# PROOF OF BANDER'S CONJECTURE CONCERNING AMBIGUITIES OF MAGNETIC FLUX

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For three or more dimensions, we prove Bander's conjecture, which says that 't Hooft's definition of the magnetic flux of an SU($N$) gauge system confined to a box with periodic boundary conditions gives a unique result for almost all configurations.

't Hooft [1], in the course of his work on electric–magnetic duality for nonabelian gauge theories, has introduced a definition of magnetic flux for gauge field configurations inside a box with periodic boundary conditions. For an SU($N$) gauge theory, this flux is quantized so that each component can take the values $0, 1, \ldots N-1$ in natural units. Unfortunately, the definition allows some ambiguity, that is to say, some field configurations can be assigned two or more different values of the magnetic flux. This was first demonstrated by Ambjørn and Flyvbjerg [2], who proved that the zero field can be assigned any of the allowed values of the flux quantum number. The flux is a gauge invariant quantity, so that configurations gauge-equivalent to zero must also have this ambiguity. It is possible to find nontrivial configurations with non-unique flux. Does this mean that the 't Hooft definition of magnetic flux is unsuitable? No, for it may so happen that the configurations where the definition leads to ambiguities are "exceptional", i.e. form a set of measure zero (in some appropriate measure). This is conjectured to be the case by Bander [3], who discusses these questions in the context of the computation of the energy of a system of specified electric and magnetic flux by functional integration.

In this letter, we prove the above conjecture. This result, however, refers only to three- or higher-dimensional boxes. For two dimensions we do not have any definite result, but it is probably untrue.

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Let us recall 't Hooft's way [1] of introducing his definition. We consider the gauge field configuration inside a rectangle with periodic boundary conditions. This will give us the magnetic flux in the direction orthogonal to the rectangle. For a box of higher dimensions, this method gives a definition for each pair of directions through the consideration of appropriate rectangle sections.

Let $(x_1, x_2) = (0,0), (a_1, 0), (a_1, a_2), (0, a_2)$ be the four corners of the rectangle. When we impose periodic boundary conditions, these refer to physical objects, so that the gauge potential $A_\mu$ is allowed to be gauge-transformed from one side to the opposite one. I.e. we say that there exist gauge group elements $\omega_1(x_2)$, $\omega_2(x_1)$, such that

$$A_\mu(x_1, a_2) = \omega_2^{-1}(x_1) A_\mu(x_1, 0)\,\omega_2(x_1) + (\mathrm{i}/g)\,\omega_2^{-1}(x_1)\,\partial_\mu\omega_2(x_1)\,,$$

$$A_\mu(a_1, x_2) = \omega_1^{-1}(x_2)\,A_\mu(0, x_2)\,\omega_1(x_2) + (\mathrm{i}/g)\,\omega_1^{-1}(x_2)\,\partial_\mu\omega_1(x_2)\,, \qquad (1)$$

$g$ being the gauge coupling constant. Now $A_\mu(a_1, a_2)$ may be related to $A_\mu(0,0)$ in two ways: the two transformations $\omega_1$, $\omega_2$ can be made in different orders. By equating the two results, we get

$$\omega_1^{-1}(a_2)\,\omega_2^{-1}(0)\,A_\mu(0,0)\,\omega_2(0)\,\omega_1(a_2) + (\mathrm{i}/g)\,\omega_1^{-1}(a_2)\,\omega_2^{-1}(0)\,\partial_\mu\omega_2(0)\,\omega_1(a_2)$$

$$+ (i/g)\omega_1^{-1}(a_2)\partial_\mu\omega_1(a_2)$$

$$= \omega_2^{-1}(a_1)\,\omega_1^{-1}(0) A_\mu(0,0)\,\omega_1(0)\,\omega_2(a_1)$$

$$+ (i/g)\,\omega_2^{-1}(a_1)\,\omega_1^{-1}(0)\,\partial_\mu\omega_1(0)\,\omega_2(a_1)$$

$$+ (i/g)\,\omega_2^{-1}(a_1)\,\partial_\mu\omega_2(a_1)\,, \qquad (2)$$

i.e.

$$[\omega_2(0)\,\omega_1(a_2)]^{-1} A_\mu(0,0)\,[\omega_2(0)\,\omega_1(a_2)]$$

$$+ (i/g)\,[\omega_2(0)\,\omega_1(a_2)]^{-1}\partial_\mu[\omega_2(0)\,\omega_1(a_2)]$$

$$= [\omega_1(0)\,\omega_2(q_1)]^{-1} A_\mu(0,0)\,[\omega_1(0)\,\omega_2(a_1)]$$

$$+ (i/g)\,[\omega_1(0)\,\omega_2(a_1)]^{-1}\partial_\mu[\omega_1(0)\,\omega_2(a_1)] \qquad (3)$$

If we define a group element $z$ by

$$\omega_2(0)\omega_1(a_2) = z\omega_1(0)\,\omega_2(a_1)\,, \qquad (4)$$

then (3) can be satisfied by having $z$ in the centre of the group. This is only a sufficient condition, but it becomes necessary too, if, as 't Hooft [4] argues, the $\omega$'s are to be so chosen that (3) is satisfied for *all* $A_\mu(0,0)$. Now, in an SU($N$) gauge theory, the centre is Z($N$), so that $z$ is restricted to be of the form $\exp(2\pi i m/N)$, where $m$ is an integer. This integer, modulo $N$, is defined by 't Hooft to be the magnetic flux, It is in principle not uniquely determined by the field configuration, because (1) could presumably be satisfied by other $\omega$'s, which could give a different value of $z$.

The trouble with nonabelian gauge transformations, as in (3), is that there is a homogeneous part and also an inhomogeneous part, so that one does not have simple relations between two gauge transformations which lead to identical results. To overcome this difficulty, we shall work with the second rank antisymmetric tensor $F_{\mu\nu}$ instead of the vector potential $A_\mu$. This object of course transforms homogeneously under gauge transformations. The boundary conditions become

$$F_{\mu\nu}(x_1,a_2) = \omega_2^{-1}(x_1) F_{\mu\nu}(x_1,0)\,\omega_2(x_1)\,,$$

$$F_{\mu\nu}(a_1,x_2) = \omega_1^{-1}(x_2) F_{\mu\nu}(0,x_2)\omega_1(x_2)\,. \qquad (5)$$

Suppose we change $\omega_1(x_2)$, $\omega_2(x_1)$ to $\bar\omega_1(x_2)\,\omega_1(x_2)$, $\bar\omega_2(x_1)\,\omega_2(x_1)$ respectively. If (5) is continued to be obeyed, we must have

$$\bar\omega_2(x_1) \in G_{\mu\nu}(x_1,0)\,, \quad \bar\omega_1(x_2) \in G_{\mu\nu}(0,x_2)\,, \qquad (6)$$

where

$$G_{\mu\nu}(x_1,x_2) = \{\omega \in \mathrm{SU}(N) \mid \omega^{-1}F_{\mu\nu}(x_1,x_2)\omega = F_{\mu\nu}(x_1,x_2)\}, \qquad (7)$$

i.e. the little group of $F_{\mu\nu}(x_1,x_2)$. Since (6) must be satisfied for each value of $\mu$ and $\nu$, we must have

$$\bar\omega_2(x_1) \in G(x_1,0)\,, \quad \bar\omega_1(x_2) \in G(0,x_2)\,, \qquad (8)$$

where

$$G(x_1,x_2) = \bigcap_{\mu,\nu} G_{\mu\nu}(x_1,x_2)\,. \qquad (9)$$

Now the new expression for $z$, which we call $z'$, is

$$z' = \bar\omega_2(0)\,\omega_2(0)\bar\omega_1(a_2)\omega_1(a_2)\omega_2^{-1}(a_1)\bar\omega_2^{-1}(a_1)$$

$$\times\,\omega_1^{-1}(0)\bar\omega_1^{-1}(0)\,. \qquad (10)$$

Clearly,

$$z' \in G(0,0)\,\omega_2(0)\,G(0,a_2)\omega_1(a_2)\,\omega_2^{-1}(a_1)G(a_1,0)$$

$$\times\,\omega_1^{-1}(0)G(0,0)\,. \qquad (11)$$

Making use of the relations

$$G(x_1,a_2) = \omega_2^{-1}(x_1)G(x_1,0)\,\omega_2(x_1)\,,$$

$$G(a_1,x_2) = \omega_1^{-1}(x_2)G(0,x_2)\omega_1(x_2)\,, \qquad (12)$$

we see that

$$z' \in G(0,0)G(0,0)\,\omega_2(0)\,\omega_1(a_2)\,\omega_2^{-1}(a_1)\omega_1^{-1}(0)$$

$$\times\,G(0,0)\,G(0,0) = G(0,0)\,zG(0,0) = zG(0,0)\,, \qquad (13)$$

since $z$ commutes with all group elements. If the elements $\bar\omega_1(0)$, $\bar\omega_1(a_2)$, $\bar\omega_2(0)$, $\bar\omega_2(a_1)$ can be chosen independently in the groups to which they are restricted by (8), they can be made to yield any desired element in $G(0,0)$ in (13), and since obviously Z($N$) $\subset G(0,0)$, $z'$ may be made to acquire any value in Z($N$). But can we choose all the four group elements independently?

First we consider two dimensions. At a given point

$(x_1, x_2)$, there is only one independent $F_{\mu\nu}(x_1, x_2)$ because of the antisymmetry in $\mu$ and $\nu$. Consequently, $G(x_1, x_2)$ includes at least the Cartan subgroup $S[U(1)^N]$. This is a connected continuous group, so that for any two elements $\bar{\omega}_1(0)$ and $\bar{\omega}_1(a_2)$ in $G(0, 0)$ and $G(0, a_2)$, it is possible to find a continuous path traced out by $\omega_1(x_2)$ in $G(0, x_2)$ as $x_2$ goes from 0 to $a_2$. Similarly, $\bar{\omega}_2(0)$ and $\bar{\omega}_2(a_1)$ can also be chosen independently. Thus it would seem that the flux can indeed be changed by choosing $\bar{\omega}_1$, $\bar{\omega}_2$ suitably. However, by choosing to work with $F_{\mu\nu}$ instead of $A_\mu$, we have ignored some information. All the $\omega$'s allowed by the periodicity of $F_{\mu\nu}$ may not be allowed by that of $A_\mu$, and it may so happen that these extra restrictions make the magnetic flux almost always unique.

For three or more dimensions, there are, at each point $(x_1, x_2)$, at least three independent $F_{\mu\nu}(x_1, x_2)$ obtained by varying $\mu$ and $\nu$. Consequently, $G(x_1, x_2)$, as defined in (9), is the intersection of the little groups of at least three vectors in the adjoint representation of $SU(N)$. While for one vector, the little group includes at least the Cartan subgroup, for two or more vectors, the intersection is almost always just $Z(N)$. This can be seen as follows. For each $F_{\mu\nu}$, the little group depends on equality relations between its eigenvalues. When all eigenvalues are different, the little group is the Cartan subgroup $S[U(1)^N]$ and is generated by the $N-1$ diagonal generators of $SU(N)$ if $F_{\mu\nu}$ is diagonalized. If we want the intersection of the centralizers of two $F_{\mu\nu}$'s, we have to find out which of the above generators commute with the second $F_{\mu\nu}$. It is easy to see that unless some of the nondiagonal elements of this second matrix vanish, none of the diagonal generators of $SU(N)$ will commute with it, so that the intersection of the little groups will be just $Z(N)$. Thus, barring special cases where some of the eigenvalues of one of the $F_{\mu\nu}(x_1, x_2)$ coincide, or when some of the nondiagonal elements of an $F_{\mu\nu}(x_1, x_2)$ vanish in a representation where another is diagonal, $G(x_1, x_2)$ will be $Z(N)$. But if for each $x_2$, $\bar{\omega}_1(x_2) \in G(0, x_2) = Z(N)$, a discrete group, then by continuity one must have $\bar{\omega}_1(0) = \bar{\omega}_1(a_2)$. In order to have a $\bar{\omega}_1(x_2)$ that passes from one element of $Z(N)$ to another as $x_2$ changes, it is necessary to have the group $G(0, x_2)$ to be a continuous group [note that it always contains $Z(N)$] over a range of values of $x_2$. This would require $F_{\mu\nu}$ to have, on a finite part of the boundary $x_1 = 0$, the exceptional kind of behaviour described above (some eigenvalues equal, or some matrix elements vanishing). Such configurations clearly form a set of measure zero. In almost all cases, therefore, we shall have $\bar{\omega}_1(0) = \bar{\omega}_1(a_2)$. Similarly, again in almost all cases, $\bar{\omega}_2(0) = \bar{\omega}_2(a_1)$. As these elements lie in $Z(N)$ and therefore commute with all group elements, (10) reduces to

$$z' = z \,. \tag{14}$$

So, apart from a zero-measure set of configurations, the magnetic flux cannot be changed by altering the gauge transformations $\omega_1$, $\omega_2$. This is precisely what was conjectured by Bander [3].

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*References*

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