonal elements real positive.

Now it is only possible to obtain $|A_{ij}|^2$, $i < N-m$ and $\sum_{i=N-m}^{N} |A_{ij}|^2$ from eq. (A.4). We may now repeat the previous arguments for the special form of A_{ij} we have, to show that A_{ij} is fixed (up to at worst some discrete ambiguities in the phases).

Appendix B

MINIMAL COMPLETE SET FOR THE CLASSICAL THEORY

In the classical theory the configuration at any instant of time is described by the potentials $A_i(x, t)$ and their conjugates $E_i(x, t)$ (in the continuum case). However, configurations which are gauge equivalent are physically indistinguishable. Therefore we have again to describe the configuration by quantities that are locally gauge invariant. In contrast to the quantized theory, both the electric and magnetic field variables have to be simultaneously specified to describe the state completely. As in the previous sections we will make the problem simpler and well defined by discretizing the space. We will fix a gauge completely as in sect. 3. Now in addition to the free link variables, the electric field variables $E(n)_i = E(n)_i^\alpha T^\alpha$ have to be specified at each lattice site. For this we simply specify the minimal Wilson loop variables considered in sect. 3 and in addition the gauge-invariant set $\mathrm{tr}\,(U_1^r U_2^s E(ni))$, $0 \leq r, s \leq (N-1)$, $r+s \neq 0$ (in the generic case). Here U_1 and U_2 are two distinct local loop variables beginning and ending at the site n. This additional set of variables correspond to quantities like $\mathrm{tr}\,(E(x)_i B(x)_j^{r_1} D_k^{r_2} B(x)_l^{r_3})$ in the continuum theory.

Since any gauge-invariant quantity can be expressed in terms of our local set, it is possible to obtain a closed set of first-order evolution equations for our gauge-invariant variables – in principle. However, the usefulness of such equations is questionable.

References

[1] J. Fröhlich, Lecture at Colloquium on random fields, Esztergom, Hungary, 1979;
B. Durhuus, Lett. Math. Phys. 4 (1980) 515;
R. Giles, M.I.T. preprint CPT 920 (March, 1981)
[2] J. Kogut and L. Susskind, Phys. Rev. D10 (1974) 3468
[3] J. Goldstone and R. Jackiw, Phys. Lett. 74B (1978) 81;
A.G. Igerzin, Y.E. Korepin, M.A. Semenov-Tyan-Shansvili and L.D. Fadeev, Theor. Math. Phys. 38 (1979) 1
[4] G. Mack, Fortschr. Phys. 29 (1981) 135
[5] M.A. Mostow, Comm. Math. Phys. 78 (1981) 137
[6] N. Christ, Phys. Rev. Lett. 34 (1979) 355;
P. Goddard and D. Olive, Rep. Prog. Phys. 41 (1978) 1357;
I. Ya. Aref'eva, Univ. of Wroclaw preprint no. 480 (Nov., 1979);
P.M. Fishbane, S. Gasiorowicz and P. Kaus, Minnesota Univ. preprint COO-1764-405 (1980)
[7] M. Creutz, Phys. Rev. D15 (1977) 1128
[8] V. Aharanov and D. Bohm, Phys. Rev. 115 (1959) 485;
T.T. Wu and C.N. Yang, Phys. Rev. D12 (1975) 3845
[9] P. Olesen, Lectures at 18th Winter School of Theoretical physics, Feb., 1981, Karpacz, Poland
[10] G. 't Hooft, Nucl. Phys. B190 [FS3] (1981) 455
[11] T.T. Wu and C.N. Yang, Phys. Rev. D12 (1975) 3843