NUMERICAL STUDIES OF PHASE TRANSITIONS IN $SU(N)/Z_N$ LATTICE GAUGE THEORIES

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The average action per plaquette is calculated for the pure $SU(N)/Z_N$, $N = 2, 3, 4, 5$ and 6 gauge groups using strong coupling expansions up to 13th order for euclidean lattice gauge theory in four space–time dimensions. These expansions are compared with Monte Carlo generated data and agreement is found to be excellent for $0 \leqslant \beta \leqslant \beta_c$, where β_c is the critical inverse temperature for the appropriate gauge group considered.

Lattice gauge theories with the gauge groups $SU(N)$ and $SU(N)/Z_N$, respectively, are believed to have the same continuum limit, while for finite lattice spacing, i.e. finite bare coupling constant, they may behave differently. Both models possess Z_N monopoles [1]. In the $SU(N)$ model the monopoles are attached to strings that carry energy, whereas in the $SU(N)/Z_N$ model the strings are invisible [1–5]. Arguments have been given that this difference leads to a phase transition in the $SU(N)/Z_N$ theory, which separates a strong coupling phase with condensed monopoles from a weak coupling phase without condensed monopoles [6]. For $N = 2$ this expectation has been confirmed by Monte Carlo calculations [5–10].

In this letter we present the results of a numerical study of $SU(N)/Z_N$ lattice gauge theories for $N = 2, 3, 4, 5$ and 6. We investigate the average action per plaquette by means of the strong coupling expansion and the Monte Carlo method. The models are defined on a hypercubical lattice in four dimensions. To each link b an element $U(b)$ of $SU(N)$ is attached. The plaquette term is denoted by $U(p)$. Let

$$\chi_A(U) = \mathrm{tr}\, U \,\mathrm{tr}\, U^\dagger - 1 , \qquad (1)$$

be the character of the adjoint representation of $SU(N)$, which is also a faithful representation of $SU(N)/Z_N$. Its dimension is

$$d_A = N^2 - 1 . \qquad (2)$$

The Wilson form of the action of euclidean $SU(N)/Z_N$ lattice gauge theory is

$$S = \beta \sum_p [1 - (N^2 - 1)^{-1} \chi_A(U(p))] . \qquad (3)$$

The sum goes over all unoriented plaquettes and β is related to the bare coupling constant g by

$$\beta (N^2 - 1)^{-1} = (g^2 N)^{-1} \equiv X . \qquad (4)$$

The path integral for the partition function reads

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$$Z = \int \prod_b \mathrm{d}U(b) \exp(-S) , \tag{5}$$

and the average action per plaquette is defined as

$$\langle E \rangle = (1/6V)\, \partial \log Z / \partial\beta$$
$$= 1 - (N^2 - 1)^{-1} \langle \chi_A(U(p)) \rangle , \tag{6}$$

where V is the total number of lattice points.

Using standard methods [11,12] [‡1] we derived the first thirteen terms of the strong coupling expansion of $\langle E \rangle$ for $2 \leqslant N \leqslant 6$. They are of the form

$$\langle E \rangle = 1 - (N^2 - 1)^{-1} \sum_{k=1}^{\infty} g_k X^k . \tag{7}$$

The coefficients g_k are listed in table 1. A computer program [13] was used to generate the 10th to the 13th order terms. We would like to add a remark about the limit $N \to \infty$ with $g^2 N$ fixed. In this limit the model approaches a trivial ultralocal theory at strong coupling. The average action per plaquette becomes equal to 1 identically and the correlation length goes to zero like

$$\xi \sim 1/\log[(N^2 - 1)(1 - X)/X] . \tag{8}$$

On the other hand we expect a nontrivial large-N limit at weak coupling, where the model should behave in a similar manner to the standard SU(N) model.

The weak coupling expansion of $\langle E \rangle$ has the leading-order behaviour

$$\langle E \rangle = \tfrac{1}{4} g^2 N + O(g^4) . \tag{9}$$

The Monte Carlo simulation data was generated on 4^4 lattices. For the strong-coupling region $0.0 \leqslant \beta \leqslant \beta_c$ we carried out 100 iterations through the lattice and averaged over the last 20 iterations. Disordered starting lattices, along with periodic boundary conditions, were used throughout our calculations. The method of Metropolis et al. [14] was used to achieve statistical equilibrium with five Monte Carlo upgrades per link of the lattice. Further details of the calculational techniques can be found in ref. [15]. The critical inverse temperatures were found to be 2.50 [6–9], 6.40 [15], 12.0 [15], 19.5 [15] and 32.0 [15] for $SU(2)/Z_2$, $SU(3)/Z_3$, $SU(4)/Z_4$, $SU(5)/Z_5$ and $SU(6)/Z_6$, respectively.

‡1 We use eq. (2) of ref. [12]. This equation is incorrect in the paper because the term $3\theta_{664}/4$ is missing. However, this term does not contribute to terms up to 14th order.

In figs. 1a, 1b, 1c, 1d and 1e the results of our Monte Carlo simulation on 4^4 lattices are shown for $SU(N)/Z_N$, N = 2, 3, 4, 5 and 6, respectively, along with the strong-coupling expansions of eq. (7). For accuracy the results were plotted by computer [16]. We can see clearly that the strong-coupling expansions and the Monte Carlo generated data agree with an error of less than 1% over the whole range $0.0 \leqslant \beta \leqslant \beta_c$ where the error is due entirely to the statistical fluctuation in the Monte Carlo generated data. These results are unexpectedly good but similar good results for the strong-coupling expansions have been found for the gauge groups SU(4) [17] and SU(5) [18]. Note also that the transitions are first order ones; therefore, the series are not expected to be singular at $\beta = \beta_c$ and give good results, even near the transition point. We can see from fig. 1 that the convergence properties of the strong-coupling expansion could not be improved by the Padé continued fraction expansion or otherwise.

The Padé continued fraction approximation for any function $E(\beta)$ can be written as

$$E(\beta) = \cfrac{a_0}{1 + \cfrac{a_1\beta}{1 + \cfrac{a_2\beta}{1 + \cfrac{a_3\beta}{\ddots\, \overline{1 + a_n\beta}}}}} . \tag{10}$$

The poles of this truncated continued fraction are expected to simulate the singularities. To have confidence in this method, poles and residues should be stable as the order varies. This analysis is also performed on the logarithmic derivative of the function, for which the method is particularly efficient when the singularity obeys a power law. We have carried out the Padé approximant analysis in the following stages: (1) the direct analysis of the series in X for $\langle E \rangle$ [eq. (7)], (2) the corresponding analysis for the one-plaquette world. This analysis is interesting because there are no singularities on the real axis but the complex singularities remain and can cause trouble in a four-dimensional lattice. The coefficients for the expansion of $\log Z/6V$ in a one-plaquette world are presented in table 2. (3) In order to improve the convergence, the series is reexpressed in terms of the energy in the one-plaquette world. By this means we expect to lower the influence of the complex singularities.

Table 1
The coefficients for the strong-coupling expansion of eq. (7).

	N = 2	3	4	5	6	∞
g_1	1	1	1	1	1	1
g_2	$\frac{1}{2}$	1	1	1	1	1
g_3	0	$\frac{5}{6}$	1	1	1	1
g_4	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{23}{24}$	1	1	1
g_5	$-\frac{11}{324}$	$\frac{129}{1024}$	$\frac{84383}{101250}$	$\frac{411269}{414720}$	$\frac{1500629}{1500625}$	1
g_6	$\frac{251}{1296}$	$-\frac{1345}{9216}$	$\frac{124987}{202500}$	$\frac{79495}{82944}$	$\frac{30827701}{30870000}$	1
g_7	$\frac{187}{648}$	$-\frac{677}{2880}$	$\frac{69073}{202500}$	$\frac{18295}{20736}$	$\frac{5102509}{5145000}$	1
g_8	$\frac{89}{720}$	$-\frac{281}{1920}$	$\frac{33193}{540000}$	$\frac{26003}{34560}$	$\frac{10004429}{10290000}$	1
g_9	$-\frac{29801}{174960}$	$-\frac{30478313}{566231040}$	$-\frac{1769596619}{10935000000}$	$\frac{36631323683}{64210599936}$	$\frac{90605581104467}{97281016875000}$	1
g_{10}	$-\frac{30367}{116640}$	$\frac{79018037}{283115520}$	$-\frac{18152274163}{63787500000}$	$\frac{94120900997}{267544166400}$	$\frac{4466839574109893}{5188320900000000}$	1
g_{11}	$\frac{8763799}{295245000}$	$\frac{70837404501257987}{156031757844480000}$	$-\frac{81940181554948931}{283553298000000000}$	$\frac{453319101414366632309}{3716589787545600000000}$	$\frac{294069028551914823772551109}{389108263459472928000000000}$	1
g_{12}	$\frac{7191661849}{16533720000}$	$\frac{82136569967392079}{156031757844480000}$	$-\frac{209392923895113467}{1134213192000000000}$	$-\frac{329541418413494279983}{3716589787545600000000}$	$\frac{239510406197975877837049537}{389108263459472928000000000}$	1
g_{13}	$\frac{6037948171}{13226976000}$	$\frac{18805124785279466903}{39944130008186880000}$	$-\frac{10619540406553207}{6751269000000000000}$	$-\frac{34187299674292118326289}{135920997944524800000000}$	$\frac{151779313388087322042214500269}{340469730527038812000000000000}$	1

Table 2
The coefficients of the one-plaquette world strong-coupling expansion for $\log Z/6V$.

	N					
	2	3	4	5	6	∞
g_1	0	0	0	0	0	0
g_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
g_3	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
g_4	0	$\frac{5}{24}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
g_5	$-\frac{1}{30}$	$\frac{1}{10}$	$\frac{23}{120}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
g_6	$-\frac{1}{72}$	$\frac{1}{48}$	$\frac{5}{36}$	$\frac{119}{720}$	$\frac{1}{6}$	$\frac{1}{6}$
g_7	$\frac{1}{336}$	$-\frac{11}{504}$	$\frac{37}{420}$	$\frac{23}{168}$	$\frac{719}{5040}$	$\frac{1}{7}$
g_8	$\frac{1}{192}$	$-\frac{47}{1440}$	$\frac{61}{1440}$	$\frac{127}{1152}$	$\frac{119}{960}$	$\frac{1}{8}$
g_9	$\frac{1}{720}$	$-\frac{13}{540}$	$\frac{79}{12960}$	$\frac{541}{6480}$	$\frac{1633}{15120}$	$\frac{1}{9}$
g_{10}	$-\frac{7}{7200}$	$-\frac{817}{86400}$	$-\frac{57}{3200}$	$\frac{229}{4032}$	$\frac{7037}{75600}$	$\frac{1}{10}$
g_{11}	$-\frac{41}{47520}$	$\frac{199}{95040}$	$-\frac{9661}{332640}$	$\frac{52351}{1663200}$	$\frac{7703}{98560}$	$\frac{1}{11}$
g_{12}	$-\frac{1}{12960}$	$\frac{10387}{1451520}$	$-\frac{4483}{151200}$	$\frac{19163}{2073600}$	$\frac{50621}{806400}$	$\frac{1}{12}$
g_{13}	$\frac{191}{786240}$	$\frac{157307}{23587200}$	$-\frac{208061}{8985600}$	$-\frac{26389}{3144960}$	$\frac{6651217}{141523200}$	$\frac{1}{13}$
g_{14}	$\frac{457}{3386880}$	$\frac{235369}{67737600}$	$-\frac{683153}{50803200}$	$-\frac{12489041}{609638400}$	$\frac{529567}{16934400}$	$\frac{1}{14}$
g_{15}	$-\frac{109}{5806080}$	$\frac{2387}{10368000}$	$-\frac{89617}{23328000}$	$-\frac{650347661}{24216192000}$	$\frac{2660866333}{163459296000}$	$\frac{1}{15}$
g_{16}	$-\frac{2483}{46448640}$	$-\frac{3398837}{2090188800}$	$\frac{1207757}{348364800}$	$-\frac{58644549763}{2092278988800}$	$\frac{11610419683}{4184557977600}$	$\frac{1}{16}$

We obtain the following results. The direct analysis of the series in X for $\langle E \rangle$ [eq. (7)] suggests a singularity at about $X \approx 0.90$ which is not very clear for $SU(2)/Z_2$ while for the other gauge groups this approach does not give any result. For the corresponding analysis for the one-plaquette world we have the exact result for $SU(2)/Z_2$ for the derivative of $\log[I_0(2X) - I_1(2X)]$ of pairs of poles at $0.6398 \pm 1.4902i$, $0.8094 \pm 3.0876i$, $0.9094 \pm 4.6710i$, ... lying near the imaginary axis. The first pair of poles is well produced by our analysis. For the other gauge groups the poles come nearer and nearer the real axis (near point 1). This accumulation of poles lowers the accuracy as the order of the group increases. It is reasonably good up to $SU(5)/Z_5$. Other complex singularities may also appear and are the reflection of the transition line end points in the phase diagram of the $SU(N)$–$SU(N)/Z_N$ extended model. This is the case for the SU(2) model, where an adapted treatment [19] improves the result. However, the number of complex singularities for large N forbids such a technique in our case.

When we reexpress our series in terms of the energy in the one-plaquette world we find a very stable pole at $X \approx 1.045$ probably corresponding to the end of the metastable region, beyond the first-order transition. In $SU(3)/Z_3$ we find a stable singularity at $X \approx 0.925$

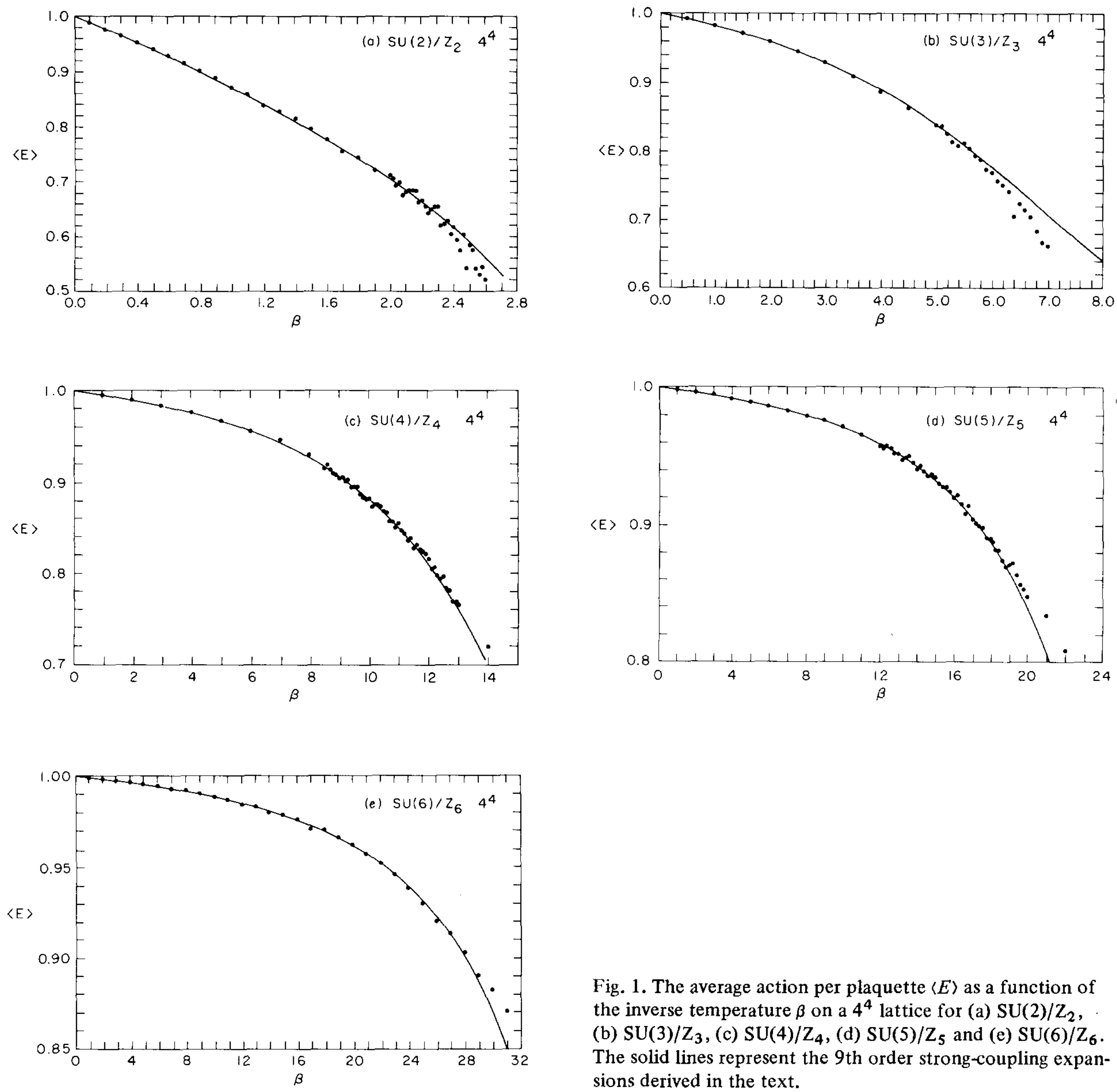

Fig. 1. The average action per plaquette $\langle E\rangle$ as a function of the inverse temperature β on a 4^4 lattice for (a) $SU(2)/Z_2$, (b) $SU(3)/Z_3$, (c) $SU(4)/Z_4$, (d) $SU(5)/Z_5$ and (e) $SU(6)/Z_6$. The solid lines represent the 9th order strong-coupling expansions derived in the text.

which is beyond the first-order transition found near 0.80 in the Monte Carlo simulation. This approach seems to be insufficient for larger gauge groups because complex poles have completely washed out the accuracy of the real singularity.

Up to the (first-order) transition, Padé approximants reproduce the Monte Carlo results quite well. The agreement suddenly breaks down at the transition as is expected: Padé approximants continue to correctly describe the metastable phase into which it smoothly enters without hitting a singularity, while the Monte Carlo simulation does not remain in this phase but jumps to the low-temperature regime. As done here, the analysis of Padé singularities can localize and analyze a second (or higher) order transition, but remains totally blind to the first-order transition. One localizes the nearest singularity, but cannot decide if this is a second-order transition or the end of a metastable region, after a first-order transition. To deal with such a first-order transition, one needs more information. It is, for instance, possible to use both strong-coupling and weak-coupling expansions and to construct type 2 Padé approximants which reproduce both expansions and which will then locate the transition. However, this analysis is beyond the scope of this letter.

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