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The Dirac-Kähler Equation and Fermions on the Lattice

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Abstract. The geometrical description of spinor fields by E. Kähler is used to formulate a consistent lattice approximation of fermions. The relation to free simple Dirac fields as well as to Susskind's description of lattice fermions is clarified. The first steps towards a quantized interacting theory are given. The correspondence between the calculus of differential forms and concepts of algebraic topology is shown to be a useful method for a completely analogous treatment of the problems in the continuum and on the lattice.

1. Introduction

Lattice approximations to quantum chromodynamics [1] seem to be a promising method to approach the problem of calculating those physical quantities which are mainly determined by the strong effective interaction at low energies. The calculations of the string constant, of the glue ball mass [2] etc. in pure lattice gauge theories, as well as the first attempts to calculate the hadron spectrum in lattice QCD with fermions [3] justify such an opinion.

However, all formulations of lattice QCD with fermions may be considered as not yet satisfactory. This problem originates in the fact that the naive transcription of the Dirac equation shows a higher degeneracy of the energy spectrum on the lattice than in the continuum [4]. There are essentially three proposals to overcome this problem of the additional degrees of freedom of the "naive" Dirac field on the lattice. In the solution of Wilson [5], the lattice action of the Dirac field gets modified in such a way that the superfluous degrees of freedom get masses which become infinite in the formal continuum limit. Hence one has the correct spin degrees of a Dirac field in the continuum. The main disadvantage of this approach is that it violates chiral symmetry on the lattice for massless quarks. Susskind [6] gives a procedure to reduce the additional degrees of freedom to a "minimal" number. He suggests an interpretation of the remaining degrees of freedom as "flavour" spins. His procedure does not include a formal continuum limit and hence obscures the geometrical origin of his flavour spins. The SLAC-group [7] tries to settle the problem by brute force. Working in the momentum space of the lattice, one can construct a Dirac equation with a correct spectrum and correct chiral invariance. The price one has to pay is a non-locality in the form of long range interactions which are difficult to control [8]. In a way all these solutions of the spectrum problem of lattice Dirac fields look somewhat arbitrary and it is difficult to judge how much this restricts the possibility and the reliability of calculations in complete lattice QCD. It is the aim of this paper to contribute to the clarification of the problem of fermions on a lattice. Our starting point is the geometric content of gauge theories [9]. The geometric interpretation of the gluon field, describing infinitesimal parallel transports of the local colour spaces, plays an important part in the formulation of the Wilson action for pure lattice gauge theories [1]. Gauge fields are represented as finite parallel transports along lattice links. However, the geometric properties of spinors are completely disregarded in the formulation of the "naive" Dirac field on a lattice. Here one associates spinors rather arbitrarily with quantities defined on lattice points.

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In order to find a more consistent procedure, one should start with a differential geometric formulation of the Dirac equation. For this we use the "well-known" fact [10] that Dirac fields are cross-sections of an Atiyah-Kähler bundle on a manifold, and we consider the generalization of the Dirac equation, first proposed by Kähler [11]:

$$(d-\delta+m)\Phi=0, \tag{1.1}$$

as very well suited for this purpose, [12].

In this equation Φ denotes a general differential form, d the operator of exterior differentiation and δ its adjoint operator, m the mass parameter; we shall give a more detailed description of (1.1) in the following section. Now we consider the lattice as a sort of triangulation of the space-time manifold. Then there is a standard procedure [13] by which cochains ("functions on the lattice elements: points, links, plaquettes, ...") are associated with differential forms. By this prescription the lattice approximation to the Dirac-Kähler equation (1.1) becomes straightforward. The Φ turns into a general cochain, d and δ become the dual boundary operator $\check{\Delta}$ and the dual coboundary operator $\check{\nabla}$, respectively. The Dirac field is therefore no longer associated with point functions. According to its geometrical meaning it is described as a superposition of functions defined on points, links, plaquettes, ... etc.

What do we gain by this systematic geometrical approach? First we find that the energy degeneracy of the Dirac-Kähler equation and of its lattice approximation is the same. However, we should realize that the multiplicity of states of given momentum of the Dirac-Kähler equation in four dimensions is four times that of the Dirac equation. As a matter of fact, it has the same multiplicity as the Susskind formulation. Indeed we can show that the Susskind description of Dirac fields is equivalent to the lattice approximation of the Dirac-Kähler equation. In this sense we have found the correct formal continuum limit for the Susskind fermions. This result is of practical and theoretical interest. It helps to control the continuum limit in Monte Carlo calculations with Susskind fermions. Theoretically, the reduction of the Dirac-Kähler equation to the Dirac equation in the continuum gives a hint at the meaning of the additional degrees of freedom on the lattice. There is a strictly local integral of motion which in the continuum case supplies the subsidiary conditions allowing an easy elimination of the additional degrees of freedom. In the lattice case, this integral is no longer strictly local, but involves nearest neighbours in a twisted way. From our point of view this is the origin of the problem of the lattice formulation for simple Dirac fields.

In this paper, we describe the Dirac-Kähler equation in detail in the next section. This includes the reduction to the Dirac equation, the discussion of its symmetries and conservation laws, the coupling to gauge fields and its quantization according to the Lagrangean approach. In Sect. 3 we treat the corresponding problems for the lattice formulation of the Dirac-Kähler equation and we prove the equivalence to the Susskind reduction. In this paper, we give only a short discussion of interacting Dirac-Kähler fields on the lattice. We add an Appendix on the fundamental formulas of differential forms and its analogues from algebraic topology on the lattice. It is supposed to be helpful for those unfamiliar with these mathematical notions. We consider this Appendix as an integral part of the paper, because it underlines the geometrical basis of our approach.

2. The Dirac-Kähler Equation

We consider the Dirac-Kähler equation as the key for the understanding of the lattice formulation of fermions. The following description of the Dirac-Kähler equation is not given in the most general setting of an Atiyah-Kähler bundle on a manifold. In view of our intended applications, we rather restrict ourselves to an elementary consideration of the case of Euclidean space time. The extension to Minkowski space or even general manifolds is at this elementary level straightforward. However, the use of the calculus of differential forms is crucial for our applications. A glossary of the most important notations and formulas together with the relations to the corresponding lattice concepts is given in an Appendix.

2.1. In *the Dirac-Kähler equation*

$$(d-\delta+m)\Phi=0, \tag{2.1}$$

the Dirac field is described by a general real or complex differential form

$$\begin{aligned}\Phi=&\,\mathring{\varphi}(x)+\varphi_\mu(x)\,dx^\mu+\frac{1}{2!}\varphi_{\mu\nu}(x)\,dx^\mu\wedge dx^\nu\\&+\frac{1}{3!}\varphi_{\mu\nu\rho}(x)\,dx^\mu\wedge dx^\nu\wedge dx^\rho\\&+\varphi_{1234}(x)\,dx^1\wedge dx^2\wedge dx^3\wedge dx^4\\\equiv&\sum_H\varphi(x,H)\,dx^H.\end{aligned} \tag{2.2}$$

In the second line we introduced the multi-index notation: $\mathring{\varphi}(x)\equiv\varphi(x,\emptyset)$, $\varphi_{12}(x)\equiv\varphi(x,12)$, $\varphi_{\mu\nu\rho}(x)\equiv\varphi(x,(\mu\nu\rho))\equiv\varphi(x,H)$, H: ordered set of indices, which we explain in more detail at the beginning of the Appendix and in (A.19)ff. The exterior differentiation d can be written as

$$d\Phi = dx^\mu \wedge \partial_\mu \Phi, \tag{2.3}$$

where ∂_μ denotes the partial differentiation of the coefficients $\varphi_{\mu\ldots\nu}(x)$. The adjoint δ of d is defined with help of the ☆-operation

$$\delta = -☆^{-1} d ☆. \tag{2.4}$$

With the appropriate sign convention, the action of the linear ☆-operator on the basis differentials is given with help of the completely antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$, $\varepsilon^{1234} = +1$:

$$☆\,1 = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4,$$

$$☆\,dx^\mu = \frac{1}{3!}\varepsilon^\mu{}_{\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma,$$

$$☆\,dx^\mu \wedge dx^\nu = -\frac{1}{2!}\varepsilon^{\mu\nu}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma, \text{ etc.}, \quad ☆☆ = 1. \tag{2.5}$$

From the antisymmetry of the wedge product in (2.3) for d and hence for δ, it follows:

$$d^2 = \delta^2 = 0. \tag{2.6}$$

After this formal definition of the Dirac-Kähler equation (DKE), we want to make a first remark on the relation of the DKE to the conventional Dirac equation [10]. It is

$$(d-\delta)^2 = -(d\delta + \delta d) = \Box = \partial_\mu \partial^\mu. \tag{2.7}$$

In this sense $(d-\delta)$ is a square root of the Laplacean $\Box$. The operator $(d-\delta)$ shares this property with the Dirac-operator $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. As it is well-known, the construction of $\Box^{1/2}$ played a decisive rôle in the original derivation of the Dirac equation [14].

Following E. Kähler, we introduce a Clifford product in the space of differential forms [11, 15], ((A.58)ff.). This distributive and associative product of differential forms is defined by that of the generating elements dx^μ:

$$1 \vee 1 = 1, \quad 1 \vee dx^\mu = dx^\mu \vee 1 = dx^\mu,$$

$$dx^\mu \vee dx^\nu = g^{\mu\nu} \cdot 1 + dx^\mu \wedge dx^\nu, \tag{2.8}$$

($g^{\mu\nu} = \delta^{\mu\nu}$ = metric tensor). With help of the Clifford product we can write the differentiation $(d-\delta)$ in a form similar to (2.3), (A.74):

$$(d-\delta)\,\Phi = dx^\mu \vee \partial_\mu \Phi. \tag{2.9}$$

Since the defining relations (2.8) imply those of the γ-matrices: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, (2.9) clarifies the relation between the DKE and the Dirac equation: The mapping

$$\gamma^\mu \mapsto dx^\mu \vee, \tag{2.10}$$

$dx^\mu \vee \equiv$ Clifford left multiplication, defines a representation of the algebra of γ-matrices in the 16-dimensional space of complex differential forms. Because all the representations of the complex Clifford algebra of the 4-dimensional euclidean space can be decomposed into 4-dimensional irreducible representations [16] equivalent to those generated by the standard γ-matrices, we can decompose the space of differential forms $\mathscr{D} = \{\Phi\}$ into 4-dimensional invariant subspaces $\mathscr{D} = \bigoplus_{b=1}^{4} \mathscr{D}^{(b)}$, on which the DKE implies the Dirac equation:

$$(d-\delta+m)\,\Phi \overset{(2.9)}{=} (dx^\mu \vee \partial_\mu + m)\,\Phi = 0$$

$$\xrightarrow{(2.10)} (\gamma^\mu \partial_\mu + m)\,\Psi = 0 \quad \text{for } \Psi \in \mathscr{D}^{(b)}. \tag{2.11}$$

(For a complete description see (2.25–27).) Equipped with the operations $\wedge$, $\vee$ and ☆, the local cotangent space spanned by the differentials dx^μ has the structure of an Atiyah-Kähler algebra [17]. In this sense one may consider Dirac fields as cross-sections of an Atiyah-Kähler bundle on a manifold.

2.2. For the following it is important to make the relation between the DKE and the Dirac equation more explicit. Therefore we study *the decomposition of the Clifford algebra* into subspaces invariant under left $\vee$-multiplication, i.e. the decomposition into left ideals. For simplicity we consider the complex DKE, the real DKE will be discussed later. Following the usual techniques of representation theory [18], we introduce a new basis in $\mathscr{D}: Z = (Z_{ab})$, a, b $= 1, 2, 3, 4$, written in matrix form, with the property

$$dx^\alpha \vee (Z_{ab}) = \Big(\sum_c (\gamma^{\alpha T})_{ac} Z_{cb}\Big) \equiv \gamma^{\alpha T} Z. \tag{2.12}$$

This basis allows the decomposition of the Clifford algebra. Here the γ^μ denote Dirac matrices in any irreducible representation. For some explicit considerations we use the euclidean γ-matrices in the Weyl basis:

$$\gamma^i = \gamma_i = i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_i,\ i = 1, 2, 3, \text{ Pauli matrices}, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad \gamma_\mu^* = \gamma_\mu^T. \tag{2.13}$$

In the following we shall use the multiindex notation also for products of γ-matrices

$$\gamma^H \equiv \gamma^{\mu_1}\gamma^{\mu_2} \cdot \ldots \cdot \gamma^{\mu_h}, \quad \mu_1 < \mu_2 < \ldots < \mu_h \in H, \tag{2.14}$$

h: number of elements of H, $\gamma^\emptyset = 1$ for $h = 0$.

We shall show that

$$Z=1+\gamma_\mu^T dx^\mu+\frac{1}{2!}\gamma_\mu^T\gamma_\nu^T dx^\mu\wedge dx^\nu$$
$$+\frac{1}{3!}\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T dx^\mu\wedge dx^\nu\wedge dx^\rho$$
$$+\frac{1}{4!}\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T\gamma_\sigma^T dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma$$
$$\equiv\sum_H(-1)^{\binom{h}{2}}(\gamma_H)^T dx^H \tag{2.15}$$

satisfies (2.12). For the proof we need the following formula for the Clifford product of differential forms, [11], Appendix (A.58):

$$dx^\mu\vee\Phi=dx^\mu\wedge\Phi+e^\mu\lrcorner\Phi,$$
$$\Phi\vee dx^\mu=\Phi\wedge dx^\mu-e^\mu\lrcorner\mathscr{A}\Phi. \tag{2.16}$$

$e^\mu\lrcorner$ is the contraction operator, (A.49), which formally differentiates a differential form with respect to the differentials. It is defined as a linear operator by the following relations, including the product rule (γ):

$$e^\mu\lrcorner 1=0,\qquad e^\mu\lrcorner dx^\nu=g^{\mu\nu}, \tag{α}$$
$$e^\mu\lrcorner(\Phi+\Xi)=e^\mu\lrcorner\Phi+e^\mu\lrcorner\Xi, \tag{β}$$
$$e^\mu\lrcorner(\Phi\wedge\Xi)=(e^\mu\lrcorner\Phi)\wedge\Xi+(\mathscr{A}\Phi)\wedge e^\mu\lrcorner\Xi. \tag{γ}$$

The main automorphism $\mathscr{A}$, $\mathscr{A}(\Phi\wedge\Xi)=\mathscr{A}\Phi\wedge\mathscr{A}\Xi$, is defined by

$$\mathscr{A}({}^0\Phi+{}^1\Phi+{}^2\Phi+{}^3\Phi+{}^4\Phi)={}^0\Phi-{}^1\Phi+{}^2\Phi-{}^3\Phi+{}^4\Phi, \tag{2.18}$$

${}^p\Phi$ are the homogeneous parts of degree p of the general differential Φ. Similarly we have the main antiautomorphism

$$\mathscr{B}({}^0\Phi+{}^1\Phi+{}^2\Phi+{}^3\Phi+{}^4\Phi)={}^0\Phi+{}^1\Phi-{}^2\Phi-{}^3\Phi+{}^4\Phi. \tag{2.19}$$

$\mathscr{B}$ applied to the γ-matrices with the hermitecity condition (2.13) gives

$$(\gamma_H)^\dagger=\mathscr{B}\gamma_H=(-1)^{\binom{h}{2}}\gamma_H,\qquad(\gamma_H)^T=(-1)^{\binom{h}{2}}\gamma_H^*. \tag{2.20}$$

Further useful formulas on these morphisms are found in the Appendix, (A.32)ff.

Now we use (2.16) and the "differentiation rules" (2.17) in order to calculate

$$(dx^\alpha\vee Z_{ab})=(dx^\alpha\wedge+e^\alpha\lrcorner)\,Z=\sum_H(-1)^{\binom{h}{2}}(\gamma_H)^T(dx^\alpha\wedge+e^\alpha\lrcorner)\,dx^H=+\gamma^{\alpha T}$$
$$+dx^\alpha+\frac{1}{2!}(\gamma^{\alpha T}\gamma_\mu^T-\gamma_\mu^T\gamma^{\alpha T})\,dx^\mu+\gamma_\mu^T dx^\alpha\wedge dx^\mu+\frac{1}{3!}(\gamma^{\alpha T}\gamma_\mu^T\gamma_\nu^T-\gamma_\mu^T\gamma^{\alpha T}\gamma_\nu^T+\gamma_\mu^T\gamma_\nu^T\gamma^{\alpha T})\,dx^\mu\wedge dx^\nu$$
$$+\frac{1}{2!}\gamma_\mu^T\gamma_\nu^T dx^\alpha\wedge dx^\mu\wedge dx^\nu+\frac{1}{4!}(\gamma^{\alpha T}\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T-\gamma_\mu^T\gamma^{\alpha T}\gamma_\nu^T\gamma_\rho^T+\gamma_\mu^T\gamma_\nu^T\gamma^{\alpha T}\gamma_\rho^T-\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T\gamma^{\alpha T})\,dx^\mu\wedge dx^\nu\wedge dx^\rho$$
$$+\frac{1}{3!}\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T dx^\alpha\wedge dx^\mu\wedge dx^\nu\wedge dx^\rho=\gamma^{\alpha T}Z. \tag{2.12'}$$

This is the desired result, (2.12, 15). In this calculation we have used repeatedly the relation $\gamma_\mu^T\gamma_\nu^T=-\gamma_\nu^T\gamma_\mu^T+2g_{\mu\nu}$; f.i. in order to evaluate the homogeneous part of degree one:

$$dx^\alpha+\frac{1}{2!}(\gamma^{\alpha T}\gamma_\mu^T-\gamma_\mu^T\gamma^{\alpha T})\,dx^\mu=\gamma^{\alpha T}\gamma_\mu^T dx^\mu.$$

For the term of highest degree, we used that it is proportional to $dx^1\wedge dx^2\wedge dx^3\wedge dx^4$, and hence we get with help of $\gamma_\alpha^T\gamma^{\alpha T}=1$:

$$\frac{1}{3!}\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T dx^\alpha\wedge dx^\mu\wedge dx^\nu\wedge dx^\rho$$
$$=\gamma^{\alpha T}\left(\frac{1}{4!}\gamma_\mu^T\gamma_\nu^T\gamma_\rho^T\gamma_\sigma^T dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma\right).$$

By a similar calculation we can derive with help of $\Phi\vee dx^\alpha=\Phi\wedge dx^\alpha-e^\alpha\lrcorner\mathscr{A}\Phi$, (2.16), the formula for right $\vee$-multiplication:

$$Z\vee dx^\alpha=Z\gamma^{\alpha T}=(\sum_c Z_{ac}\gamma_{bc}^\alpha). \tag{2.21}$$

The transformation (2.15) can be inverted with help of the well-known trace and completeness relations [19] of the γ-matrices, (2.13):

$$\text{Trace }(\gamma^H(\gamma^K)^\dagger)=4\delta^{H,K}, \tag{2.22}$$

$$\sum_H\gamma_{ab}^H\gamma_{cd}^{H*}=4\delta_{ac}\delta_{bd}. \tag{2.23}$$

Applied to (2.15), this gives:

$$dx^H=\tfrac{1}{4}(-1)^{\binom{h}{2}}\text{Trace }(\gamma^{H*}Z). \tag{2.24}$$

The equations (2.12, 15, 21–23) allow a complete and explicit description of the relation between the DKE and the Dirac equation. For this we expand the general differential form with respect to the basis Z:

$$\Phi=\sum_H\varphi(x,H)\,dx^H=\sum_{a,b}\psi_a^{(b)}(x)\,Z_{ab}. \tag{2.25}$$

Then it follows immediately from (2.12) that the left ideal $\mathscr{D}^{(b)}$ is spanned by

$$\Phi^{(b)}=\sum_a\psi_a^{(b)}(x)\,Z_{ab}\in\mathscr{D}^{(b)},\ b\text{ fixed}, \tag{2.26}$$

and that the DKE for Φ implies the Dirac equation for $\psi^{(b)}(x)$:

$$(\gamma^\mu \partial_\mu + m)\,\psi^{(b)}(x) = 0,$$
$$\psi^{(b)}(x) = \begin{pmatrix} \psi_1^{(b)}(x) \\ \vdots \\ \psi_4^{(b)}(x) \end{pmatrix}, \qquad b = 1, \ldots, 4. \tag{2.26'}$$

This is the explicit interpretation of (2.11). We call the $(\psi_a^{(b)}(x)) = \psi(x)$ the Dirac components of the differential form Φ. The equations (2.22–25) give the transformation between the Dirac components and the cartesian components:

$$\varphi(x, H) = \mathrm{Trace}(\psi(x)(\gamma_H)^\dagger) \equiv \sum_{a,b} \psi_a^{(b)}(x)(\gamma_H)^\dagger_{ba}, \tag{2.27}$$
$$\psi_a^{(b)}(x) = \tfrac{1}{4} \sum_H \varphi(x, H)\, \gamma^H_{ab}.$$

Right $\vee$-multiplication transforms the different subspaces $\mathscr{D}^{(b)}$. This follows from the associative law and repeated application of (2.21). The following matrix representation of the right $\vee$-multiplication describes this fact most explicitly:

$$(Z_{ab} \vee C) = (\sum_d Z_{ad} \hat{C}_{db}) \equiv \hat{C} \cdot Z. \tag{2.28}$$

If $C = \sum_H C(H)\, dx^H$, then it follows from the formulas above:

$$\hat{C}_{ab} = \sum_H C(H)(\gamma^H)^T_{ab},$$
$$C(H) = \tfrac{1}{4}\mathrm{Trace}((\hat{C}_{ab})\,\gamma^{H*}). \tag{2.29}$$

Shifted on the Dirac components, right $\vee$-multiplication $\Phi' = \Phi \vee C$ induces the transformation

$$\psi_a'^{(b)}(x) = \sum_d \hat{C}_{bd}\, \psi_a^{(d)}(x). \tag{2.30}$$

The group of unitary transformations $\hat{C} \in SU(4)$ is called the (global) "flavour" group, following a suggestion of Susskind [20] for the equivalent lattice case. Equations (2.28, 29) define the representation of the flavour group by right $\vee$-multiplication.

We want to give some applications of these notions. It follows from straightforward calculation that the $\star$-operator, (2.5), can be written as right $\vee$-multiplication:

$$\star \Phi = \Phi \vee \varepsilon, \qquad \varepsilon := dx^{(1234)} \equiv dx^1 \vee dx^2 \vee dx^3 \vee dx^4. \tag{2.31}$$

Hence, according to (2.29, 13):

$$\star Z = Z\gamma^5, \qquad \gamma^5 := \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\star Z_{ai} = -Z_{ai}, \ i = 1, 2; \ \star Z_{ai} = +Z_{ai}, \ i = 3, 4. \tag{2.32}$$

The forms $\Phi \in \mathscr{D}^{(i)}$, $i = 3, 4$, are dual with respect to the $\star$-operation, those of $\mathscr{D}^{(i)}$, $i = 1, 2$, are antidual – in the Weyl representation of the γ-matrices.

Another application of these formulas is the construction of a spectral set $\{P^{(b)}\}$ of $\vee$-idempotent elements which characterize the minimal left ideals $\mathscr{D}^{(b)}$ [10]. The matrix which projects on the $\Phi \in \mathscr{D}^{(b)}$ is

$$(\hat{P}^{(b)}_{cd}) = \delta_{cb}\delta_{db}. \tag{2.33}$$

It is represented by the right $\vee$-multiplication with $P^{(b)}$, (2.28, 29):

$$Z_{ac} \vee P^{(b)} = Z_{ab}\delta_{bc},$$
$$P^{(b)} = \tfrac{1}{4}(1 + i\,\mathrm{sign}(12) \cdot dx^1 \wedge dx^2) \vee (1 + \mathrm{sign}(1234) \cdot \varepsilon),$$

with the sign combinations

$$(\mathrm{sign}\ (12), \mathrm{sign}\ (1234)) = (--), (+-), (-+), (++)$$
$$\text{for } b = 1, 2, 3, 4. \tag{2.34}$$

The $P^{(b)}$ have the property of a spectral set

$$P^{(b)} \vee P^{(\bar{b})} = \delta_{b\bar{b}} P^{(b)}, \qquad \sum_b P^{(b)} = 1. \tag{2.35}$$

$P^{(b)}$ projects on the irreducible subspace $\mathscr{D}^{(b)}$, i.e. on the minimal left ideal $\mathscr{D}^{(b)}$. All elements $\Phi \in \mathscr{D}^{(b)}$ are characterized by

$$\Phi \vee P^{(b)} = \Phi. \tag{2.36}$$

Combining the different results, we can make the following statement: The Dirac-Kähler equation (2.1) for differential forms, together with the "subsidiary conditions", (2.36) with b fixed, is equivalent to the conventional Dirac equation, according to (2.27).

For the discussion of the DKE on the lattice, it is useful to give the "subsidiary condition" (2.36) the form of a symmetry property. For this, we consider the group $\mathscr{R}$, called the reduction group, which is generated by $\vee$-multiplication of the elements $\tau := i dx^1 \wedge dx^2$ and ε:

$$\mathscr{R} = \{1, \tau, \varepsilon, \varepsilon \vee \tau\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{2.37}$$

If we regard $\mathscr{R}$ as a sub-group of the flavour group, i.e. acting on Φ by right $\vee$-multiplication

$$\hat{\tau} \cdot \Phi = \Phi \vee \tau, \qquad \hat{\varepsilon} \cdot \Phi = \Phi \vee \varepsilon, \tag{2.38}$$

then the subsidiary conditions (2.36) are equivalent to the symmetry properties

$$\hat{t}\cdot\Phi^{(b)}=\mathrm{sign}(12)\,\Phi^{(b)},\qquad \hat{\varepsilon}\cdot\Phi^{(b)}=\mathrm{sign}(1234)\,\Phi^{(b)} \tag{2.39}$$

with the sign combinations given in (2.34).

It should be evident how these results on the relation of the DKE to the conventional Dirac equation can be generalized to arbitrary even dimensions, in particular for the case of dimension 2, which is important for model considerations. (In odd dimensions the representation theory of the Clifford algebra is somewhat more involved [21].)

2.3. In the following we discuss the *symmetries of the Dirac-Kähler equation* and the *conserved currents* related to them.

The first symmetry we want to consider, is the global "flavour" symmetry described by the "flavour" group $SU(4)$ mentioned in the preceding section (2.28)ff.. It is a simple consequence of the associativity of the $\vee$-multiplication, that right $\vee$-multiplication with a constant differential (f.i. $U\in SU(4)$) transforms a solution of the DKE into another solution:

$$\begin{aligned}&\Phi'=\Phi\vee U,\\&(d-\delta+m)\,\Phi'\equiv(dx^\mu\vee\partial_\mu+m)\,(\Phi\vee U)\\&=((dx^\mu\vee\partial_\mu+m)\,\Phi)\vee U=0.\end{aligned} \tag{2.40}$$

Next we study the symmetry of the Euclidean DKE under the 4-dim. rotation group $SO(4)$ with help of a general formula which extends the definition of a Lie derivative L_α acting on functions, i.e. 0-forms

$$L_\alpha\overset{\circ}{\varphi}(x)=\alpha^\mu(x)\,\partial_\mu\overset{\circ}{\varphi}(x) \tag{2.41}$$

to one which is defined on general differential forms

$$L_\alpha\Phi=\alpha^\mu(x)\,\partial_\mu\Phi+\tfrac{1}{4}(d\alpha\vee\Phi-\Phi\vee d\alpha) \tag{2.42}$$

(see [11] for the assumption which is necessary for this procedure). $\alpha=\alpha_\mu(x)\,dx^\mu=\alpha^\mu(x)\,g_{\mu\nu}dx^\nu$ denotes the Killing form associated with the Lie derivative L_α. Commutator relations between different Lie derivatives are conserved by this extension. This procedure is illustrated by an application to the infinitesimal rotations. The Lie derivative of an infinitesimal rotation in the $\mu\nu$-plane, i.e. the operator of the orbital angular momentum (multiplied by i), is

$$L^{\mu\nu}\overset{\circ}{\varphi}(x)=(x^\mu\partial^\nu-x^\nu\partial^\mu)\,\overset{\circ}{\varphi}(x). \tag{2.43}$$

Its extension is

$$L^{\mu\nu}\Phi=(x^\mu\partial^\nu-x^\nu\partial^\mu)\,\Phi+\tfrac{1}{2}(S^{\mu\nu}\vee\Phi-\Phi\vee S^{\mu\nu}), \tag{2.44}$$

$$S^{\mu\nu}=\tfrac{1}{2}d\alpha^{\mu\nu}=\tfrac{1}{2}d(x^\mu dx^\nu-x^\nu dx^\mu)=dx^\mu\wedge dx^\nu. \tag{2.44'}$$

The "Clifford commutator", $[S^{\mu\nu},\Phi]_\vee:=S^{\mu\nu}\vee\Phi-\Phi\vee S^{\mu\nu}$ in (2.44) describes correctly the infinitesimal rotations of the differentials, f.i.:

$$\tfrac{1}{2}[S^{\mu\nu},dx^\rho]_\vee=g^{\nu\rho}dx^\mu-g^{\mu\rho}dx^\nu\qquad\text{etc.} \tag{2.45}$$

From this, it follows that the extended Lie derivative commutes with the DK operator

$$L^{\mu\nu}(d-\delta+m)\,\Phi=(d-\delta+m)\,L^{\mu\nu}\Phi \tag{2.46}$$

thus describing the rotational symmetry of the DKE. The same discussion for translation invariance expressed by the "Lie derivative" $P^\mu=\partial^\mu$, ($\rightarrow d\alpha=0$), is trivial.

What is the relation of the rotation symmetry of the DKE, (2.44–46), to the well-known rotation symmetry of the Dirac equation? This question is somewhat intriguing, because (2.44) describes representations with integral spin, whereas the symmetry transformations of the Dirac equation contain half integral spins [22]. Formally the answer to this question is the following. The minimal left ideals $\mathcal{D}^{(b)}$, (2.26, 36), are not invariant under the infinitesimal rotations $L^{\mu\nu}$, (2.44). However, there are transformations $J^{\mu\nu}$ combining the infinitesimal rotations $L^{\mu\nu}$ with infinitesimal "flavour" transformations $\hat{S}^{\mu\nu}$

$$\begin{aligned}&\delta\Phi=J^{\mu\nu}\Phi\cdot\delta\beta_{\mu\nu}=(L^{\mu\nu}+\hat{S}^{\mu\nu})\,\Phi\cdot\delta\beta_{\mu\nu}\\&=(x^\mu\partial^\nu-x^\nu\partial^\mu+\tfrac{1}{2}S^{\mu\nu}\vee)\,\Phi\cdot\delta\beta_{\mu\nu},\\&\hat{S}^{\mu\nu}\cdot\Phi=\tfrac{1}{2}\Phi\vee S^{\mu\nu},\end{aligned} \tag{2.47}$$

which

(i) leave the $\mathcal{D}^{(b)}$ invariant (2.12, 26),

(ii) commute with the DK operator (2.40, 46) and

(iii) satisfy the commutation relations of $SO(4)$

$$[J^{\mu\nu},J^{\rho\sigma}]=-\delta^{\mu\rho}J^{\nu\sigma}+\delta^{\mu\sigma}J^{\nu\rho}-\delta^{\nu\sigma}J^{\mu\rho}+\delta^{\nu\rho}J^{\mu\sigma}.$$

These $J^{\mu\nu}$ are the well-known 4-dim. angular momentum operators of a Dirac field (as can be seen from (2.12, 25) and (2.44')). We may interprete this result by the following statement: "The half-integral spin of Dirac fields can be described by a coherent superposition of differential forms. The coherence is formulated with help of an irreducibility condition applied to the representation of the Clifford algebra on differential forms via left $\vee$-multiplication". Such a form of a coherence condition is common in quantum mechanics [23].

The chiral symmetry of massless Dirac fields plays an important rôle in the discussion of lattice formulations of fermions. Hence we consider this symmetry here for the DKE. According to (2.12, 25), the infinitesimal chiral transformation of a Dirac field: $\delta\psi = i\gamma^5\psi \cdot \delta\beta$ corresponds to the "chiral transformation" of the differential forms:

$$\delta\Phi = \varepsilon \vee \Phi \cdot \delta\beta \equiv (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4) \vee \Phi \cdot \delta\beta. \quad (2.48)$$

It can be easily seen that $\varepsilon \vee (dx^\mu \vee \partial_\mu) = -(dx^\mu \vee \partial_\mu)\,\varepsilon\,\vee$, and hence solutions of the massless DKE $(d-\delta)\,\Phi=0$ remain solutions under chiral transformations. The main automorphism $\mathscr{A}$, (2.18), transforms solutions of the massless DKE into solutions, too:

$$(d-\delta)\,\mathscr{A}\Phi = -\mathscr{A}(d-\delta)\,\Phi = 0. \quad (2.49)$$

Since we can write $\mathscr{A}$ in the form $\mathscr{A}\Phi = \varepsilon \vee \Phi \vee \varepsilon$, the relation between $\mathscr{A}$ and the chiral transformation equation (2.48) is similar to that between the infinitesimal rotations $L^{\mu\nu}$ and $J^{\mu\nu}$ according to (2.44, 47).

Symmetries imply conserved currents. In order to relate the symmetries discussed above to currents, we may use the following identity for differential forms [11]

$$d(\Phi, \Xi)_1 = (\Phi, (d-\delta)\,\Xi)_0 + ((d-\delta)\,\Phi, \Xi)_0. \quad (2.50)$$

Here we used expressions $(\Phi, \Xi)_p$ called scalar products by Kähler. $(\Phi, \Xi)_p$ is a $(d-p)$-form, $(d=4)$, constructed from the forms $\Phi, \Xi \in \mathscr{D}$ in the following manner:

$$(\Phi, \Xi)_0 := ((\mathscr{B}\Phi) \vee \Xi) \wedge \varepsilon$$
$$= (\mathring{\varphi}\mathring{\xi} + \varphi_\mu \xi^\mu + \frac{1}{2!}\varphi_{\mu\nu}\xi^{\mu\nu}$$
$$+ \frac{1}{3!}\varphi_{\mu\nu\rho}\xi^{\mu\nu\rho} + \frac{1}{4!}\varphi_{\mu\nu\rho\sigma}\xi^{\mu\nu\rho\sigma})\,\varepsilon$$
$$= (\sum_H \varphi(x,H)\,\xi(x,H))\,\varepsilon, \quad (2.51)$$

$$(\Phi, \Xi)_1 := e_\alpha \lrcorner (dx^\alpha \vee \Phi, \Xi)_0 = e_\alpha \lrcorner \{(dx^\alpha \vee \Phi \vee \mathscr{B}\Xi) \wedge \varepsilon\}$$
$$= \Big\{\mathring{\varphi}\xi_\alpha + \varphi_\alpha\mathring{\xi} + \varphi^\mu \xi_{\alpha\mu} + \varphi_{\alpha\mu}\xi^\mu + \frac{1}{2!}(\varphi_{\alpha\mu\nu}\xi^{\mu\nu} + \varphi^{\mu\nu}\xi_{\alpha\mu\nu})$$
$$+ \frac{1}{3!}(\varphi_{\alpha\mu\nu\rho}\xi^{\mu\nu\rho} + \varphi^{\mu\nu\rho}\xi_{\alpha\mu\nu\rho})\Big\}\frac{1}{3!}\varepsilon^\alpha{}_{\beta\gamma\delta}\,dx^\beta \wedge dx^\gamma \wedge dx^\delta, \quad (2.51')$$

$$(\Phi, \Xi)_p := e_{\mu_1} \lrcorner e_{\mu_2} \lrcorner \ldots e_{\mu_p} \lrcorner (dx^{\mu_p} \vee \ldots \vee dx^{\mu_1} \vee \Phi, \Xi)_0$$
$$= \ldots. \quad (2.51'')$$

These scalar products are bilinear and symmetric: $(\Phi, \Xi)_p = (-1)^{\binom{p}{2}}(\Xi, \Phi)_p$. (For additional formulas see [11]). The complexified scalar products expressed in Dirac components, (2.23, 27), get a familiar form

$$(\Phi^\dagger, \Phi)_0 = 4 \cdot \sum_{a,b} \psi_a^{(b)}(x)^* \psi_a^{(b)}(x) \cdot \varepsilon, \quad (2.52)$$

$$(\Phi^\dagger, \Phi)_1 = e_\alpha \lrcorner \{(dx^\alpha \vee \Phi^\dagger \vee \mathscr{B}\Phi) \wedge \varepsilon\}$$
$$= 4 \cdot \sum_{a,a',b} (\psi_a^{(b)}(x)^* \gamma^\alpha_{aa'} \psi_{a'}^{(b)}(x))\, e_\alpha \lrcorner \varepsilon, \quad (2.52')$$

where we easily calculate: $\star e_\alpha \lrcorner \varepsilon = dx_\alpha$. Similarly, higher products correspond to higher tensor products, like T, A, P.

From Kähler's formula (2.50), conservation laws follow easily. If Φ is a solution of the DKE then $\bar{\Phi} := \mathscr{A}\Phi^\dagger$ is a solution of the adjoint DKE:

$$(d-\delta-m)\,\bar{\Phi} = 0, \qquad \bar{\Phi} := \mathscr{A}\Phi^\dagger. \quad (2.53)$$

Furthermore, if Φ' and $\bar{\Phi}$ are solutions of the DKE and its adjoint equation, respectively, then

$$j = j_\mu dx^\mu = \star^{-1}(\bar{\Phi}, \Phi')_1 \quad (2.54)$$

satisfies a conservation law, as it is seen from (2.50):

$$\delta j = -\star^{-1} d \star j = -\star^{-1} d(\bar{\Phi}, \Phi')_1$$
$$= -\star^{-1}\{((d-\delta)\,\bar{\Phi}, \Phi')_0 + (\bar{\Phi}, (d-\delta)\,\bar{\Phi})_0\} = 0, \quad (2.54')$$

or

$$\partial^\mu j_\mu = 0.$$

These currents get a familiar form, if we express the Dirac components of the adjoint form $\bar{\Phi}$ by

$$\bar{\psi}_a^{(b)}(x) = \tfrac{1}{4}\sum_H \bar{\varphi}(x,H)\,\gamma^{H*}_{ab} \quad (2.55)$$

instead of (2.27). Then $\bar{\psi}^{(b)}(x) = (\bar{\psi}_1^{(b)}(x), \ldots, \bar{\psi}_4^{(b)}(x))$ satisfies the adjoint Dirac equation

$$(\gamma^{\mu T}\partial_\mu - m)\,\bar{\psi}^{(b)}(x)^T = 0$$

as a consequence of the adjoint DKE (2.53). Similar to (2.52), the expressions for the most familiar conserved currents related to symmetries become:

– from reality of the DKE:

$$j = \tfrac{1}{4}\star^{-1}(\bar{\Phi}, \Phi)_1 \to j_\mu(x) = \sum_b \bar{\psi}^{(b)}(x)\,\gamma_\mu \psi^{(b)}(x), \quad (2.56)$$

– from flavour invariance:

$$j = \tfrac{1}{4}\star^{-1}(\bar{\Phi}, \Phi \vee C)_1 \to j_\mu(x) = \sum_{b,\bar{b}} \bar{\psi}^{(b)}(x)\gamma_\mu \hat{C}_{b\bar{b}}\psi^{(\bar{b})}(x), \quad (2.56')$$

– from γ^5 invariance of the massless DKE:

$$j = \tfrac{1}{4}\star^{-1}(\bar{\Phi}, \varepsilon \vee \Phi)_1 \to j_\mu(x) = \sum_b \bar{\psi}^{(b)}(x)\,\gamma_\mu \gamma^5 \psi^{(b)}(x). \quad (2.56'')$$

Finally we would like to have a look on the real solutions of the DKE. These seem even more natural from a geometrical point of view. The reality of the DK field: $\Phi^\dagger = \Phi$ might be expressed in the Dirac components. Of course, this form of the reality condition depends on the representation of the Dirac matrices. In our Weyl basis, (2.13), we have for the complex conjugate of Dirac components of real Dirac-Kähler forms

$$(\psi_a^{(b)*}) = -\gamma^{13}(\psi_{\cdot}^{(\cdot)})\,\gamma^{13}, \qquad \gamma^{13} = -\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \tag{2.57}$$

In order to get the physical interpretation of this relation, we have to consider the charge conjugation of Dirac fields. For euclidean fields, this is somewhat involved [24]. Therefore we repeat the essential points. We start from the well-known definition of charge conjugation of Minkowski fields

$$(\mathscr{C}\psi)(x) = C\bar{\psi}(x)^T = C\gamma^0\psi(x)^*, \qquad \mathscr{C}^2 = +1. \tag{2.58}$$

In Weyl basis, (2.13), $\gamma^0 = \frac{1}{i}\gamma^4$, it is

$$C = -i\gamma^{02} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

By the continuation to the euclidean region, the relation $\bar{\psi}(x) = \psi(x)^*\gamma^0$ becomes $\bar{\psi}_{\text{eucl}}(x^0, \vec{x}) = \psi_{\text{eucl}}(-x^0, \vec{x})^\dagger\gamma^0$. It relates fields at different space-time points. Therefore in a consistent treatment of local euclidean Dirac fields, one has to consider $\bar{\psi}_{\text{eucl}}(x)$ as an independent field, which satisfies the adjoint Dirac equation. The charge conjugate field is then defined by ${}^c\psi(x) = C\bar{\psi}(x)^T$, $C = \gamma^{24}$. It satisfies the Dirac equation. We use this formula in order to define the charge conjugation of a general complex euclidean Dirac-Kähler field by that of its Dirac components ${}^c\psi^{(b)}(x) = C\bar{\psi}^{(b)}(x)^T$:

$$\begin{aligned} {}^c\Phi &= \sum_{a,b} {}^c\psi_a^{(b)}(x)\,Z_{ab} = \sum_{a,b} (C\bar{\psi}^{(b)}(x))_a Z_{ab} \\ &= \sum_{a,b} \bar{\psi}_a^{(b)}(x)\,(\gamma^{24}\gamma^{13}Z^*\gamma^{13})_{ab} \qquad Z_{ab} \text{ with } (2.57) \\ &= \mathscr{A}\bar{\Phi} \vee dx^{24} \qquad \text{with } (2.12, 49\text{f}). \end{aligned}$$

Although there are no real euclidean Dirac fields, there are real DK fields. Following (2.53), we may set for real DK fields

$$\bar{\Phi} = (\mathscr{A}\Phi) \vee dx^4. \tag{2.60}$$

Then we get

$${}^c\Phi = \Phi \vee dx^2. \tag{2.60'}$$

The right-$\vee$-factor dx^4 in (2.60) is necessary to get ${}^c({}^c\Phi) = \Phi$. Since in our basis:

$$\gamma^2 = \begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & -1 & & \\ 1 & 0 & & \end{pmatrix},$$

the flavour transformation related to charge conjugation interchanges the (1–4) and (2–3) flavours. This means that for real DK fields, the different flavours represent pairwise charge conjugate fields.

2.4. The *Dirac-Kähler equation in the presence of a gauge field* follows straightforward from free DKE (2.1). This procedure is particularly simple for the coupling of a complex DK field to a $U(1)$-gauge field $A_\mu(x)$. In this case one has only to substitute the derivative ∂_μ by the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ in (2.9):

$$(dx^\mu \vee D_\mu + m)\,\Phi \equiv (dx^\mu \vee (\partial_\mu - ieA_\mu) + m)\,\Phi = 0, \tag{2.61}$$

or, with the differential form $A = eA_\mu(x)\,dx^\mu$:

$$(d - \delta + m)\,\Phi = iA \vee \Phi. \tag{2.62}$$

The resulting DKE with minimal coupling to an electromagnetic field is then invariant against local gauge transformations

$$\begin{aligned} &\Phi(x) \mapsto e^{i\theta(x)}\Phi(x), \\ &A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x). \end{aligned} \tag{2.63}$$

In the interesting case of a non-abelian gauge theory like QCD, the differential forms get an additional colour index α or, in general, they become vectors with respect to a representation of the gauge group G. The DKE with gauge interaction then looks like (2.61) with D_μ the covariant derivative with respect to G. In the case of QCD the field equation for quarks becomes

$$(d - \delta)\,\Phi = iA \vee \Phi \tag{2.64}$$

with the Lie-algebra valued differential form [25] $A = g\frac{\lambda_a}{2}A_\mu^a(x)\,dx^\mu$. In mathematical terms $\Phi_\alpha = \sum_H \varphi_\alpha(\mathrm{x}, H)\,\mathrm{d}x^H$ is a cross section in the Whitney product of the Atiyah-Kähler bundle of differential forms with the vector bundle of local colour spaces [10]. This concept is particularly useful for the consideration of gauge interactions in curved space-time manifolds as in general relativity.

The proof of the flavour invariance of the DKE, (2.40), goes through for the DKE with gauge interaction (2.64). Hence, the decomposition of this equation into elementary Dirac equations according to (2.36)ff. is possible, too. However, it is not in the spirit of the geometrical approach to consider the DKE always in its Dirac decomposition. The powerful calculus with differential forms allows often a direct solution of this equation. The treatment of the electron in a Coulomb field by Kähler [11] may serve as an example for such a procedure.

On the other hand, one should not forget that the DKE describes a multiplet of Dirac fields with global flavour symmetry, (2.30, 40). The discussion of the lattice formulation of the DKE will give some arguments (Sect. 3.5d) in favour of the consideration of the flavour symmetry as a local symmetry. For this one has to insert in (2.61) the covariant derivative D_μ in the form

$$D_\mu \Phi = (\partial_\mu + \mathscr{A}_\mu)\,\Phi,\qquad \mathscr{A}_\mu \Phi = \sum_i A^i_\mu \Phi \vee \tau_i. \tag{2.65}$$

τ_i is the differential form which generates infinitesimally the flavour group $\mathscr{G}=\{C\}$ defined in (2.30). The DKE with minimal gauge invariant flavour coupling

$$(dx^\mu \vee D_\mu + m)\,\Phi = (dx^\mu \vee \partial_\mu + m)\,\Phi + A^i_\mu dx^\mu \vee \Phi \vee \tau_i = 0 \tag{2.66}$$

is invariant under local gauge transformations

$$\Phi \mapsto \Phi \vee C(x), \qquad \mathscr{A}_\mu \mapsto C(x) \vee (\mathscr{A}_\mu + \partial_\mu) \vee C^{-1}(x). \tag{2.67}$$

This can be shown in the usual way by a simple straightforward calculation. We would like to illustrate this scheme by a simple example, which will be used in a later discussion. The group $\mathscr{G}$ is the Abelian group generated by $\tau = dx^{12}$.

It follows from (2.28, 30):

$$\hat{C} \cdot \Phi \equiv \Phi \vee C = \Phi e^{-\vee \theta\tau} = \Phi \vee (\cos\theta - \sin\theta\, dx^1 \wedge dx^2). \tag{2.68}$$

Hence, the gauge transformations and covariant derivatives of such a DK field are

$$\Phi'(x) = \Phi(x) \vee (\cos\theta(x) - \sin\theta(x)\, dx^1 \wedge dx^2),\qquad A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x),\qquad D_\mu \Phi = \partial_\mu \Phi + A_\mu(x)\,\Phi \vee dx^{12}. \tag{2.69}$$

For Dirac components this corresponds to

$$\psi^{(b)}_a(x) \mapsto \psi^{(b)}_a(x)\, e^{\mp i\theta(x)},\ \mp\Big\}\text{for } b = \begin{cases}1,3\\ 2,4\end{cases} \tag{2.70}$$

as might be seen from (2.29, 30). If we consider this flavour gauge coupling to a real DK field, then the charge generator τ anticommutes with charge conjugation ${}^c\Phi = \Phi \vee dx^2$; (2.60'). This model describes correctly two charged fields coupled to an abelian gauge field.

2.5. Finally we discuss the *Lagrange function* of the Dirac-Kähler equation and its *formal quantization* with help of the Feynman path integral formula. Let us consider first the complex Dirac-Kähler field Φ. The Dirac-Kähler equation might be derived from the following action:

$$S_m[\bar\Phi, \Phi] = \tfrac{1}{4}\int (\bar\Phi, (d_A - \delta_A + m)\,\Phi)_0, \tag{2.71}$$

where the gauge invariant coupling is described with help of the covariant exterior differentiation d_A and its adjoint δ_A, formed with help of the covariant derivative D_μ:

$$d_A \Phi = dx^\mu \wedge D_\mu \Phi, \qquad \delta_A \Phi = -e^\mu \lrcorner D_\mu \Phi. \tag{2.72}$$

This action becomes

$$S_m = \sum_b \int d^4x\, \bar\psi^{(b)}(x)\,(\gamma^\mu D_\mu + m)\,\psi^{(b)}(x)$$

when we express Kähler's scalar product $(\bar\Phi, \Phi')_0$, (2.51, 52) by the Dirac components $\psi^{(b)}_a(x)$, $\bar\psi^{(b)}_a(x)$, (2.27, 55). It is a sum of the Dirac actions of the independent flavours, in agreement with our interpretation of the DK field Φ. In deriving the Dirac-Kähler equation from the euclidean S by the action principle, we have to consider Φ and $\bar\Phi$ as independent fields [26]. By the same reason we have to integrate over the independent Grassmann fields Φ and $\bar\Phi$ in the path integral formula for the generating functional of the Schwinger functions of the Dirac-Kähler fields [27]:

$$T\{\bar\eta, \eta\} = \mathscr{Z}\{\bar\eta, \eta\}/\mathscr{Z}\{0, 0\},\qquad \mathscr{Z}\{\bar\eta, \eta\} = \int \mathscr{D}[A] \int \mathscr{D}[\Phi] \int [\bar\Phi] \cdot e^{S_m[\bar\Phi, \Phi] + S_G[A] + \int (\bar\eta, \Phi)_0 + (\bar\Phi, \eta)_0}. \tag{2.73}$$

For the Grassmann integration $\int \mathscr{D}[\Phi] \int \mathscr{D}[\bar\Phi]$ we adopt the sign convention "$\int \mathscr{D}[\Phi] \int \mathscr{D}[\bar\Phi]\, \bar\Phi\Phi = +1$". The action of the gauge field is called $S_G[A]$; we suppress all details related to the gauge field integration. The action of the DK field has the bilinear form $S_m = (\bar\Phi, \Gamma\Phi)$. Therefore "Gaussian integration" allows the evaluation of the Grassmann integral [28]:

$$\int \mathscr{D}[\Phi] \int \mathscr{D}[\bar\Phi]\, e^{S_m[\bar\Phi, \Phi] + \int (\bar\eta, \Phi)_0 + (\bar\Phi, \eta)_0} = e^{-4\int (\bar\eta, (d_A - \delta_A + m)^{-1}\eta)_0} \cdot \det\left[\tfrac{1}{4}(d_A - \delta_A + m)\right]. \tag{2.74}$$

The formal expressions (2.73, 74) are the starting points for the derivation of Feynman rules for DK fields, effective interactions etc. This is not subject of this paper. We only want to add the expression of the propagator for DK fields, which we get from evaluating (2.74) for the free case:

$$\sum_{H,K} \int d^4x \int d^4y \bar{\eta}(x,H) \langle \varphi(x,H) \bar{\varphi}(y,K) \rangle \eta(y,K)$$
$$= -4 \int (\bar{\eta}, (d-\delta+m)^{-1} \eta)_0.$$

Using $(d-\delta+m)^{-1} = (d-\delta-m)(\Box - m^2)^{-1}$, with $d - \delta = dx^\mu \vee \partial_\mu$ and $(\Box - m^2)\, \Delta_S(x-y) = -\delta(x-y)$, as well as the definition of the scalar product, (2.51), leads to

$$\langle \varphi(x,H) \bar{\varphi}(y,K) \rangle$$
$$= 4\Big(\sum_\mu [H;\mu;K]\, \partial_\mu - m\,\delta^{H,K}\Big)\, \Delta_S(x-y) \tag{2.75}$$

with

$$\mathscr{B} dx^H \vee dx^\mu \vee dx^K) \wedge \varepsilon = [H;\mu;K]\, \varepsilon \tag{2.76}$$
$$[H;\mu;K] = \begin{cases} \rho_{\{\mu\}, K\setminus\{\mu\}} \cdot \delta^{H, K\setminus\{\mu\}} & \text{if } \mu \in K, \\ \rho_{\{\mu\}, K} \cdot \delta^{H, K \cup \{\mu\}} & \text{if } \mu \notin K. \end{cases}$$

For the notation see the Appendix.

The decomposition of the $\varphi(x,H)$, $\bar{\varphi}(x,K)$ in Dirac components shows that (2.75) is in agreement with the vacuum expectation values for euclidean Dirac fields

$$\langle \psi_a^{(b)}(x) \bar{\psi}_{a'}^{(b')}(y) \rangle = \delta^{b,b'} S_{aa'}(x-y),$$
$$S(x) = (\gamma^\mu \partial_\mu - m)\, \Delta_S(x). \tag{2.77}$$

We add a short remark on the action of a free real Dirac field. Following (2.60, 71) we set

$$S[\Phi] = \tfrac{1}{8} \int ((\mathscr{A}\Phi \vee dx^4, (d-\delta+m)\,\Phi)_0. \tag{2.78}$$

With the interpretation of the real DK field Φ as a pair of charge conjugate Dirac fields $\psi^{(1)}(x)$, $\psi^{(2)}(x)$, $\psi^{(3)}(x) = -{}^c\psi^{(2)}(x)$, $\psi^{(4)}(x) = +{}^c\psi^{(1)}(x)$, the action S is the appropriate Dirac action

$$S = \sum_{b=1}^{2} \int d^4x\, \bar{\psi}^{(b)}(x) (\gamma^\mu \partial_\mu + m)\, \psi^{(b)}(x).$$

The form $\{\Phi, \Xi\}$ is antisymmetric:

$$\{\Phi, \Xi\} := ((\mathscr{A}\Phi) \vee dx^4, \Xi)_0 = -\{\Xi, \Phi\}. \tag{2.79}$$

This follows directly from the properties of the Dirac-Kähler product $(\Phi, \Xi)_0 = (\Xi, \Phi)_0 = (\mathscr{A}\Phi, \mathscr{A}\Xi)_0$, (2.51), and $(\mathscr{A}\Phi) \vee \mathrm{d}x^4 = -\mathscr{A}(\Phi \vee \mathrm{d}x^4)$. Therefore the path integral for the generating functional (2.73) can be evaluated by Gaussian integration. It leads to a 2-point function of the form (2.75), with $\bar{\Phi}$ set equal to $(\mathscr{A}\Phi) \vee dx^4$.

In this discussion of the Feynman path integral formulas for DK fields, the action S_m could be always separated in the actions of the Dirac components; $S_m = \sum_b S^{(b)}$. From this we conclude that we can represent, formally, the Green's functions of Dirac fields by path integrals over DK fields which satisfy the subsidiary conditions (2.36).

Let us shortly summarize the results of this section. We discussed Dirac spinors in terms of differential forms Φ. The general solutions of the DKE are equivalent to four (complex Φ) or two (real Φ) Dirac fields. Then we studied several physical concepts like symmetries, currents, minimal coupling to gauge fields and quantization by Grassmann path integrals for Dirac-Kähler fields. The necessary tools for relating them to Dirac fields were given. This differential geometric description of fermions might be a basis for the construction of different kinds of field theoretical models. For such purposes we plan to develop further the Feynman rules for interacting DK fields in a forthcoming paper. Our main interest here is to analyse with help of the DKE the problems of the lattice approximation of the Dirac equation.

3. The Dirac-Kähler Equation on the Lattice

3.1. There is a natural way to find a formal lattice approximation to a field theory which is formulated in terms of differential forms. In order to express such a *correspondence between continuum and lattice*, one considers the lattice embedded in the euclidean space-time and the p-forms mapped on functions depending on p-dimensional lattice elements:

$${}^p\Phi \xrightarrow{\Sigma} \overline{{}^p\Phi}({}^pC_i) = \int_{{}^pC_i} {}^p\Phi. \tag{3.1}$$

We use sometimes the language of algebraic topology [13], and consider the lattice as a cell complex Γ with its elements ${}^pC_i$ as cells: points, links, plaquettes, cubes, super-cubes for $p=0, 1, 2, 3, 4$. In this spirit we regard (3.1) as the definition of a "cochain", i.e. a linear functional $\overline{{}^p\Phi}({}^pC)$ defined on the "p-chains": ${}^pC = \sum_i \alpha_i {}^pC_i$ of the lattice Γ. The well-known boundary Δ and co-boundary ∇ of a cell ${}^pC$ have an intuitive geometric meaning (for examples see Fig. 1). They are extended as linear operators acting on chains C of any dimension p. The "dual boundary operator $\check{\Delta}$" and "dual co-boundary operator $\check{\nabla}$" acting on co-chains are defined by

$$(\check{\Delta}\bar{\Phi})(C) = \bar{\Phi}(\Delta C), \quad (\check{\nabla}\bar{\Phi})(C) = \bar{\Phi}(\nabla C). \tag{3.2}$$

It follows from Stokes' theorem, that the mapping Σ, (3.1), transforms the exterior differential d into the

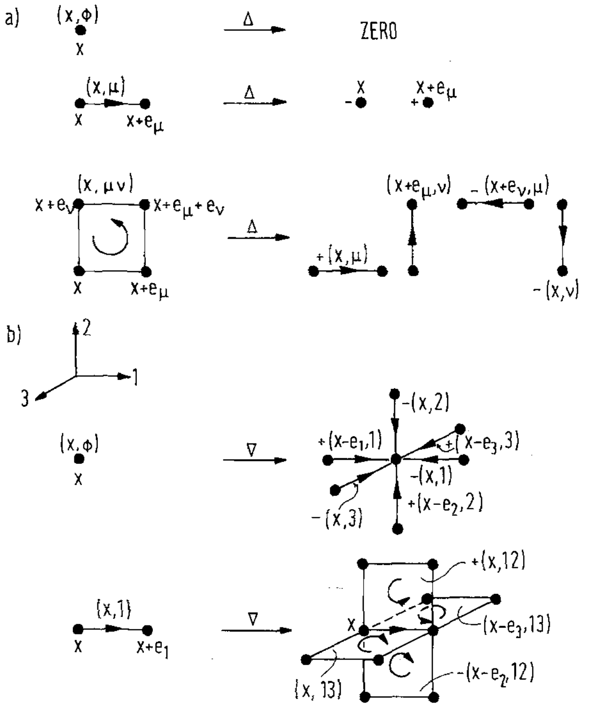

Fig. 1. a The boundary Δ of elementary zero-, one- and two-chains. b The co-boundary ∇ of elementary zero- and one-chains in three dimensions

dual boundary-operator $\check{\Delta}: \Sigma d = \check{\Delta}\Sigma$, i.e.

$$\int_{{}^pC} d({}^{p-1}\Phi) = \int_{\Delta\,{}^pC} {}^{p-1}\Phi = \bar{\Phi}(\Delta\,{}^pC) = (\check{\Delta}\bar{\Phi})({}^pC). \tag{3.3}$$

This is an example of a correspondence between objects and operations on the lattice and on the space-time manifold. Another example is the correspondence between δ and $\check{\nabla}$ which might be based on formula (2.4). The $\star$-operation maps the lattice on its dual lattice such that $\nabla = -\star^{-1}\Delta\star$. Transferred to the co-chains gives $\delta \sim \check{\nabla}$. Our considerations up to now lead to the upper half of Table 1:

Continuum	Lattice
path	chain C
boundary Δ	boundary Δ
differential forms Φ	co-chains $\bar{\Phi}$
exterior differentiation d	dual boundary operator $\check{\Delta}$
$d^2=0$	$\check{\Delta}^2=0$
co-differential δ	dual co-boundary operator $\check{\nabla}$
$\delta^2=0$	$\check{\nabla}^2=0$
Laplace operator $-(d\delta+\delta d)$	Laplace operator $-(\check{\Delta}\check{\nabla}+\check{\nabla}\check{\Delta})$
wedge product $\wedge$	cup product $\wedge$
contraction operator $e^\mu \lrcorner$	contraction operator $e^\mu \lrcorner$
Clifford product $\vee$	Clifford product $\vee$

The Appendix contains a glossary of all these different notions which puts its emphasis on the continuum lattice correspondence.

Applying these correspondences to linear field equations in differential form results in a natural lattice approximation. This procedure gives for the DKE (2.1) the Dirac-Kähler equation on the lattice

$$(\check{\Delta} - \check{\nabla} + m)\,\bar{\Phi} = 0. \tag{3.4}$$

For the further analysis of this equation, the possibility of extending these correspondences to operations like $\star$, $\wedge$, $\vee$, ... will be decisive. In the following we shall consider only cubic lattices. This allows us to introduce the following notations (see Fig. 1):

$$\begin{aligned} &\text{points } {}^0C_i := x = a(n^1, \ldots, n^4), \\ &\text{links } {}^1C_i := (x, x+e_\mu) \equiv (x,\mu), \\ &\quad\vdots \\ &{}^pC_i := (x, x+e_{\mu_1}, x+e_{\mu_2}, \ldots, x+e_{\mu_p}) \equiv (x, H). \end{aligned} \tag{3.5}$$

a is the lattice constant set equal to 1 in most of the general consideration, e_μ the free unit vector in μ-direction, $H = \{\mu_1, \ldots, \mu_p\}$ as in (2.2). The basis of co-chains dual to the cells (3.5) is defined by

$$d^{x,H}((x', H')) = \delta^x_{x'}\,\delta^H_{H'}. \tag{3.6}$$

Hence, a general co-chain can be written in the form

$$\bar{\Phi} = \sum_{x,H} \varphi(x, H)\, d^{x,H}. \tag{3.7}$$

The action of the dual boundary and dual co-boundary operator on the basis vectors $d^{x,H}$ is expressed by the following formulas:

$$\begin{aligned} \check{\Delta} d^{x,H} &= \sum_{\mu\in\mathscr{C}H} \rho_{\{\mu\},H}\,\partial^+_\mu\, d^{x,H\cup\{\mu\}}, \\ \check{\nabla} d^{x,H} &= \sum_{\mu\in H} \rho_{\{\mu\},H\setminus\{\mu\}}(-\partial^-_\mu)\, d^{x,H\setminus\{\mu\}}. \end{aligned} \tag{3.8}$$

Here, ∂^+_μ and ∂^-_μ denote the forward and backward difference operators, respectively:

$$\begin{aligned} \partial^+_\mu\, d^{x,H} &:= d^{x-e_\mu,H} - d^{x,H}, \\ \partial^-_\mu\, d^{x,H} &:= d^{x,H} - d^{x+e_\mu,H}. \end{aligned} \tag{3.9}$$

Applied to a general co-chain (3.7) yields

$$\begin{aligned} \partial^+_\mu \bar{\Phi} &= \sum_{x,H} (\varphi(x+e_\mu, H) - \varphi(x,H))\, d^{x,H}, \\ \partial^-_\mu \bar{\Phi} &= \sum_{x,H} (\varphi(x,H) - \varphi(x-e_\mu, H))\, d^{x,H}. \end{aligned} \tag{3.9'}$$

This justifies the notation. $\rho_{H,K}$ is a sign function, which is $(-1)^\nu$, $\nu \equiv$ number of pairs (i,j) with $i\in H, j\in K$ and $i>j$, $\mathscr{C}H := \{1, \ldots, 4\}\setminus H$ is the complementary (ordered) set of H.

In cubical homology theory [29] a cup product is defined which corresponds to the wedge product of differential forms. It is bilinear, associative and therefore determined by

$$d^{x,H}\wedge d^{y,K}=\begin{cases}\rho_{H,K}\delta^{x+e_H,y}d^{x,H\cup K} & \text{if } H\cap K=\emptyset,\\ 0 & \text{otherwise,}\end{cases}\qquad(3.10)$$

$e_H:=\sum_{\mu\in H}e_\mu$. Intuitively, $\wedge$ describes the product of elementary co-chains on cells with matching boundaries (see Fig. 2). By this matching condition the cup product is not local, but combines nearest neighbours as expressed by the arguments of the Kronecker symbol $\delta^{x+e_H,y}$. The product rule holds with respect to the dual boundary operator:

$$\check{\Delta}(\bar\Phi\wedge\bar\Xi)=(\check{\Delta}\bar\Phi)\wedge\bar\Xi+(\mathscr{A}\bar\Phi)\wedge\check{\Delta}\bar\Xi,\qquad(3.11)$$

$\mathscr{A}^p\bar\Phi=(-1)^p\overline{{}^p\Phi}$ like in (2.18).

A lattice analogue of the Clifford product does not have all the nice properties one would like to have. However, the following procedure leads to a useful definition. First, we define a lattice correspondence of the contraction operator (2.17):

$$e^K\lrcorner d^{x,H}:=\begin{cases}\rho_{K,H\setminus K}d^{x,H\setminus K} & \text{if } K\subset H,\\ 0 & \text{otherwise,}\end{cases}\qquad(3.12)$$

$$e^\mu\lrcorner(d^{x,H}\wedge d^{y,K})=(e^\mu\lrcorner d^{x,H})\wedge d^{y-e_\mu,K}+(\mathscr{A}d^{x,H})\wedge e^\mu\lrcorner d^{y,K},\quad e^\mu\equiv e^{\{\mu\}}.\qquad(3.13)$$

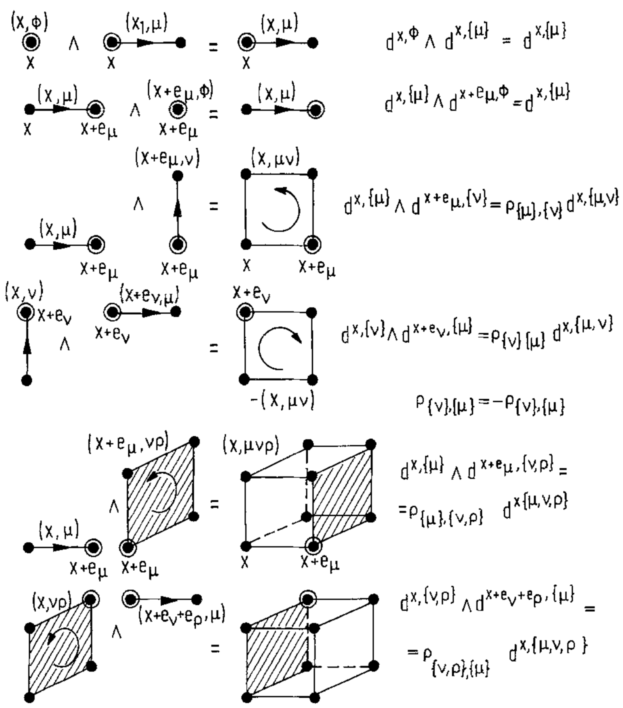

Fig. 2. The exterior product of elementary co-chains. The Figure illustrates the matching condition for the elementary chains, on which the respective co-chains are different from zero

This allows the generalization of the definition of the Clifford product in the continuum

$$\Phi\vee\Xi=\sum_{p\geqq 0}\frac{(-1)^{\binom{p}{2}}}{p!}(\mathscr{A}^p e_{\mu_1}\lrcorner\dots e_{\mu_p}\lrcorner\Phi)\wedge(e^{\mu_1}\lrcorner\dots e^{\mu_p}\lrcorner\Xi)\qquad(3.14)$$

to the lattice:

$$\bar\Phi\vee\bar\Xi=\sum_L(-1)^{\binom{l}{2}}(\mathscr{A}^l T_{-e_L}e^L\lrcorner\bar\Phi)\wedge(e^L\lrcorner\bar\Xi),\qquad(3.15)$$

$T_{e_H}d^{x,K}:=d^{x-e_H,K}$, $l\equiv$ number of elements of L. The $\vee$-product is non-local like the cup product; it is even non-associative in general; however, we shall show that right-Clifford multiplication with constant cochains transforms the solutions of the Dirac-Kähler equation on the lattice similar to (2.40).

3.2. Now we have sufficiently extended the correspondences between continuum and lattice for a *discussion of the Dirac-Kähler equation on the lattice*

$$(\check{\Delta}-\check{\nabla}+m)\,\Phi=0\qquad(3.4)$$

along the lines of Sect. 2. First we want to check the energy-momentum spectrum. For this, we multiply by the adjoint operator

$$\begin{aligned}&(-(\check{\Delta}-\check{\nabla})+m)(\check{\Delta}-\check{\nabla}+m)\,\Phi\\&=(\check{\Delta}\check{\nabla}+\check{\nabla}\check{\Delta}+m^2)\,\Phi=(-\partial^+_\mu\partial^{-,\mu}+m^2)\,\Phi=0.\end{aligned}\qquad(3.16)$$

The iterated DK equation is indeed the correct Klein-Gordon equation on the lattice. If we go to momentum space, we consider plane wave solutions on the lattice

$$\varphi(x,H)=u(p,H)\cdot e^{-ip_\mu x^\mu},\qquad -\frac{\pi}{a}<p_\mu\leqq\frac{\pi}{a}.\qquad(3.17)$$

With this ansatz (3.16) becomes

$$\left(\sum_\mu\left(\frac{2}{a}\sin\frac{p_\mu a}{2}\right)^2+m^2\right)u(p,H)=0.\qquad(3.18)$$

Because $\frac{2}{a}\sin\frac{p_\mu a}{2}$ is monotoneous in the cut-off momentum range $-\frac{\pi}{a}<p_\mu\leqq\frac{\pi}{a}$, the energy-momentum spectrum is the same as that of the Dirac-Kähler equation in the continuum. This is in contrast to the spectrum problem, which arises from the naive lattice approximation of the Dirac equation. In this sense, the Dirac-Kähler equation is a realization of the most general first order linear lattice formulation without spectrum degeneracy of the form discussed in [12]. This result was already anticipated in this paper. Of course, the DKE on the lattice has a 4-fold multiplicity of Dirac components as discussed for the continuum in (2.25)ff.

In order to study this multiplicity we have to discuss the lattice analogue of the reducing group $\mathscr{R}$ (2.37). For this, we bring the Dirac-Kähler operator with help of the Clifford product in a form similar to (2.9). By comparison of (3.8) with (3.10, 12) we observe that

$$\check{\Delta}=d^\mu\wedge\partial_\mu^-, \quad d^\mu:=\sum_x d^{x,\mu}, \qquad \check{V}=-e^\mu\lrcorner\,\partial_\mu^-; \tag{3.19}$$

hence, with help of (3.15):

$$\check{\Delta}-\check{V}=(d^\mu\wedge+e^\mu\lrcorner)\,\partial_\mu^-=d^\mu\vee\partial_\mu^-. \tag{3.20}$$

This formula allows an easy calculation leading from the first to the second line of (3.16):

$$\begin{aligned}&-(\check{\Delta}\check{V}+\check{V}\check{\Delta})\\&=(d^\mu\vee\partial_\mu^-)(d^\nu\vee\partial_\nu^-)=(d^\mu\vee d^\nu\vee)\,\partial_\mu^-\partial_\nu^-\\&=\tfrac{1}{2}(d^\mu\vee d^\nu\vee+d^\nu\vee d^\mu\vee)\,\partial_\mu^-\partial_\nu^-\\&=\delta^{\mu\nu}T_{e_\mu}\partial_\mu^-\partial_\nu^-=\partial_\mu^+\partial^{-,\mu}.\end{aligned} \tag{3.21}$$

Here, we have used

$$d^\mu\vee d^\nu\vee+d^\nu\vee d^\mu\vee=2\delta^{\mu\nu}T_{e_\mu} \tag{3.22}$$

and $T_{e_\mu}\partial_\mu^-=\partial_\mu^+$. Now we get the invariance of the DKE under lattice flavour transformations by $\vee$-multiplication from the right with constant cochains $C=\sum_{x,H}C(H)\,d^{x,H}$. Using the definition of the $\vee$-product, (3.15), one can show, that

$$d^\mu\vee(d^{x,H}\vee C)=(d^\mu\vee d^{x,H})\vee C \tag{3.23}$$

i.e. the $\vee$-product is associative in this special case. From (3.20, 23) it follows that the linear transformation

$$\hat{C}\cdot\Phi=\Phi\vee C \tag{3.24}$$

transforms the solutions of the DKE similar to (2.40). The linear transformation of the components $\varphi(x,H)$ of Φ might be easily calculated from the $\vee$-product definition (3.15). However, the resulting general formula is too clumsy. Therefore, we restrict ourselves to calculate the transformations of the lattice analogue $\hat{\mathscr{R}}=\{1,\hat{\tau},\hat{\varepsilon},\hat{\tau\varepsilon}\}$ to the reduction group $\mathscr{R}$, (2.37):

$$\hat{\tau}\cdot\Phi=i\Phi\vee d^{\{12\}}, \quad \hat{\varepsilon}\cdot\Phi=\Phi\vee d^{\{1234\}}, \qquad \hat{\tau\varepsilon}\cdot\Phi=-i\Phi\vee d^{\{34\}} \tag{3.25}$$

with $d^H:=\sum_x d^{x,H}$. This means for the components:

$$(\hat{\tau}\cdot\Phi)(x,H)=-i\cdot(-1)^{\binom{l}{2}}\rho_{L,\{12\}\setminus L}\,\varphi(x-e_{L\vartriangle\{12\}},H\vartriangle\{12\}),$$

$$(\hat{\varepsilon}\cdot\Phi)(x,H)=(-1)^{\binom{h}{2}}\rho_{H,\mathscr{C}H}\,\varphi(x-e_{\mathscr{C}H},\mathscr{C}H), \tag{3.26}$$

$$(\hat{\tau\varepsilon}\cdot\Phi)(x,H)=i\cdot(-1)^{\binom{k}{2}}\rho_{K,\{34\}\setminus K}\,\varphi(x-e_{K\vartriangle\{34\}},H\vartriangle\{34\}),$$

where $L:=H\cap\{12\}$, $K:=H\cap\{34\}$ and l, k, h are the numbers of elements in L, K, H, respectively. $H\vartriangle H'$ denotes the symmetric difference $(H\setminus H')\cup(H'\setminus H)$. Because of the nonlocality in the definition of the $\vee$-product (3.15), the action of the $\hat{\mathscr{R}}$-transformations on Φ is non-local, too. This means in contrast to the continuum, the transformations of the components at fixed x are combined with translations. Therefore the operations of $\hat{\mathscr{R}}$ don't close under multiplication. Rather, $\hat{\tau}$ and $\hat{\varepsilon}$ generate a group $\bar{\mathscr{R}}$ which contains the translations along e_1+e_2 and $e_1+e_2+e_3+e_4$ as a subgroup. $\bar{\mathscr{R}}$ is defined by the generators $\hat{\tau}$ and $\hat{\varepsilon}$, which satisfy the following relations:

$$\hat{\tau}^2=T_{-(e_1+e_2)}, \quad \hat{\varepsilon}^2=T_{-(e_1+e_2+e_3+e_4)}, \qquad \hat{\tau}\vee\hat{\varepsilon}=\hat{\varepsilon}\vee\hat{\tau}=T_{e_1+e_2}\hat{\tau\varepsilon}. \tag{3.27}$$

This abelian group contains the translations generated by $T_{-(e_1+e_2)}$ and $T_{-(e_1+\ldots+e_4)}$ as invariant subgroup $\mathscr{T}$. The factor group $\bar{\mathscr{R}}/\mathscr{T}$ is isomorphic to the reducing group $\mathscr{R}$ in the continuum. The reducing group on the lattice $\bar{\mathscr{R}}$ forms the basis for the reduction of the Dirac-Kähler equation to Dirac equations. However, because the flavour transformations are intertwined with the translations, this reduction is more involved on the lattice.

We start the further reduction of the DKE with a short remark on the representation theory of the group $\bar{\mathscr{R}}$. Since $\bar{\mathscr{R}}$ is abelian, its irreducible representations are one-dimensional and can be expressed by exponentials:

$$\hat{\tau}\mapsto\pm e^{i\beta_{12}/2}, \quad \hat{\varepsilon}\mapsto\pm e^{i\beta_{1234}/2}. \tag{3.28}$$

These irreducible representations are characterized by $[b,\beta_{12},\beta_{1234}]$, i.e. the "momenta" β_{12}, β_{1234}, $-\pi<\beta_{12}$, $\beta_{1234}\leqq\pi$, and the 4 different sign combinations of $\hat{\tau}$ and $\hat{\varepsilon}$: $(--)$, $(+-)$, $(-+)$, $(++)$ for $b=1,2,3,4$, respectively. Since in physical momenta $\exp(i\beta_{12}/2)=\exp(i(p_1+p_2)\,a/2)$ and $\exp(i\beta_{1234}/2)=\exp(i(p_1+p_2+p_3+p_4)\,a/2)$, one sees that in the formal continuum limit $\hat{\tau}$ and $\hat{\varepsilon}$ approach the representation of the reducing group on the Dirac components $\psi^{(b)}$ in the continuum, (2.34, 39).

The symmetry group $\bar{\mathscr{R}}$ allows the decomposition of the lattice DKE. However, in contrast to the continuum case, the intertwining of $\bar{\mathscr{R}}$ with the translations requires the transition to the momentum space. Following the usual group theoretical procedure, we make for $\varphi(x,H)$ the ansatz

$$\varphi(x,H)=e^{-i\frac{\beta}{2}e_H}\,\mathrm{Trace}(\gamma_H^\dagger\psi(\beta))\,e^{-i\beta x}. \tag{3.29}$$

$\psi(\beta)$ denotes a matrix of Dirac components like that in (2.27). Now, the equations (3.8, 9) allow the straightforward calculation

$$((\check{\Delta}-\check{\nabla})\Phi)(x,H)$$
$$=\sum_{\mu\in H}\rho_{\{\mu\},H\backslash\{\mu\}}e^{-i\frac{\beta}{2}e_{H\backslash\{\mu\}}}\operatorname{Trace}((\gamma_{H\backslash\{\mu\}})^{\dagger}\psi(\beta))\,\Delta_{\mu}^{+}\,e^{-i\beta x}$$
$$+\sum_{\mu\in\mathscr{C}H}\rho_{\{\mu\},H}\,e^{-i\frac{\beta}{2}e_{H\cup\{\mu\}}}\operatorname{Trace}((\gamma_{H\cup\{\mu\}})^{\dagger}\psi(\beta))\,\Delta_{\mu}^{-}\,e^{-i\beta x}$$
$$=e^{-i\frac{\beta}{2}e_H}\operatorname{Trace}((\gamma_H)^{\dagger}\sum_{\mu}\left(-2i\sin\frac{\beta_{\mu}}{2}\right)\gamma_{\mu}\psi(\beta)\Big)\,e^{-i\beta x}. \tag{3.30}$$

We used the formulas

$$\Delta_{\mu}^{+}\,e^{-i\beta x}=-2i e^{-i\beta_{\mu}/2}\sin\frac{\beta_{\mu}}{2}\,e^{-i\beta x},$$
$$\Delta_{\mu}^{-}\,e^{-i\beta x}=-2i e^{i\beta_{\mu}/2}\sin\frac{\beta_{\mu}}{2}\,e^{-i\beta x}, \tag{3.30'}$$
$$(\gamma_{H\cup\{\mu\}})^{\dagger}=(\gamma_H)^{\dagger}\gamma_{\mu}^{\dagger}\rho_{\{\mu\},H},$$
$$(\gamma_{H\backslash\{\mu\}})^{\dagger}=(\gamma_H)^{\dagger}\gamma_{\mu}^{\dagger}\rho_{\{\mu\},H\backslash\{\mu\}},\qquad \gamma_{\mu}^{\dagger}=\gamma_{\mu}$$

and the definition (A.17) of the Appendix for $\Delta_{\mu}^{\pm}$. The completeness relations of the Fourier transforms and of (2.22, 23) allow to conclude from (3.30) that the Dirac components $\psi^{(b)}(\beta)$ of solutions of the DKE: $(\check{\Delta}-\check{\nabla}+m)\,\Phi=0$ satisfy the Dirac equation in momentum space

$$\left(\sum_{\mu}\left(-2i\sin\frac{\beta_{\mu}}{2}\right)\gamma_{\mu}+m\right)\psi^{(b)}(\beta)=0. \tag{3.31}$$

This result is very similar to that of the continuum case, (2.27). It is also possible to characterize the different columns $\psi^{(b)}(\beta)$ in the matrix of Dirac components by the symmetry properties with respect to the reduction group $\bar{\mathscr{R}}$. A direct calculation, similar to that of (3.30), leads from the definition (3.26) of $\hat{\tau}$ and $\hat{\varepsilon}$ to

$$(\hat{\tau}\cdot\Phi)(x,H)=e^{i\frac{\beta}{2}e_{12}}\,e^{-i\frac{\beta}{2}e_H}\operatorname{Trace}((\gamma_H)^{\dagger}\psi(\beta)\,i\gamma^{12})\,e^{-i\beta x},$$
$$(\hat{\varepsilon}\cdot\Phi)(x,H)=e^{i\frac{\beta}{2}e_{1234}}\,e^{-i\frac{\beta}{2}e_H}\operatorname{Trace}((\gamma_H)^{\dagger}\psi(\beta)\,\gamma^5)\,e^{-i\beta x}. \tag{3.32}$$

The sign combinations (b), which characterize the different representation of $\bar{\mathscr{R}}$ according to (3.28), are produced by right multiplication of $\psi(\beta)$ with the diagonal matrices $i\gamma^{12},\gamma^5$.

There is an important difference to the continuum case. The lattice subsidiary conditions analogous to (2.36, 38), are in momentum space:

$$\hat{\tau}\cdot\psi(\beta)=\operatorname{sign}(12)\,e^{i\beta_{12}/2}\,\psi(\beta),$$
$$\hat{\varepsilon}\cdot\psi(\beta)=\operatorname{sign}(1234)\,e^{i\beta_{1234}/2}\,\psi(\beta). \tag{3.33}$$

They cannot be expressed by local relations between lattice fields. This is a consequence of the factors $\exp(i\beta_{12}/2)$, $\exp(i\beta_{1234}/2)$, which describe in coordinate space a translation by half a lattice link.

Thus we have given a complete analysis of the DKE on the lattice, which leads to similar results as in the continuum. However, the fact that the subsidiary conditions which reduce the DKE to the Dirac equation hold only for fields extrapolated from the lattice points, will make the description of interacting Dirac fields by Dirac-Kähler fields on the lattice much more involved.

3.3 In the following, we shall clarify the relation between the DKE and the naive Dirac equation on the lattice. Our result will be that the *Susskind reduction* of the naive Dirac equation is equivalent to the *Dirac-Kähler equation*. The naive lattice approximation of the Dirac equation consists in the substitution of the partial differential operator by the symmetric difference operator

$$(\gamma^{\mu}\bar{\Delta}_{\mu}+m)\,\psi(y)=0, \tag{3.34}$$
$$(\bar{\Delta}_{\mu}\psi)(y)\equiv(\Delta_{\mu}^{+}+\Delta_{\mu}^{-})\,\psi(y)=\frac{1}{2a}(\psi(y+e_{\mu})-\psi(y-e_{\mu})).$$

There is an algebra of symmetry transformations $\{\hat{M}^H\}$ [30], (the "spectrum doubling group") which commutes with the Dirac operator $\gamma^{\mu}\bar{\Delta}_{\mu}$ and causes a 16-fold degeneracy in the energy-momentum spectrum compared to the continuum. $\hat{M}^H$ is defined as

$$(\hat{M}^H\psi)(y)=e^{iy\pi_H}M^H\psi(y). \tag{3.35}$$

Here M^H denotes

$$M^{\mu}=i\gamma^5\gamma^{\mu},\quad M^H=M^{\mu_1}\dots M^{\mu_h},\quad \mu_i\in H \tag{3.36}$$

and

$$(\pi_H)^{\mu}=\begin{cases}\pi/a & \text{for } \mu\in H,\\ 0 & \text{otherwise.}\end{cases} \tag{3.37}$$

It is easy to check, that M^{μ} and $\hat{M}^{\mu}$ satisfy the defining anticommutation relations of the Clifford algebra of γ-matrices. Because of the factor $\exp(iy\pi_H)$ in (3.35), the symmetry transformation shifts the momentum in the solution by π_H/a and produces in this way the energy momentum degeneracy.

In order to prove

$$(\hat{M}\gamma^{\mu}\bar{\Delta}_{\mu}-\gamma^{\mu}\bar{\Delta}_{\mu}\hat{M})\,\psi(y)=0, \tag{3.38}$$

we first state a remarkable symmetry between the Clifford algebras $\{\gamma^H\}$ and $\{M^H\}$:

$$M^{\mu}=i\gamma^5\gamma^{\mu},\quad M^5=\gamma^5,\quad \gamma^{\mu}=-iM^5M^{\mu},$$
$$(M^H)^{-1}\gamma^K M^H=e^{ie_K\pi_H}\gamma^K,\quad (\gamma^K)^{-1}M^H\gamma^K=e^{ie_K\pi_H}M^H. \tag{3.39}$$

This means that the M^H act on the γ's as a group of equivalence transformations isomorphic to $(\mathbb{Z}_2)^4$. The formulas are proved by iterated application of

$$(M^\mu)^{-1}\gamma^\nu M^\mu = e^{ie_\nu \pi_\mu}\gamma^\nu, \qquad \pi_\mu \equiv \pi_{\{\mu\}}.$$

Further we have

$$\bar{\Delta}_\mu e^{iy\pi_H}\psi(y) = e^{ie_\mu \pi_H} e^{iy\pi_H}\bar{\Delta}_\mu \psi(y). \tag{3.40}$$

Combining this with the relations (3.39) yields immediately (3.38).

Our aim is to decompose the naive Dirac equation by use of the symmetry transformations $\{\hat{M}^H\}$. These transformations do not commute with general lattice translations but only with those by a multiple of two lattice constants. This suggests considering the lattice of blocks with points x:

$$y = x + e_H, \qquad x = 2a(n_1, \ldots, n_4), \tag{3.41}$$

H and e_H defined as usual. In this block notation we get:

$$\psi(y) \equiv \psi(x, H), \qquad e^{iy\pi_K}\psi(y) \equiv e^{ie_H \pi_K}\psi(x, H), \tag{3.42}$$

$$(\bar{\Delta}_\mu \psi)(x, H) = \begin{cases} (\Delta_\mu^+ \psi)(x, H\setminus\{\mu\}) & \text{if } \mu \in H, \\ (\Delta_\mu^- \psi)(x, H\cup\{\mu\}) & \text{if } \mu \in \mathscr{C}H. \end{cases} \tag{3.42'}$$

On the four components of $\psi(x, H)$, (x, H) fixed, the Clifford algebra $\hat{M}^K$ is represented by 4×4 matrices. This representation differs by an H-dependent sign factor $\exp(ie_H\pi_K)$ from the representation of the M^K. Because all irreducible representations of these Clifford algebras are equivalent we can remove this by an equivalence transformation

$$\psi(x, H) \stackrel{T}{\longmapsto} \varphi(x, H) = (\gamma^H)^\dagger \psi(x, H). \tag{3.43}$$

We use (3.39) in order to show

$$(\gamma^H)^\dagger \hat{M}^K \psi(x, H) = e^{ie_H\pi_K}(\gamma^H)^\dagger M^K \psi(x, H) = M^K \varphi(x, H),$$

which means $T\hat{M} = MT$. In the basis $\varphi(x, H)$ the spectrum doubling group acts particularly simply: $\hat{M}^K \stackrel{T}{\longmapsto} M^K$. Since the Dirac operator commutes with this group, $\gamma^\mu \bar{\Delta}_\mu$ decomposes in the φ-basis. This means:

$$\psi_a(x, H) = \gamma^H_{ai}\varphi_i(x, H), \qquad \begin{matrix} i = 1, 2, 3, 4, \\ i \text{ fixed} \end{matrix} \tag{3.44}$$

is invariant under the application of $\gamma^\mu \bar{\Delta}_\mu$. Equation (3.44) is called the Susskind reduction of the naive Dirac equation [31].

Now we can formulate our main result relating the naive lattice Dirac equation to the Dirac-Kähler equation [32]: The co-chain

$$\Phi = \sum_{x,H} \varphi_i(x, H)\, d^{x,H} \tag{3.45}$$

satisfies the lattice Dirac-Kähler equation iff $\psi_a(x, H)$ according to (3.42), is a solution of the naive Dirac equation. It is

$$T\gamma^\mu \bar{\Delta}_\mu \psi = (\check{\Delta} - \check{\nabla})\, T\psi. \tag{3.46}$$

In order to prove this, we have to perform the following calculation:

$$\begin{aligned}
&(\gamma^H)^\dagger (\gamma^\mu \bar{\Delta}_\mu \psi)(x, H) \\
&= \sum_{\mu\in H} (\gamma^H)^\dagger \gamma^\mu \Delta_\mu^+ \psi(x, H\setminus\{\mu\}) \\
&+ \sum_{\mu\in\mathscr{C}H} (\gamma^H)^\dagger \gamma^\mu \Delta_\mu^- \psi(x, H\cup\{\mu\}) \quad (3.42') \\
&= \sum_{\mu\in H} \rho_{\{\mu\}, H\setminus\{\mu\}} \Delta_\mu^+ \varphi(x, H\setminus\{\mu\}) \\
&+ \sum_{\mu\in\mathscr{C}H} \rho_{\{\mu\}, H} \Delta_\mu^- \varphi(x, H\cup\{\mu\}) \quad (3.30') \\
&= ((\check{\Delta} - \check{\nabla})\,\Phi)(x, H). \quad (3.8)
\end{aligned}$$

Because of this equivalence we also may say that the Dirac-Kähler equation is the formal continuum limit for the Susskind formulation of Dirac fields on the lattice.

3.4. Although the DKE on the lattice is equivalent to the Susskind formulation of lattice fermions, we have the strong opinion, that the description in the framework of Kähler's formalism is the superior one. It is geometrically more intuitive. The close correspondence to the continuum makes many of the lattice definitions and manipulations more transparent. This is illustrated by the definition of *currents and* by the derivation of *their conservation laws.* For this we define scalar products analogously to the continuum, (2.51), using the lattice correspondences, (3.10), (A.35, 42):

$$\begin{aligned}
&(\Phi, \Xi)_0 := \sum_p ({}^p\Phi, {}^p\Xi)_0, \\
&({}^p\Phi, {}^p\Xi)_0 := (\mathscr{B}\,{}^p\Phi) \wedge \star\,{}^p\Xi = {}^p\Phi \wedge *\,{}^p\Xi.
\end{aligned} \tag{3.47}$$

We get for elementary cochains

$$(d^{x,H}, d^{y,K})_0 = \delta^{H,K}\delta^{x,y} d^{x,\{1234\}} \tag{3.48}$$

and therefore in general

$$(\Phi, \Xi)_0 = \sum_{x,H} \varphi(x, H)\, \xi(x, H)\, d^{x,\{1234\}} = (\Xi, \Phi)_0. \tag{3.49}$$

The first derived scalar product is defined as

$$(\Phi, \Xi)_1 := e_\mu \lrcorner (T_{-\mu} d^\mu \vee \Phi, \Xi)_0 \\ = \sum_p {}^p\Phi \wedge *^{p+1}\Xi + {}^p\Xi \wedge *^{p+1}\Phi. \tag{3.50}$$

A direct calculation using the product rule, (A.37), leads to

$$(\check{\Delta}\Phi, \Xi)_0 - (\Phi, \check{\nabla}\Xi)_0 = \check{\Delta}(\sum_p {}^p\Phi \wedge *^{p+1}\Xi), \\ (\check{\nabla}\Phi, \Xi)_0 - (\Phi, \check{\Delta}\Xi)_0 = -\check{\Delta}(\sum_p {}^p\Xi \wedge *^{p+1}\Phi). \tag{3.51}$$

From (3.50, 51) the lattice correspondence (A.81) of the Green's formula (2.50) follows:

$$((\check{\Delta} - \check{\nabla})\Phi, \Xi)_0 + (\Phi, (\check{\Delta} - \check{\nabla})\Xi)_0 = \check{\Delta}(\Phi, \Xi)_1. \tag{3.52}$$

By the same reasoning as in (2.54) ff. we can define conserved currents in the form of 1-cochains

$$j = \text{☆}^{-1}(\bar{\Phi}, \Phi')_1. \tag{3.53}$$

From solutions of the DKE and its adjoint

$$(\check{\Delta} - \check{\nabla} + m)\Phi' = 0 \quad \text{and} \quad (\check{\Delta} - \check{\nabla} - m)\bar{\Phi} = 0 \tag{3.54}$$

current conservation follows

$$\check{\nabla} j = -\text{☆}^{-1}\check{\Delta}\text{☆}\text{☆}^{-1}(\bar{\Phi}, \Phi')_1 = 0. \tag{3.55}$$

Examples of such conserved currents are the lattice correspondences of (2.56):

Dirac current: $j = \frac{1}{4}\text{☆}^{-1}(\bar{\Phi}, \Phi)_1$, (3.56)

flavour current: $j = \frac{1}{4}\text{☆}^{-1}(\bar{\Phi}, \Phi \vee C)_1$, (3.57)

chiral current: $j = \frac{1}{4}\text{☆}^{-1}(\bar{\Phi}, \varepsilon \vee \Phi)_1$. (3.58)
(for $m = 0$)

3.5. Finally we want to make some first remarks on the problem of the *minimal coupling* of a DK field *to a gauge field*. As in the case of the continuum, the first steps look straightforward within the well-known scheme of lattice gauge theories. We assume that the coefficients of the cochains $\varphi(x, H)$ allow the transformations of a symmetry group $G = \{g\}$. Then we can define the local gauge transformations of $\varphi(x, H)$ and of the lattice gauge field $U(x, e_\mu)$:

$$\varphi(x, H) \mapsto g(x)\,\varphi(x, H), \\ U(x, e_\mu) \mapsto g(x + e_\mu)\,U(x, e_\mu)\,g(x)^{-1}. \tag{3.59}$$

Geometrically, $U(x, e_\mu)$ describes the parallel transport of the local symmetry transformation along the link (x, μ). Hence, we define the positive and negative covariant lattice derivatives D_μ^+ and D_μ^- as the gauge covariant generalizations of $\partial_\mu^\pm$, (3.9'):

$$(D_\mu^+ \varphi)(x, H) := U(x, e_\mu)^{-1}\,\varphi(x + e_\mu, H) - \varphi(x, H), \\ (D_\mu^- \varphi)(x, H) := \varphi(x, H) - U(x - e_\mu, e_\mu)\,\varphi(x - e_\mu, H). \tag{3.60}$$

This allows us to define the covariant dual boundary and dual coboundary operator

$$\check{\Delta}_A \Phi := \sum_{x,H} (\sum_{\mu \in H} \rho_{\{\mu\}, H\setminus\{\mu\}} (D_\mu^+ \varphi)(x, H\setminus\{\mu\}))\, d^{x,H}, \\ \check{\nabla}_A \Phi := -\sum_{x,H} (\sum_{\mu \notin H} \rho_{\{\mu\}, H} (D_\mu^- \varphi)(x, H \cup \{\mu\}))\, d^{x,H}. \tag{3.61}$$

The gauge invariant DK equation on the lattice becomes

$$(\check{\Delta}_A - \check{\nabla}_A + m)\,\Phi = 0. \tag{3.62}$$

This equation is related to an action which is similar to (2.71):

$$S = \tfrac{1}{4}(\bar{\Phi}, (\check{\Delta}_A - \check{\nabla}_A + m)\,\Phi)_0(V). \tag{3.63}$$

The volume V is represented by the constant 4-chain

$$V = \sum_x (x, (1234)). \tag{3.64}$$

This action is the starting point for the calculation of the euclidean Green's functions by the path integral formula. As only one illustration of the special feature of the quantized DK field on the lattice we give its free propagator:

$$\langle \varphi(x, H)\,\bar{\varphi}(y, K) \rangle \\ = 4(\sum_{\mu \in K} [H; \mu; K]\,\Delta_\mu^- + \sum_{\mu \notin K} [H; \mu; K]\,\Delta_\mu^+ \\ - m\delta^{H,K})\,\Delta_S(x - y)$$

with

$$(\Delta_\mu^+ \Delta^{\mu,-} - m^2)\,\Delta_S(x - y) = -\delta_{x,y},$$

$$\Delta_S(x) = (2\pi)^{-4} \int_{-\pi}^{\pi} d^4\beta\, e^{-i\beta x} \left(\sum_\mu \left(2 \sin\frac{\beta_\mu}{2}\right)^2 + m^2\right)^{-1}$$

and $\Delta_\mu^\pm$, $[H; \mu; K]$ according to (A.17), (2.76), respectively. The calculation is completely analogous to that of the continuum, (2.74–76). The propagator does not suffer from a spectrum degeneracy.

These simple-minded remarks don't even touch the most fundamental problems of interacting DK fields on the lattice. We mention some questions which should be pursued:

a) In the continuum we discussed a large variety of interacting DK fields: real or complex fields coupled to abelian or non-abelian external or flavour-type gauge groups. Our first attempts to translate these models to the lattice showed that it is difficult to treat them in the same way. The geometrical interpretation of the differently interacting fields must be considered separately. In this sense, (3.59–63) describe only the simplest case of the coupling of an external gauge field to a complex DK field.

b) The currents defined in (3.56–58) show nearest-neighbour point splitting. For interacting DK fields they must be formulated gauge invariant. It is an important question to answer whether these fields lead to the correct anomalies [33] like for example the point-split current of the Susskind field of Sharatchandra, Thun and Weisz [31].

c) The reduction of the DKE to a simple Dirac equation can be performed for free fields with help of the reduction group or subsidiary conditions. On the lattice, the reduction group $\hat{\mathscr{R}}$, (3.26) ff., gets intertwined with the translation group; the subsidiary conditions, (3.33), are non-local. This fact makes it difficult to find a gauge invariant flavour separation. Therefore, the problem of the description of a simple interacting Dirac field on the lattice is still open

d) As an approach to this problem one can consider in the continuum a model where only one flavour is coupled to a $U(1)$-flavour gauge field. The other Dirac components are free. The coupling of a real DK field to the flavour charge $dx^{12} \vee \frac{1}{2}(1-\varepsilon)$ similar to (2.69) is such an example. The lattice version of this model might be considered as the lattice approximation to a simple interacting Dirac field. It will be accompanied by auxiliary Dirac fields which decouple in the continuum limit. The complete exposition of this model still requires the solution of the problem of flavour gauging mentioned in point a).

4. Conclusions

It was the aim of this paper to discuss the problem of the lattice approximation of Dirac fields from a geometric point of view. For this, we used an extended correspondence between the calculus of differential forms and lattice concepts known from algebraic topology. A suitable basis was given by Kähler's formulation of a generalized Dirac equation. After having discussed the open problems of interacting Dirac fields on the lattice at the end of the last sections, we want to conclude with a summary of the results of our approach:

a) The lattice approximation of the Dirac-Kähler equation is straightforward. There is no spectrum degeneracy problem caused by the lattice approximation.

b) We describe explicitly the decomposition of real or complex Dirac-Kähler forms into simple Dirac fields. The application of the corresponding methods to the lattice leads to a decomposition of the Dirac-Kähler equation into Dirac equations in momentum space.

c) The decomposition of Dirac-Kähler forms into Dirac fields sheds some new light on the geometric meaning of spinor fields as coherent superpositions of differential forms.

d) We gave a coordinate-free description of the Susskind reduction of the naive lattice Dirac equation and established their equivalence with the lattice Dirac-Kähler equation. This implies that we found the formal continuum limit for Susskind fermions.

e) The construction of conserved currents from symmetries was given in the continuum and on the lattice in complete analogy.

f) We considered the quantization of Dirac-Kähler fields by the path integral formula. In particular, we calculated the DK propagator in the continuum and on the lattice. Because of its simple form, we expect that perturbation theory for Sussking fermions becomes more straightforward in the framework of DK fields.

g) We gave first hints on gauge interactions of Dirac-Kähler fields in the continuum and on the lattice.

h) The description of matter by Dirac-Kähler fields opens the possibility of new types of models which are inspired by geometric intuitions. In this spirit one should look for generalizations of gauge theories with fermions and supersymmetric theories in the continuum and on the lattice.

At the end we want to state again that we were inspired by the special feature of "differential geometry as a field in which geometry is expressed in analysis, algebra and calculations and in which analysis and calculations are sometimes understood in intuitive steps that could be called geometric" (Kuiper [34]). We believe that the relevance of this geometric viewpoint for physics is revealed by the success of quantized gauge theories and the increasing understanding of renormalized field theory as the description of the dynamical continuum structure of physical spacetime.

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Appendix

The Dirac-Kähler spinor field is a purely geometric object. The essential point of our approach to lattice fermions is the systematic transcription of this object to the lattice. The possibility to do so is illustrated best by de Rham's theorem which states the equivalence of the de Rham cohomology theory (in terms of differential forms) with the cubical cohomology theory of any smoothly triangulated manifold [13, 29]. In this Appendix we supplement our discussion by a short continuum-lattice glossary. For this, we restrict ourselves to the flat, 4 dimensional euclidean space-time manifold and to a hypercubic lattice. Also, for the reader's convenience we do not try to formulate coordinate free, we introduce cartesian coordinates from the very beginning. For generalizations we refer to the literature.

a) *Cubes.* In the language of algebraic topology the hypercubic lattice is a geometric cubical complex which defines a triangulation of the space-time manifold. The complex consists of *p-cubes* $(x, \mu_1 \dots \mu_p)$, $\mu_1, \dots, \mu_p = 1, \dots, 4$, $\mu_1 < \mu_2 < \dots < \mu_p$, $p = 0, 1, \dots, 4$. The 0-cubes x are the lattice points with $x^\mu = a \cdot n^\mu$, n^μ integer, a the lattice constant. For $p \geqq 1$, a p-cube is the set

$$\begin{aligned}&(x, \mu_1 \dots \mu_p)\\&= \{y^\mu | y^{\mu_i} = x^{\mu_i} + \xi^{\mu_i} \text{ and } y^\nu = x^\nu \text{ for } \nu \neq \mu_i,\\&\xi^{\mu_i} \in I, i = 1, \dots, p\},\end{aligned} \tag{A.1}$$

I the open interval $(0, a)$. We introduce the following

b) *Multiindex-Notation* (see f.i. [35]): we write generically $H, K, \dots$ for the index set $\{\mu_1 \dots \mu_p\}$, $\mu_1 < \mu_2 < \dots < \mu_p$, such that $x \equiv (x, \emptyset)$, $\emptyset$ the empty set and $(x, \mu_1 \dots \mu_p) \equiv (x, H)$. For any two ordered index sets H, K we define the union $H \cup K$, the intersection $H \cap K$, the difference $H \backslash K$, the symmetric difference $H \vartriangle K := (H \backslash K) \cup (K \backslash H)$ and the complement $\mathscr{C}H := \{1234\} \backslash H$. These new index sets are again taken to be in their natural order. In order to account for permutations which are necessary for this ordering, we introduce a sign function $\rho_{H,K}$ if $H \cap K = \emptyset$. $\rho_{H,K}$ is equal to $(-1)^\nu$, where ν is the number of pairs $(i, j) \in H \times K$ with $i > j$, and for the trivial case $\rho_{\emptyset, H} = \rho_{H, \emptyset} = +1$.

c) A *chain* is an element of the vector space $\mathscr{C}$ of formal linear combinations

$$C = \sum_{x, H} \alpha(x, H) \cdot (x, H) \tag{A.2}$$

of cubes, where the coefficients $\alpha(x, H)$ are real or complex numbers. $\sum_H$ is the sum over all ordered index subsets of $\{1, 2, 3, 4\}$. The chains with $\alpha(x, H) = \delta^{x,y} \delta^{H,K}$ are called *elementary*. They can be identified with the oriented cubes (y, K). If all coefficients $\alpha(x, H)$ in (A.2) are zero except those with cardinality of H equal to p, the chains are called *p-chains.* They constitute the subspace ${}^p\mathscr{C}$ of $\mathscr{C}$: $\mathscr{C} = \bigoplus_p {}^p\mathscr{C}$.

d) The *boundary operator* Δ and the *co-boundary operator* ∇ are linear operators on $\mathscr{C}$:

$$\Delta : {}^p\mathscr{C} \to {}^{p-1}\mathscr{C}, \qquad \nabla : {}^p\mathscr{C} \to {}^{p+1}\mathscr{C}. \tag{A.3}$$

For elementary chains they are defined according to

$$\Delta(x, H) = \sum_{\mu \in H} \rho_{\{\mu\}, H \backslash \{\mu\}} [(x + e_\mu, H \backslash \{\mu\}) - (x, H \backslash \{\mu\})], \tag{A.4}$$

$$\nabla(x, H) = \sum_{\mu \in \mathscr{C}H} \rho_{\{\mu\}, H} [(x - e_\mu, H \cup \{\mu\}) - (x, H \cup \{\mu\})]. \tag{A.5}$$

Here, e_μ is a 'free' vector of length a in μ-direction and $\sum_{\mu \in H} = 0$ if $H = \emptyset$. It follows

$$\Delta^2 \equiv \Delta \Delta = 0, \qquad \nabla^2 \equiv \nabla \nabla = 0. \tag{A.6}$$

(Hence, $(\mathscr{C}, \Delta)$ is a cubical chain complex Γ.)

e) The dual spaces $\check{\mathscr{C}}$ and ${}^p\check{\mathscr{C}}$ of $\mathscr{C}$ and ${}^p\mathscr{C}$ are the spaces of *cochains* and *p-cochains*, respectively. The basis vectors $d^{x,H}$ of $\check{\mathscr{C}}$ dual to the basis vectors (x, H) of $\mathscr{C}$ are defined by

$$d^{x,H}((x', H')) = \delta^x_{x'} \delta^H_{H'}. \tag{A.7}$$

They are called *elementary cochains*. The most general cochain $\Phi \in \check{\mathscr{C}}$ is given by

$$\Phi = \sum_{x, H} \varphi(x, H) \, d^{x,H}, \tag{A.8}$$

$\varphi(x, H)$ real or complex. If $\sum_H$ runs only over H with cardinality p, $\Phi \in {}^p\check{\mathscr{C}}$ is called *homogeneous* of degree p. For constant $\varphi(x, H) = 1$ we introduce

$$d^H := \sum_x d^{x,H}, \qquad d^\mu \equiv d^{\{\mu\}} = \sum_x d^{x,\{\mu\}} \tag{A.9}$$

for the resulting constant cochain. The boundary and co-boundary operators can be dualized by the Stokes formulas

$$(\check{\Delta}\Phi)(C) = \Phi(\Delta C) \quad \text{and} \quad (\check{\nabla}\Phi)(C) = \Phi(\nabla C) \tag{A.10}$$

with $\Phi \in \check{\mathscr{C}}$ and $C \in \mathscr{C}$ according to (A.8, 2). $\check{\Delta}$ is the *dual boundary operator*, $\check{\nabla}$ is the *dual co-boundary operator*. They are linear operators on $\check{\mathscr{C}}$:

$$\check{\Delta} : {}^p\check{\mathscr{C}} \to {}^{p+1}\check{\mathscr{C}}, \qquad \check{\nabla} : {}^p\check{\mathscr{C}} \to {}^{p-1}\check{\mathscr{C}}. \tag{A.11}$$

In order to express the action of $\check{\Delta}$ and $\check{\nabla}$ on elementary cochains, we introduce the linear translation operator T_{e_H} on $\check{\mathscr{C}}$:

$$T_{e_H} d^{x,K} := d^{x-e_H,K}, \quad e_H := \sum_{\mu \in H} e_\mu. \tag{A.12}$$

For $H=\{\mu\}$ we write $T_\mu \equiv T_{e_{\{\mu\}}}$ and we define the difference operators

$$\partial_\mu^+ := T_\mu - 1, \quad \partial_\mu^- := 1 - T_{-\mu}, \quad T_{-\mu} \equiv (T_\mu)^{-1}, \tag{A.13}$$

such that

$$\partial_\mu^+ = T_\mu \partial_\mu^-. \tag{A.14}$$

With help of (A.10, 4, 5) we calculate:

$$\check{\Delta} d^{x,H} = \sum_{\mu \in \mathscr{C}H} \rho_{\{\mu\},H} \partial_\mu^+ d^{x,H\cup\{\mu\}}, \tag{A.15}$$

$$\check{\nabla} d^{x,H} = -\sum_{\mu \in H} \rho_{\{\mu\},H\backslash\{\mu\}} \partial_\mu^- d^{x,H\backslash\{\mu\}}, \tag{A.16}$$

and, for a general cochain (A.8):

$$\check{\Delta}\Phi = \sum_{x,H} \Big(\sum_{\mu \in H} \rho_{\{\mu\},H\backslash\{\mu\}} (\Delta_\mu^+ \varphi)(x, H\backslash\{\mu\})\Big) d^{x,H}, \tag{A.15'}$$

$$\check{\nabla}\Phi = -\sum_{x,H} \Big(\sum_{\mu \in \mathscr{C}H} \rho_{\{\mu\},H} (\Delta_\mu^- \varphi)(x, H\cup\{\mu\})\Big) d^{x,H}, \tag{A.16'}$$

where

$$\begin{aligned}(\Delta_\mu^+ \varphi)(x,H) &:= \varphi(x+e_\mu, H) - \varphi(x,H),\\ (\Delta_\mu^- \varphi)(x,H) &:= \varphi(x,H) - \varphi(x-e_\mu, H).\end{aligned} \tag{A.17}$$

It follows from (A.6, 10) that

$$\check{\Delta}^2 \equiv \check{\Delta}\check{\Delta} = 0, \quad \check{\nabla}^2 \equiv \check{\nabla}\check{\nabla} = 0. \tag{A.18}$$

($(\check{\mathscr{C}}, \check{\Delta})$ is the cubical cochain complex dual to the chain complex $(\mathscr{C}, \Delta)$.)

f) The *exterior algebra* Λ *of differential forms* $\in \mathscr{D}$ over the 4 dim. flat euclidean space-time manifold is generated by the 1-forms

$$dx^\mu \in {}^1\Lambda, \quad \mu = 1, 2, 3, 4. \tag{A.19}$$

The cartesian basis on the space ${}^p\Lambda$ of p-forms is given by

$$\begin{aligned} &dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \equiv dx^H,\\ &H = \{\mu_1, \ldots, \mu_p\}, \mu_1 < \ldots < \mu_p. \end{aligned} \tag{A.20}$$

Here, we use the multiindex notation introduced above. $\wedge$ is the bilinear, alternating, associative and distributive *exterior* (*Grassmann*) *product* on $\Lambda = \bigoplus_p {}^p\Lambda$:

$$\begin{aligned} &\wedge : {}^p\Lambda \times {}^q\Lambda \to {}^{p+q}\Lambda,\\ &dx^H \wedge dx^K = \begin{cases} \rho_{H,K} dx^{H\cup K} & \text{if } H \cap K = \emptyset,\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{A.21}$$

g) A general differential form can be expanded with respect to the basis (A.20) according to

$$\Phi = \sum_H \varphi(x,H)\, dx^H. \tag{A.22}$$

A change of coordinates $x=f(y)$ transforms Φ into

$$\Phi \circ f := \sum_H \varphi_f(y,H)\, dy^H$$

with

$$\begin{aligned}&\varphi_f(y, \mu_1 \ldots \mu_p)\\ &= \sum_{\substack{\nu_1,\ldots,\nu_p\\ \nu_1<\ldots<\nu_p}} \Big(\sum_{\sigma \in \mathscr{S}_p} \operatorname{sign}(\sigma) \frac{\partial f^{\nu_{\sigma 1}}}{\partial y^{\mu_1}} \cdots \frac{\partial f^{\nu_{\sigma p}}}{\partial y^{\mu_p}}\Big)\\ &\cdot \varphi(f(y), \nu_1 \ldots \nu_p),\end{aligned} \tag{A.23}$$

$\mathscr{S}_p$ the symmetric group of p elements. p-forms ${}^p\Phi \in {}^p\Lambda$ can be integrated over p-chains ${}^pC = \sum_{x,H} \alpha(x,H) \cdot (x,H) \in {}^p\mathscr{C}$:

$$\int_{{}^pC} {}^p\Phi = \sum_{x,H} a(x,H) \int_{(x,H)} {}^p\Phi, \tag{A.24}$$

especially

$$\int_{(x',H')} \chi_{(x,H)}(y)\, dy^H = \delta_{x'}^x \delta_{H'}^H \tag{A.25}$$

if $\chi_{(x,H)}$ is the characteristic function of the elementary p-chain (x,H):

$$\chi_{(x,H)}(y) = \begin{cases} (1/a)^p & \text{if } y \in (x,H), \quad (x,H) \in {}^p\mathscr{C},\\ 0 & \text{otherwise.}\end{cases}$$

By comparison of (A.25) and (A.7), we conclude that the *differential form* $\chi_{(x,H)}(y)\, dy^H$ *corresponds to the elementary cochain* $d^{x,H}$ on the lattice-triangulation of space-time.

The integration of p-forms over more general p-dimensional domains S can be done by mapping the standard p-cube $(x=0,H)$ onto $S: f:(x=0,H) \to S$. S is called a *singular p-cube*. The integral over S is then given by

$$\int_S {}^p\Phi := \int_{(0,H)} {}^p\Phi \circ f, \tag{A.26}$$

where ${}^p\Phi \circ f$ is the f-transform of ${}^p\Phi$ according to (A.22). This definition with help of the transformation of variables can be readily extended to the more general cubical singular chains as integration domains.

h) The *exterior differential* d is a linear operator on Λ:

$$d: {}^p\Lambda \to {}^{p+1}\Lambda \tag{A.27}$$

defined by

$$d\Phi := dx^\mu \wedge \partial_\mu \Phi, \tag{A.27'}$$

where

$$\partial_\mu \Phi := \sum_H (\partial_\mu \varphi)(x, H)\, dx^H \tag{A.28}$$

for the general differential form (A.22). It follows that

$$d^2 \equiv dd = 0. \tag{A.29}$$

With help of (A.21) we calculate

$$\begin{aligned} d\Phi &= \sum_H (\partial_\mu \varphi)(x, H)\, dx^\mu \wedge dx^H \\ &= \sum_H \sum_{\mu \in \mathscr{C}H} \rho_{\{\mu\}, H} (\partial_\mu \varphi)(x, H)\, dx^{H \cup \{\mu\}} \\ &= \sum_H \Big(\sum_{\mu \in H} \rho_{\{\mu\}, H \setminus \{\mu\}} (\partial_\mu \varphi)(x, H \setminus \{\mu\})\Big)\, dx^H. \end{aligned} \tag{A.30}$$

Comparing (A.30) and (A.15') we conclude that the lattice analogue of the exterior differential d is the dual boundary operator $\check{\Delta}$. This can also be seen from the *Stokes formulas*

$$\int_C d\Phi = \int_{\Delta C} \Phi, \qquad (\check{\Delta}\Phi)(C) = \Phi(\Delta C), \tag{A.31}$$

which are valid in the continuum and on the lattice, respectively.

i) It is convenient to define *the main automorphism* $\mathscr{A}$ and *the main antiautomorphism* $\mathscr{B}$ on Λ and $\check{\mathscr{C}}$:

$$\left.\begin{aligned} \mathscr{A}\Phi &:= (-1)^p \Phi \\ \mathscr{B}\Phi &:= (-1)^{\binom{p}{2}} \Phi \end{aligned}\right. \quad \text{if } \Phi \in {}^p\Lambda \text{ or } \Phi \in {}^p\check{\mathscr{C}}. \tag{A.32}$$

Some properties of $\mathscr{A}$ and $\mathscr{B}$ are

$$\mathscr{A}^2 = \mathscr{B}^2 = 1, \qquad \mathscr{A}\mathscr{B} = \mathscr{B}\mathscr{A}, \tag{A.33}$$

$$\begin{aligned} \mathscr{A} d &= -d\mathscr{A}, \qquad \mathscr{A}\check{\Delta} = -\check{\Delta}\mathscr{A}, \\ \mathscr{B} d &= d\mathscr{A}\mathscr{B}, \qquad \mathscr{B}\check{\Delta} = \check{\Delta}\mathscr{A}\mathscr{B}. \end{aligned} \tag{A.34}$$

k) *The exterior product* of forms has a lattice analogue, which is bilinear and associative *on the space* $\check{\mathscr{C}}$ *of cochains:*

$$\begin{aligned} &\wedge : {}^p\check{\mathscr{C}} \times {}^q\check{\mathscr{C}} \to {}^{p+q}\check{\mathscr{C}}, \\ &d^{x,H} \wedge d^{y,K} = \begin{cases} \rho_{H,K}\, \delta^{x+e_H, y}\, d^{x, H \cup K} & \text{if } H \cap K = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{A.35}$$

(compare (A.21)). The definition is such that the *product formula*

$$d(\Phi \wedge \Xi) = (d\Phi) \wedge \Xi + (\mathscr{A}\Phi) \wedge d\Xi, \tag{A.36}$$

which holds for differential forms Φ, Ξ in the continuum is also valid for cochains Φ, Ξ on the lattice:

$$\check{\Delta}(\Phi \wedge \Xi) = (\check{\Delta}\Phi) \wedge \Xi + (\mathscr{A}\Phi) \wedge \check{\Delta}\Xi. \tag{A.37}$$

The formula

$$\mathscr{A}(\Phi \wedge \Xi) = (\mathscr{A}\Phi) \wedge (\mathscr{A}\Xi) \tag{A.38}$$

holds in the continuum *and* on the lattice, whereas the continuum equation

$$\mathscr{B}(\Phi \wedge \Xi) = (\mathscr{B}\Xi) \wedge (\mathscr{B}\Phi) \tag{A.39}$$

has no simple lattice analogue.

Remarks. 1) The exterior product $\wedge$ on the lattice is not alternating in the simple form as it is in the continuum. On the other hand, for constant 1-cochains we have

$$d^\mu \wedge d^\nu \wedge + d^\nu \wedge d^\mu \wedge = 0. \tag{A.40}$$

2) In the literature on algebraic topology the exterior product is usually called "cup product", in notation "$\cup$". We do not adopt this convention in order to make the analogy between continuum and lattice notions more striking and in order to avoid confusions in the notation with respect to the Clifford product "$\vee$".

l) *The Hodge star operator* $\star$ is a linear operator on Λ and $\check{\mathscr{C}}$:

$$\star : {}^p\Lambda \to {}^{4-p}\Lambda, \qquad \star : {}^p\check{\mathscr{C}} \to {}^{4-p}\check{\mathscr{C}} \tag{A.41}$$

defined by

$$\begin{aligned} &\star := *\mathscr{B}, \\ &*dx^H := \rho_{H, \mathscr{C}H}\, dx^{\mathscr{C}H}, \qquad *d^{x,H} := \rho_{H, \mathscr{C}H}\, d^{x+e_H, \mathscr{C}H} \end{aligned} \tag{A.42}$$

in the continuum and on the lattice, respectively. Some properties of this operation are

$$\star\star = 1 \text{ on } \Lambda, \qquad \star\star = \prod_{\mu=1}^{4} T_{-\mu} \text{ on } \check{\mathscr{C}}, \tag{A.43}$$

$$\left.\begin{aligned} \star\mathscr{A} &= \mathscr{A}\star \\ \star\mathscr{B} &= \mathscr{A}\mathscr{B}\star \end{aligned}\right\} \text{ on } \Lambda \text{ or } \check{\mathscr{C}} \text{ and for space-time dimension 4.} \tag{A.44}$$

From (A.43):

$$\begin{aligned} &\star^{-1} = \star \text{ on } \Lambda \text{ and} \\ &\star^{-1} d^{x,H} = \rho_{\mathscr{C}H, H}\, \mathscr{B}\, d^{x - e_{\mathscr{C}H}, \mathscr{C}H} \text{ on } \check{\mathscr{C}}. \end{aligned} \tag{A.45}$$

The dual boundary and the dual co-boundary operator on the lattice are related by

$$\check{\nabla} = -\star^{-1} \check{\Delta} \star. \tag{A.46}$$

In the continuum, the analogous relation defines the *co-differential or generalized divergence* δ:

$$\delta = -\star^{-1} d \star. \tag{A.47}$$

The co-differential δ and dual co-boundary $\check{\nabla}$ are corresponding operations. It follows from (A.29) that

$$\delta^2 \equiv \delta\delta = 0$$

and from (A.47, 46, 44, 34):

$$\mathscr{A}\delta = -\delta\mathscr{A}, \quad \mathscr{A}\check{\nabla} = -\check{\nabla}\mathscr{A}, \quad \mathscr{B}\delta = -\delta\mathscr{A}\mathscr{B}, \quad \mathscr{B}\check{\nabla} = -\check{\nabla}\mathscr{A}\mathscr{B}. \tag{A.48}$$

m) The *differentiation with respect to a differential* $e^K \lrcorner$ is a linear operation on Λ and on $\check{\mathscr{C}}$ defined by

$$e^K \lrcorner\, dx^H = \begin{cases} \rho_{K,H\setminus K}\, dx^{H\setminus K} & \text{if } K \subset H, \\ 0 & \text{otherwise} \end{cases} \tag{A.49}$$

and

$$e^K \lrcorner\, d^{x,H} = \begin{cases} \rho_{K,H\setminus K}\, d^{x,H\setminus K} & \text{if } K \subset H, \\ 0 & \text{otherwise,} \end{cases} \tag{A.50}$$

respectively. For $K=\{\mu\}$ we write $e^{\{\mu\}}\lrcorner \equiv e^\mu \lrcorner$. The comparison of (A.50) with (A.16) yields

$$\check{\nabla} = -e^\mu \lrcorner\, \partial_\mu^- \tag{A.51}$$

on the lattice; similarly we get

$$\check{\Delta} = d^\mu \wedge \partial_\mu^- \tag{A.52}$$

from (A.15, 13, 35, 9). The corresponding continuum formulas are

$$\delta = -e^\mu \lrcorner\, \partial_\mu \tag{A.53}$$

and

$$d = dx^\mu \wedge \partial_\mu. \tag{A.27}$$

Note the properties

$$(e^\mu \lrcorner)\mathscr{A} = -\mathscr{A}(e^\mu \lrcorner), \quad (e^\mu \lrcorner)\mathscr{B} = \mathscr{A}\mathscr{B}(e^\mu \lrcorner) \tag{A.54}$$

on Λ and on $\check{\mathscr{C}}$. The product rules for $e^\mu \lrcorner$ are

$$e^\mu \lrcorner (dx^H \wedge dx^K) = (e^\mu \lrcorner\, dx^H) \wedge dx^K + (\mathscr{A}\, dx^H) \wedge (e^\mu \lrcorner\, dx^K) \tag{A.55}$$

in the continuum and

$$e^\mu \lrcorner (d^{x,H} \wedge d^{y,K}) = (e^\mu \lrcorner\, d^{x,H}) \wedge T_\mu d^{y,K} + (\mathscr{A}\, d^{x,H}) \wedge (e^\mu \lrcorner\, d^{y,K}) \tag{A.56}$$

on the lattice. Successive derivations are antisymmetric:

$$e^\mu \lrcorner\, e^\nu \lrcorner + e^\nu \lrcorner\, e^\mu \lrcorner = 0 \quad \text{on } \Lambda \text{ and on } \check{\mathscr{C}}. \tag{A.57}$$

n) The *Clifford product* of differential forms is a bilinear, associative and distributive mapping of $\Lambda \times \Lambda$ into Λ defined by

$$\Phi \vee \Xi := \sum_{p \geq 0} \frac{(-1)^{\binom{p}{2}}}{p!} (\mathscr{A}^p e_{\mu_1} \lrcorner \ldots e_{\mu_p} \lrcorner\, \Phi) \wedge (e^{\mu_1} \lrcorner \ldots e^{\mu_p} \lrcorner\, \Xi). \tag{A.58}$$

For the generating 1-forms this yields

$$dx^\mu \vee \Phi = dx^\mu \wedge \Phi + e^\mu \lrcorner\, \Phi, \quad \Phi \vee dx^\mu = \Phi \wedge dx^\mu - e^\mu \lrcorner\, \mathscr{A}\Phi \tag{A.59}$$

and

$$dx^\mu \vee dx^\nu \vee + dx^\nu \vee dx^\mu \vee = 2\delta^{\mu\nu}. \tag{A.60}$$

Equation (A.60) is the defining relation for the Clifford algebra of the metric $g^{\mu\nu} = \delta^{\mu\nu}$ of euclidean space-time.

On the lattice, it is possible to define a Clifford product as well. The lattice version of (A.58) is

$$\Phi \vee \Xi = \sum_L (-1)^{\binom{l}{2}} (\mathscr{A}^l T_{-e_L} e^L \lrcorner\, \Phi) \wedge (e^L \lrcorner\, \Xi), \tag{A.61}$$

l the cardinality of L, which can be written in the form

$$d^{x,H} \vee d^{y,K} = (-1)^{\binom{\lambda}{2}} (-1)^{\lambda(h-\lambda)} \cdot \rho_{\Lambda, H \triangle K}\, \rho_{H\setminus\Lambda, K\setminus\Lambda} \cdot \delta^{x+e_H, y} \cdot d^{x+e_\Lambda, H \triangle K} \tag{A.62}$$

for elementary cochains. Here, $\Lambda = H \cap K$ and λ and h are the cardinalities of Λ and H, respectively. For constant 1-cochains d^μ and arbitrary cochains Φ, Ξ, (A.62) yields

$$d^\mu \vee \Xi = d^\mu \wedge \Xi + e^\mu \lrcorner\, \Xi, \quad \Phi \vee d^\mu = \Phi \wedge d^\mu - T_{-\mu} e^\mu \lrcorner\, \mathscr{A}\Phi \tag{A.63}$$

and

$$d^\mu \vee d^\nu \vee + d^\nu \vee d^\mu \vee = 2\delta^{\mu\nu} T_\mu \tag{A.64}$$

as lattice analogues of (A.59) and (A.60). In general, the product (A.61) is not associative, however, the relation

$$(d^\mu \vee \Phi) \vee d^K = d^\mu \vee (\Phi \vee d^K) \tag{A.65}$$

which holds for an arbitrary cochain Φ and the constant cochains d^μ, d^K is enough to discuss the global flavour symmetry of the Dirac-Kähler equation. The $\star$-operation is a special Clifford product:

$$\star\Phi = \Phi \vee \varepsilon \tag{A.66}$$

where $\varepsilon = dx^1 \wedge \ldots \wedge dx^4$ in the continuum and $\varepsilon = \sum_x d^{x,\{1234\}}$ on the lattice. Also, the automorphism

$\mathscr{A}$ can be written with help of the Clifford product:

$$\mathscr{A}\Phi=(\varepsilon\vee\Phi)\vee\varepsilon=\varepsilon\vee(\Phi\vee\varepsilon) \tag{A.67}$$

on Λ *and* on $\breve{\mathscr{C}}$. This relation follows from

$$\varepsilon\vee\Phi=(\mathscr{A}\Phi)\vee\varepsilon \tag{A.68}$$

and

$$\varepsilon\vee\varepsilon=1 \tag{A.69}$$

in the continuum and the corresponding equations

$$\varepsilon\vee\Phi=\left(\prod_{\mu=1}^{4}T_\mu\right)((\mathscr{A}\Phi)\vee\varepsilon) \tag{A.70}$$

and

$$(\Phi\vee\varepsilon)\vee\varepsilon=\left(\prod_{\mu=1}^{4}T_{-\mu}\right)\Phi \tag{A.71}$$

on the lattice. The equations (A.38, 39) for the exterior product are valid also for the Clifford product.

$$\mathscr{A}(\Phi\vee\Xi)=(\mathscr{A}\Phi)\vee(\mathscr{A}\Xi) \tag{A.72}$$

holds in the continuum *and* on the lattice, whereas the continuum equation

$$\mathscr{B}(\Phi\vee\Xi)=(\mathscr{B}\Xi)\vee(\mathscr{B}\Phi) \tag{A.73}$$

has no simple lattice analogue. The *Dirac-Kähler operator* $d-\delta$ and its lattice version $\breve{\Delta}-\breve{\nabla}$ can be written with help of the Clifford product, too. For this, we use (A.27, 53, 59) and (A.51, 52, 63), respectively. The result is

$$d-\delta=dx^\mu\vee\partial_\mu \quad \text{on } \Lambda, \tag{A.74}$$

$$\breve{\Delta}-\breve{\nabla}=d^\mu\vee\partial_\mu^- \quad \text{on } \breve{\mathscr{C}}. \tag{A.75}$$

The differentials dx^μ and the cochains d^μ satisfy the Clifford relations (A.60) and (A.64), respectively.

o) The *scalar product* between two differential forms or cochains Φ, Ξ is given by

$$(\Phi,\Xi)_0:=\sum_p(\mathscr{B}^p\Phi)\wedge\star^p\Xi=(\Xi,\Phi)_0, \tag{A.76}$$

where ${}^p\Phi$ is the part of Φ which is homogeneous of degree p. The *first derived scalar product* is defined according to

$$\begin{aligned}(\Phi,\Xi)_1&:=e_\mu\lrcorner(dx^\mu\vee\Phi,\Xi)_0\\&=\sum_p(\mathscr{B}^p\Phi)\wedge\star^{p-1}\Xi+(\mathscr{B}^p\Xi)\wedge\star^{p+1}\Phi\in{}^3\Lambda\end{aligned} \tag{A.77}$$

in the continuum and

$$\begin{aligned}(\Phi,\Xi)_1&:=e_\mu\lrcorner(T_{-\mu}d^\mu\vee\Phi,\Xi)_0\\&=\sum_p(\mathscr{B}^p\Phi)\wedge\star^{p+1}\Xi+(\mathscr{B}^p\Xi)\wedge\star^{p+1}\Phi\in{}^3\breve{\mathscr{C}}\end{aligned} \tag{A.78}$$

on the lattice. Higher derived scalar products are necessary for the discussion of higher tensor currents. These are, however, not considered in this paper. From the identities

$$\begin{aligned}(d\Phi,\Xi)_0-(\Phi,\delta\Xi)_0&=d(\sum_p(\mathscr{B}^p\Phi)\wedge\star^{p+1}\Xi),\\(\delta\Phi,\Xi)_0-(\Phi,d\Xi)_0&=-d(\sum_p(\mathscr{B}^p\Xi)\wedge\star^{p+1}\Phi),\end{aligned} \tag{A.79}$$

which hold also on the lattice, if we substitute $d\to\breve{\Delta}$ and $\delta\to\breve{\nabla}$ we derive the *Green's formulas*

$$((d-\delta)\Phi,\Xi)_0+(\Phi,(d-\delta)\Xi)_0=d(\Phi,\Xi)_1 \text{ on } \Lambda \tag{A.80}$$

and

$$((\breve{\Delta}-\breve{\nabla})\Phi,\Xi)_0+(\Phi,(\breve{\Delta}-\breve{\nabla})\Xi)_0=\breve{\Delta}(\Phi,\Xi)_1 \text{ on } \breve{\mathscr{C}}. \tag{A.81}$$

They are the starting point for the discussion of conserved currents.

References

1. K.G. Wilson: Phys. Rev. **D10**, 2445 (1974)
2. M. Creutz: Phys. Rev. Lett. **45**, 313 (1980); B. Berg: Phys. Lett. **97B**, 401 (1980); G. Bhanot, C. Rebbi: Nucl. Phys. **B189**, 469 (1981); K. Ishikawa, G. Schierholz, M. Teper: Phys. Lett. **110B**, 399 (1982)
3. H. Hamber, G. Parisi: Phys. Rev. Lett. **47**, 1972 (1981); E. Marinari, G. Parisi, C. Rebbi: Phys. Rev. Lett. **47**, 1795 (1981); N. Kawamoto: Nucl. Phys. **B190**, 617 (1981); Niels Bohr Institute preprint NBI-HE-82-4 (1982); D. Weingarten: Phys. Lett. **109B**, 57 (1982); A. Hasenfratz, Z. Kunszt, P. Hasenfratz, C.B. Lang: CERN preprint, Ref. TH. 3220-CERN (1981)
4. For a review of this problem see: L.H. Karsten, J. Smit: Nucl. Phys. **B183**, 103 (1981)
5. K.G. Wilson: In: New phenomena in subnuclear physics. A. Zichichi ed. ("Erice 1975") New York: Plenum 1977. P.H. Ginsparg, K.G. Wilson: Harvard Univ. preprint HUTP-81/A060 (1981)
6. L. Susskind: Phys. Rev. **D16**, 3031 (1977); H.S. Sharatchandra, H.J. Thun, P. Weisz: Nucl. Phys. **B192**, 205 (1981)
7. S.D. Drell, M. Weinstein, S. Yankielowicz: Phys. Rev. **D14**, 487, 1627 (1976)
8. J.M. Rabin: Phys. Rev. **D24**, 3218 (1981); In a recent preprint (Yale University YTP81-29), J.M. Rabin followed a suggestion of [12] and discussed the spectrum problem in the framework of the Dirac-Kähler equation
9. C.N. Yang: Phys. Rev. Lett. **33**, 445 (1974); M.E. Mayer, W. Drechsler: Fiber bundle techniques in gauge theories. Lecture Notes in Physics, 67. Berlin, Heidelberg, New York: Springer 1977; G. Mack: Fortschr. Phys. **29**, 135 (1981)
10. W. Graf: Ann. Inst. H. Poincaré. Sect. **A29**, 85 (1978)
11. E. Kähler: Rend. Mat. Ser. V, **21**, 425 (1962)
12. P. Becher: Phys. Lett. **104B**, 221 (1981)
13. I.M. Singer, J.A. Thorpe: Lecture notes on elementary topology and geometry. Glenview, ILL.: Scott, Foresman 1967 P.S. Aleksandrov: Combinatorial topology, vol. 1-3. Rochester, N.Y.: Graylock 1956/57 and Albany, N.Y. (1960)

14. P.A.M. Dirac: Proc. Royal Soc. London **A117**, 610 (1928); **A118**, 351 (1928)
15. The description of Fermi fields in terms of geometric algebras was attempted by M. Schönberg: An. Acad. Brasil. Ci. **29**, 473 (1957); **30**, **1**, 117, 259, 429 (1958); Suppl. Nuovo Cimento **VI**, Ser. X, 356 (1957)
16. W. Pauli: Die allgemeinen Prinzipien der Wellenmechanik. Handbuch der Physik 24, Berlin: Springer 1933
17. M. Atiyah: Vector fields on manifolds. Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Natur-, Ingenieur- und Gesellschaftswissenschaften, Heft 200 (1970)
18. E.P. Wigner: Group theory and its application to the quantum mechanics of atomic spectra. Expanded and improved edition, p. 210ff. New York, London: Academic Press 1959
19. Textvooks on quantum field theory; f.i.: C. Itzykson, J.-B. Zuber: Quantum field theory. New York: McGraw-Hill 1980
20. L. Susskind: Ref. [6]; T. Banks et al.: Phys. Rev. **D15**, 1111 (1977)
21. C.C. Chevalley: The algebraic theory of spinors. Morningside Heights, N.Y.. Columbia Univ. Press 1954
22. B.L. van der Waerden; Group theory and quantum mechanics. Berlin, Heidelberg, New York: Springer 1974
23. H. Weyl: The theory of groups and quantum mechanics Chap. IV, 14. New York: Dover 1931; R. Haag, D. Kastler: J. Math. Phys. **5**, 848 (1964)
24. T. Nakano: Progr. Theor. Phys. **21**, 241 (1959)
25. M.E. Mayer, et al.: Ref. [9]; W. Greub, S. Halperin, R. Vanstone: Connections, curvature, and cohomology, vol. I. New York: Academic Press 1972
26. K. Osterwalder, R. Schrader: Helv. Phys. Acta **46**, 277 (1973)
27. R.P. Feynman, A.R. Hibbs: Quantum mechanics and path integrals. New York: McGraw-Hill 1965
28. F.A. Berezin: The method of second quantization. New York, London: Academic Press 1966
29. J.-P. Serre: Ann. Math. **54**, 425 (1951); P.J. Hilton, S. Wylie: Homology theory, Cambridge: Cambridge University Press 1960
30. A. Chodos, J.B. Healy: Nucl. Phys. **B127**, 426 (1977); H.S. Sharatchandra, et al.: Ref. [6]
31. H.S. Sharatchandra, et al.: Ref. [6]
32. A similar statement in the Hamiltonian approach has recently been made by A. Dhar, R. Shankar: Tata Institute Bombay preprint TIFR/TH/82-4 (1982)
33. Some references on this much disputed problem are: H.B. Nielsen, M. Ninomiya: Phys. Lett. **105B**, 219 (1981); L. Karsten, et al.: Ref. [4]; P.H. Ginsparg, et al.: Ref. [5]; W. Kerler: Phys. Rev. **D23**, 2384 (1981)
34. N.H. Kuiper: In: The CHERN-Symposium 1979. W.-Y. Hsiang, et al., eds.. Berlin, Heidelberg, New York: Springer 1980
35. F. Flanders: Differential forms with applications to the physical sciences. New York: Academic Press 1963