# BRS TRANSFORMATION AND COLOR CONFINEMENT

K. NISHIJIMA
*Department of Physics, University of Tokyo, Tokyo, Japan*

Received 25 June 1982

The condition of confinement of quarks and gluons in QCD is derived. It is shown that color confinement is realized when there exist massless scalar color-octet bound states of two Faddeev–Popov ghosts.

The problem of confinement of quarks and gluons occupies a central position in quantum chromodynamics (QCD). In the lattice gauge theory the condition for quark confinement is given by the area law for the Wilson loop [1]. In the present paper we look for the corresponding condition within the framework of the conventional field theory. This condition is expressed in terms of certain properties of the Faddeev–Popov ghost as we shall see in what follows. The lagrangian density in QCD is given in the Pauli metric by

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}\cdot F_{\mu\nu} + A_\mu \cdot \partial_\mu B + \tfrac{1}{2}\alpha B\cdot B + \mathrm{i}\partial_\mu \bar{c}\cdot D_\mu c - \bar{\psi}(\gamma_\mu D_\mu + m)\psi\,, \tag{1}$$

where covariant derivatives $D_\mu$ are defined by

$$D_\mu c = \partial_\mu c + gA_\mu \times c, \quad D_\mu\psi = (\partial_\mu - \mathrm{i}gT\cdot A_\mu)\psi\,, \tag{2}$$

and use has been made of the abbreviations, $S\cdot T = S^aT^a$ and $(S\times T)^a = f_{abc}S^bT^c$.

Next we introduce the BRS transformation of fields [2].

$$\delta A_\mu = D_\mu c, \quad \delta B = 0\,,$$
$$\delta c = -\tfrac{1}{2}gc\times c, \quad \delta\bar{c} = \mathrm{i}B,$$
$$\delta\psi = \mathrm{i}g(c\cdot T)\psi\,. \tag{3}$$

In terms of the generator $Q_\mathrm{B}$ they can be expressed as

$$\delta O = \mathrm{i}[Q_\mathrm{B}, O]_\pm$$

where we choose the $-(+)$ sign when $O$ involves an even (odd) number of the hermitian ghost fields $c$ and $\bar{c}$. Kugo and Ojima [3] have also introduced another charge $Q_\mathrm{c}$ satisfying

$$\mathrm{i}[Q_\mathrm{c}, c(x)] = c(x), \quad \mathrm{i}[Q_\mathrm{c}, \bar{c}(x)] = -\bar{c}(x)\,. \tag{4}$$

It commutes with all other fields, and it defines the ghost number, namely (+1) for $c$ and (−1) for $\bar{c}$. These charges are known to satisfy the relations.

$$\mathrm{i}[Q_\mathrm{c}, Q_\mathrm{B}] = Q_\mathrm{B}, \quad Q_\mathrm{B}^2 = 0\,. \tag{5}$$

The second relation is equivalent to $\delta^2 = 0$, so that the BRS transformation is nilpotent.

We can also define the BRS transformation for asymptotic fields. Due to infrared singularities in QCD their presence is doubtful, nevertheless we shall simply assume that they exist in this paper. Then the BRS transformation for the asymptotic fields is linear. When $\delta a(x) = b(x) \neq 0$, $\{a(x), b(x)\}$ is called a BRS doublet. Notice that $\delta b(x) = \delta^2 a(x) = 0$. The asymptotic fields $a(x)$ and $b(x)$ carry the same set of quantum numbers but for the ghost number. When $\delta a(x) = 0$ but its parent $f(x)$, defined by $\delta f(x) = a(x)$, does not exist, $a(x)$ is called a BRS singlet.

In what follows we shall assume the naive asymptotic completeness and introduce the state vector space $\mathcal{V}$ for QCD. By applying only singlet operators to the vacuum state $|0\rangle$, a subspace of $\mathcal{V}$, denoted by $\mathcal{V}_\mathrm{S}$, is generated. When $|\alpha\rangle, |\beta\rangle \in \mathcal{V}_\mathrm{S}$, the unitarity of the S matrix is expressed as

$$\langle\beta|\alpha\rangle = \langle\beta|S^+S|\alpha\rangle = \langle\beta|S^+P(\mathcal{V}_\mathrm{S})S|\alpha\rangle\,, \tag{6}$$

and similarly for $SS^+$. This relation is a consequence of the Kugo–Ojima theorem [3]. $P(\mathcal{V}_\mathrm{S})$ stands for the projection operator to the subspace $\mathcal{V}_S$, so that no doublets appear in the intermediate states. In this sense, doublets in QCD are analogous to longitudinal and scalar photons in QED, and doublets are confined consequently. This defines confinement in the present paper. Interpreting that singlets represent hadrons, the problem of color confinement reduces to that of demonstrating that both quarks and gluons are BRS doublets.

We shall further assume that the BRS invariance is exact, so that $Q_B|0\rangle = 0$ and utilize the BRS identity,

$$\langle 0|\delta T(...)|0\rangle = 0 . \tag{7}$$

We shall abbreviate $\langle 0|T(...)|0\rangle$ as $\langle ...\rangle$ below. Then by making use of the BRS identities we find the following Ward–Takahashi (W–T) identities [‡1]

$$\partial_\lambda\langle (D_\lambda\bar{c})^a(x), \delta\psi(y), \bar{\psi}(z)\rangle + \partial_\lambda\langle (D_\lambda\bar{c})^a(x), \psi(y), \delta\bar{\psi}(z)\rangle = \mathrm{i}gT^a[\delta^4(x-y) - \delta^4(x-z)]S_F(y-z) , \tag{8}$$

$$\partial_\lambda\langle (D_\lambda\bar{c})^a(x), \delta A^b_\mu(y), A^c_\nu(z)\rangle + \partial_\lambda\langle (D_\lambda\bar{c})^a(x), A^b_\mu(y), \delta A^c_\nu(z)\rangle = \mathrm{i}gM^a{}_{bc}[\delta^4(x-y) - \delta^4(x-z)]D_{F\mu\nu}(y-z) , \tag{9}$$

where $M^a{}_{bc} = \mathrm{i}f_{bac}$, and $S_F$ and $D_F$ denote propagators of the quark and gluon fields, respectively. Then we write

$$\langle (D_\lambda\bar{c})^a(x), \delta\psi(y), \bar{\psi}(z)\rangle = \int \mathrm{d}^4z'\, gG^a_\lambda(yz' : x)S_F(z'-z) ,$$

$$\langle (D_\lambda\bar{c})^a(x), \psi(y), \delta\bar{\psi}(z)\rangle = \int \mathrm{d}^4y' gS_F(y-y')\bar{G}^a_\lambda(y'z : x) . \tag{10}$$

The Fourier-transform of eq. (8) can be expressed as

$$(p-q)_\lambda \cdot G^a_\lambda(p,q)S_F(q) + S_F(p)(p-q)_\lambda \cdot \bar{G}^a_\lambda(p,q) = \mathrm{i}T^a[S_F(p) - S_F(q)] . \tag{11}$$

In this equation we can replace $G^a_\lambda$ and $\bar{G}^a_\lambda$ by their spin 0 projection defined by

$$G^a_\lambda(p,q)^{(0)} = [(p-q)_\lambda(p-q)_\mu/(p-q)^2]G^a_\mu(p,q) . \tag{12}$$

[‡1] A similar relationship has been derived by Hata in a different context [4].

In order to simplify our argument we shall choose the Landau gauge ($\alpha = 0$) in what follows. In this gauge we have $\partial_\lambda D_\lambda\bar{c} = 0$, and possible poles in $G_\lambda$ due to massless vector particles will disappear in the projection. A pole due to massless scalar particles is still present as is clear from the W–T identity

$$\langle (D_\lambda\bar{c})^a(x), c^b(y)\rangle = -\mathrm{i}\delta_{ab}\partial_\lambda D_F(x-y) , \tag{13}$$

where $D_F$ denotes the free massless propagator. Then this equation shows that $D_\lambda\bar{c}$ generates a massless scalar particle as

$$D_\lambda\bar{c} \to \partial_\lambda\bar{\Gamma}, \quad c \to \Gamma . \tag{14}$$

We then replace $D_\lambda\bar{c}$ by $D_\lambda\bar{c} - \partial_\lambda\bar{\Gamma}$ and write $F$ for $G$ in eq. (10). The functions $F^a_\lambda(p,q)^{(0)}$ and $\bar{F}^a_\lambda(p,q)^{(0)}$ so defined are free of the poles at $(p-q)^2 = 0$ except for the projection operator in eq. (12).

According to Nakanishi's theorem [5] the asymptotic field $\bar{\Gamma}$ carrying the ghost numer (−1) cannot be a BRS singlet, but it must be a member of a BRS doublet. Confinement is realized when $\bar{\Gamma}$ is the second generation of the doublet expressible as

$$\delta\bar{d}(x) = \bar{\Gamma}(x) . \tag{15}$$

Then the BRS identity leads to

$$\langle\partial_\lambda\bar{\Gamma}(x), \delta\psi(y), \bar{\psi}(z)\rangle + \langle\partial_\lambda\bar{\Gamma}(x), \psi(y), \delta\bar{\psi}(z)\rangle = 0 , \tag{16}$$

and by subtracting eq. (16) from eq. (8) we find

$$(p-q)_\lambda \cdot F^a_\lambda(p,q)^{(0)}S_F(q) + S_F(p)(p-q)_\lambda \cdot \bar{F}^a_\lambda(p,q)^{(0)} = iT^a[S_F(p) - S_F(q)] . \tag{17}$$

We then put $p - q = \epsilon P$ with $P^2 \neq 0$, and apply the limiting procedure $\lim_{\epsilon\to 0}\partial/\partial\epsilon$ to eq. (17). Since the individual terms on the lhs of eq. (17) are of the order of $\epsilon$ because of the absence of poles at $(p-q)^2 = 0$, we obtain

$$P_\lambda \cdot F^a_\lambda(p,p:P)^{(0)}S_F(p) + S_F(p)P_\lambda \cdot \bar{F}^a_\lambda(p,p:P) = \mathrm{i}T^aP_\lambda \cdot (\partial/\partial p_\lambda)S_F(p) . \tag{18}$$

$F_\lambda$ and $\bar{F}_\lambda$ gain a possible dependence on the direction of $P$ through the factor $P_\lambda P_\mu/P^2$ originated from the projection operator in eq. (12). Eq. (18) shows that $F_\lambda$ and/or $\bar{F}_\lambda$ must have a pole corresponding to $\mathrm{i}p\gamma + m = 0$. For the symmetry reason both must have this pole implying that both $\delta\psi$ and $\delta\bar{\psi}$ generate a pole at the quark mass. Hence $\{\psi^{\mathrm{in}}, \delta\psi^{\mathrm{in}}\}$ represents a BRS doublet, and quarks are confined. A similar argument starting from eq. (9) shows that gluons are also confined.

We shall reexpress the condition (15) in a more convenient form by using the BRS identity.

$$0 \neq \langle \bar{\Gamma}(x), \Gamma(y) \rangle = \langle \delta \bar{d}(x), \Gamma(y) \rangle = -\langle \bar{d}(x), \delta \Gamma(y) \rangle , \quad (19)$$

This implies the existence of $\delta \Gamma$. Since $\Gamma$ is the asymptotic field of $c$ and $\delta c \sim c \times c$, there must exist the asymptotic field of $c \times c$ carrying the same set of quantum numbers as that of $c$ but for the ghost number. Quarks and gluons are confined when they form bound states with the ghost $c$ as is clear from the explicit expressions for $\delta \psi$ and $\delta A_\mu$ in eq. (3). When the ghost $c$ iself forms a bound state with another ghost, the ability of forming a bound state with the ghost is communicated to other colored particles through the BRS identities.

This work has been completed while the author was visiting II. Institut für Theoretische Physik der Universität Hamburg. He would like to thank Rudolf Haag for his warm hospitality there and also for useful criticism. He is also indebted to Taichiro Kugo for stimulating correspondence which motivated the present work.

*References*

[1] K.G. Wilson, Phys. Rev. D10 (1974) 2445.
[2] C. Becchi, A. Rouet and R. Stora, Ann. Phys. (NY) 98 (1976) 287.
[3] T. Kugo and I. Ojima, Phys. Lett. 73B (1978) 459; Suppl. Prog. Theor. Phys. No. 66 (1979).
[4] H. Hata, Kyoto University preprint, KUNS 618 (1981).
[5] N. Nakanishi, Prog. Theor. Phys. 62 (1979) 1396; T. Kugo and S. Uehara, Prog. Theor. Phys. 64 (1980) 1395.