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Renormalisation Group Behaviour of 0^+ and 2^+ Glueball Masses in $SU(2)$ Lattice Gauge Theory

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Abstract. We calculate the 0^+ and 2^+ glueball masses at several values of the coupling and verify compatibility with the desired renormalisation group behaviour. The calculation uses momentum smeared glueball wave functions on a large 8^4 lattice and confirms our previous results obtained on smaller lattices.

Introduction

Recently, there have been several calculations of glueball masses in $SU(2)$ [1,2] and $SU(3)$ [3,4] non-abelian gauge theories. The calculations involved Monte Carlo simulations [5] of the lattice regulated theories [6], and all employed the variational method [7] as an important ingredient, although they differed in other important technical details.

For such a calculation to be relevant to the theory in the continuum (infinite momentum cut-off) limit, we must certainly require that the glueball size, l_G, should be (much) greater than the spatial lattice spacing, a, and (much) less than half the spatial extent of the lattice, $\frac{1}{2}La$, (half because of periodic boundary conditions):

$$a < l_G < \tfrac{1}{2}La, \tag{1}$$

To provide a connection with the parameters of perturbation theory, we should also like the lattice spacing to be small enough for the coupling to be well represented by the usual two-loop perturbative formula. Fortunately, on the basis of previous work [6, 8] in $SU(2)$ lattice gauge theory (which is the theory of interest in this paper), we already know that in the region of $\beta(=4/g^2)$ of interest to us here this will always be so.

Relation (1) is represented pictorially in Fig. 1. The lattice spacing is expressed in physical units via the two-loop formula

$$a(\beta) = \frac{57.5}{\Lambda_{mom}} e^{-(3\pi^2/11)\beta} \left(\frac{6\pi^2}{11}\beta\right)^{51/121} \tag{2}$$

as are $\frac{1}{2}La$ for $L=4$ and $L=8$. The shaded area represents a crude estimate of the glueball size, l_G, as inferred from our previous work [1]. The overall normalisation is unimportant. However, for the readers orientation we have expressed the vertical scale in "fermi", using a value of 270 MeV for Λ_{mom} as would be appropriate for a string tension of 400 MeV.

In our previous $SU(2)$ calculations [1] we worked on $L=4$ and $L=6$ lattices at $\beta=2.3$. The reason for choosing this value of β is apparent if one considers Fig. 1 in the light of (1). It is also apparent that if one wishes to calculate glueball masses over a range of β values so as to check that they are independent of β (in physical units), and hence to check that the correct continuum renormalisation group behaviour has been achieved, a larger lattice needs to be employed. Unfortunately, using the techniques of [1-4], the computing time for a given statistical error increases as L^3, and hence the use of large lattices is prohibitively costly. In this paper we calculate glueball masses using a modification of the techniques in [1], which enables us to work on a large 8^4 lattice as efficiently as on a $4^3 \cdot 8$ lattice.

In the next section we introduce our notation and the method to be used. We then calculate the 0^+ and 2^+ glueball masses using only our measurements on the 8^4 lattice, assuming that we are in the continuum limit (as we did in our previous work). The results are

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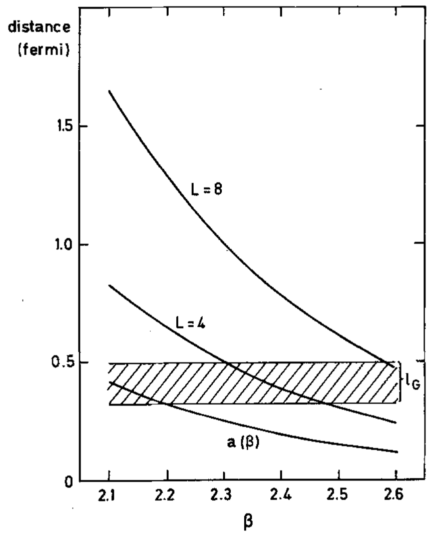

Fig. 1. The lattice spacing $a(\beta)$, the glueball width l_G and half the lattice spatial size $\frac{1}{2}La$ for $L=4$ and $L=8$ lattices are plotted against β in physical units (see text)

consistent with our previous results indicating that lattice size effects are smaller than our statistical errors. This justifies combining our data on an 8^4 lattice at $\beta=2.2$, 2.3, and 2.4 with our previous data at $\beta=2.3$. This combined data is used to calculate 0^+ and 2^+ glueball masses at the above three values of β and to compare with the desired continuum renormalisation group behaviour. We end with some wave function estimates.

Crucial to the usefullness of such an investigation is a reliable error analysis. In an appendix we outline our error analysis.

Method

Let $\phi(t,\mathbf{p})$ be a colour singlet operator at (Euclidean) time t with momentum $\mathbf{p}$. Then, using the notation $\Gamma_t = \langle \Omega|\phi(t,\mathbf{p})\phi(0,\mathbf{p})|\Omega\rangle$ (where $|\Omega\rangle$ is the vacuum), we have [1]

$$\frac{\Gamma_a}{\Gamma_0}=\frac{\langle \Omega|\phi(0,\mathbf{p})e^{-Ha}\phi(0,\mathbf{p})|\Omega\rangle}{\langle \Omega|\phi(0,\mathbf{p})\phi(0,\mathbf{p})|\Omega\rangle} = \frac{\sum_{n=0}^{\infty} e^{-E_n a}|\langle n|\phi(0,\mathbf{p})|\Omega\rangle|^2}{\sum_{n=0}^{\infty} |\langle n|\phi(0,\mathbf{p})|\Omega\rangle|^2} \leqq e^{-E_0 a} \tag{3}$$

(where $E_0 \leqq E_1 \leqq \dots$) and

$$\frac{\Gamma_{2a}}{\Gamma_a}=\frac{\langle \Omega|\phi(0,\mathbf{p})e^{-H2a}\phi(0,\mathbf{p})|\Omega\rangle}{\langle \Omega|\phi(0,\mathbf{p})e^{-Ha}\phi(0,\mathbf{p})|\Omega\rangle} = \frac{\sum_{n=0}^{\infty} e^{-E_n 2a}|\langle n|\phi(0,\mathbf{p})|\Omega\rangle|^2}{\sum_{n=0}^{\infty} e^{-E_n a}|\langle n|\phi(0,\mathbf{p})|\Omega\rangle|^2} \approx e^{-E_0 a}, \tag{4}$$

so that

$$E_0 \leqq \frac{1}{a}\left|\ln\frac{\Gamma_a}{\Gamma_0}\right| \tag{5}$$

and (for not too large $|\mathbf{p}|$)

$$E_0 \approx \frac{1}{a}\left|\ln\frac{\Gamma_{2a}}{\Gamma_a}\right|. \tag{6}$$

In [1] we have measured Γ_0, Γ_a and Γ_{2a} with operators of zero momentum, $\mathbf{p}=0$, definite quantum numbers J^P and zero projection onto the vacuum to give us the glueball *mass* estimates

$$m \leqq \frac{1}{a}\left|\ln\frac{\Gamma_a}{\Gamma_0}\right| \tag{7}$$

and

$$m \approx \frac{1}{a}\left|\ln\frac{\Gamma_{2a}}{\Gamma_a}\right|. \tag{8}$$

The variational part of the calculation consists of varying the operators ϕ such as to maximise the right-hand side of (3) (i.e. to minimise the right-hand side of (5), (7)). In the hope that the higher mass intermediate states in (3) get sufficiently suppressed by this means, the minimum value of the right-hand side of (7) can then be used as a first estimate of the glueball mass. It is clear however that Γ_{2a}/Γ_a will, for a not small (as in the case here), provide a much better estimate for m than Γ_a/Γ_0 as higher mass intermediate states in (4) are double suppressed by the variational method and the exponential. This advantage of using Γ_{2a}/Γ_a is to be set against the disadvantage of a much larger error-to-signal ratio. The situation improves a lot however, if this is supplemented by the variational calculation (which maximises the signal) as has successfully been employed in [1,3].

The difficulty in extending this approach to large lattices lies in the fact that we have one wave function at any given time irrespective of the size of the lattice, whereas computing time increases roughly as L^3. Hence, the computing time required to achieve a given signal-to-error ration also increases as L^3. A solution to this problem would be to consider wave functions not just with $\mathbf{p}=0$, but with any $\mathbf{p}^2 \leqq m^2$ (or $\mathbf{p}^2 \leqq cm^2$ for some $c \ll 1$ if one is conservative). For such a low momentum we expect the simple dispersion relation $E^2 \approx m^2+\mathbf{p}^2$, and hence we can use (4) to give us

$$m^2 \approx \left(\frac{1}{a}\ln\frac{\Gamma_{2a}}{\Gamma_a}\right)^2 - \mathbf{p}^2. \tag{9}$$

For a lattice of spatial extent La, the number of such low momentum states, and hence measurements, increases as L^3, so that one looses nothing in going to larger lattices (to the extent that the various measurements are indeed statistically independent).

In this paper we employ a slight variation of this idea. The basic components of our wave functions will be 1×1 or 2×2 plaquettes (see Fig. 2). To construct a $\mathbf{p}=0$ 0^+ wave function one would simply sum up

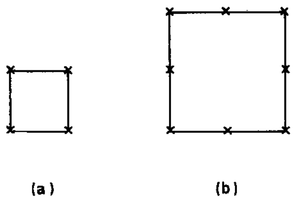

Fig. 2a and b. The basic components of our wave functions: **a** a 1×1 plaquette, **b** a 2×2 plaquette

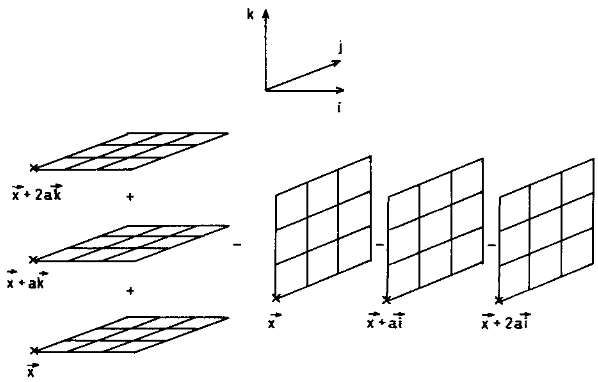

Fig. 3. A 2^+ momentum smeared wave function formed out of some of the planes of a cube (3 links each side) with a corner at site **x**

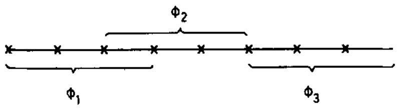

Fig. 4. The distribution of the three wave functions amongst the 8 links of a one-dimensional section of the 8^4 lattice

all the elementary plaquettes at a given time [1]; the wave function is translationally invariant and hence has zero momentum. Our wave functions will not be translationally invariant but will consist of summing up all the plaquettes contained within spatial cubes, three lattice spacings on each side. For 2^+ wave functions we take three parallel planes in such a cube and subtract an orthogonal set of three parallel planes; see Fig. 3. We construct such cube wave functions using separately the two basic operators in Fig. 2, and we restrict ourselves to 0^+ and 2^+ wave functions. At each time, in the 8^4 lattice which we shall use, we can construct 27 ($=3^3$) wave functions. These wave functions overlap very little as can be seen from the one dimensional section shown in Fig. 4. It is clear that populating an equal time slice of a lattice of spatial extent La will give us a number of wave functions increasing as L^3, so that our computing time does not grow with increasing lattice size.

Since the wave functions are not translationally invariant, they clearly involve some momentum smearing. However, we have deliberately chosen rather extended wave functions so that this momentum smearing should be small. We parametrise the effect of this smearing by writing

$$E^2 = m^2 + \overline{\mathbf{p}^2}, \tag{10}$$

where $\overline{\mathbf{p}^2}$ is some average momentum squared. Since the smearing is geometric, we expect that as we vary the lattice spacing, a,

$$\overline{\mathbf{p}^2} = \frac{\delta^2}{a^2}, \tag{11}$$

where δ^2 is independent of a. Hence, our mass estimate would be given by

$$m^2 \approx \left(\frac{1}{a}\ln\frac{\Gamma_{2a}}{\Gamma_a}\right)^2 - \frac{\delta^2}{a^2}, \tag{12}$$

where Γ_{2a}/Γ_a is obtained using our "cube" wave functions.

The above procedure should be reliable as long as $m^2 \gtrsim \delta^2/a^2$. For the 0^+ glueball this will be the case for $\beta \lesssim 2.4$. For heavier glueballs such as the 2^+ we shall always have $m^2 \gg \delta^2/a^2$, and the cube wave functions are effectively zero momentum.

The momentum smearing parameter δ^2 will be estimated from our data. It will not be determined very precisely, but it will turn out that this does not matter. Analytic estimates fall into the same ball-park.

The measurements taken will be entirely on an 8^4 lattice. At various stages we shall compare and combine these measurements with those taken previously [1] on $4^3 \cdot 8$ and 6^4 lattices. We have taken measurements at four values of β, i.e. $\beta = 2.2$, 2.3, 2.4 and 2.5. We shall disregard the $\beta = 2.5$ data for now because, as we shall see, our techniques are no longer reliable at such large β. Our old data [1] was taken at $\beta = 2.3$.

Glueball Masses on the 8^4 Lattice

We begin with a calculation of the 0^+ and 2^+ glueball masses using solely our 8^4 data. The calculation will be in the same spirit as our previous work. That is to say, we assume that if (1) is satisfied we are indeed in the continuum limit. We see from Fig. 1 that (1) is indeed satisfied for $\beta = 2.2$, 2.3 and 2.4 on an 8^4 lattice.

We average Γ_{2a}/Γ_a as obtained for our two types of wave function and present the results in Table 1. We apply (2) and express the lattice spacing $a(\beta)$ in terms of $a(\beta = 2.3)$ which we simply call a from now

Table 1. Measured values of Γ_{2a}/Γ_a for momentum smeared wave functions on the 8^4 lattice

β	Γ_{2a}/Γ_a 0^+	2^+
2.2	0.152 ± 0.015	0.054 ± 0.024
2.3	0.183 ± 0.009	0.048 ± 0.014
2.4	0.230 ± 0.025	0.123 ± 0.031

on. Using (2) at our three values of β we obtain for the 0^+ glueball

$$1.66m^2a^2+\delta^2=3.55\begin{array}{l}+0.40\\-0.35\end{array},$$

$$m^2a^2+\delta^2=2.88\begin{array}{l}+0.18\\-0.16\end{array}, \qquad (13)$$

$$0.605\,m^2a^2+\delta^2=2.16\begin{array}{l}+0.35\\-0.29\end{array}.$$

We also calculate Γ_a/Γ_0 for $\mathbf{p}=0$ wave functions at $\beta=2.3$, and this gives us a lower bound for δ^2:

$$(m^2a^2+\delta^2)^{1/2}-ma\geqq\ln\left\{\frac{\Gamma_a/\Gamma_0|_{\mathbf{p}=0}}{\Gamma_a/\Gamma_0|_{\text{cube}}}\right\}=0.44. \qquad (14)$$

Taking (13) and (14) together we obtain

$$\delta^2=1.5\begin{array}{l}+0.5\\-0.3\end{array} \qquad (15)$$

and for the 0^+ glueball mass

$$ma=1.15\begin{array}{l}+0.11\\-0.22\end{array}. \qquad (16)$$

Our previous result [1] on smaller lattices was

$$ma=1.24\begin{array}{l}+0.11\\-0.10\end{array}. \qquad (17)$$

The results agree within errors.

We now repeat the above procedure for the 2^+ glueball. Analogously to (13) we have

$$1.66\,m^2a^2+\delta^2=8.55\begin{array}{l}+3.10\\-2.00\end{array},$$

$$m^2a^2+\delta^2=9.18\begin{array}{l}+2.21\\-1.51\end{array}, \qquad (18)$$

$$0.605\,m^2a^2+\delta^2=4.39\begin{array}{l}+1.30\\-0.89\end{array}.$$

Using the value of δ^2 in (15) (although it is clear that m is insensitive to δ^2) we obtain for the 2^+ glueball mass

$$ma=2.46\begin{array}{l}+0.07\\-0.21\end{array} \qquad (19)$$

(the upper error is derived from the upper bound provided by Γ_a/Γ_0 at $\beta=2.2$; see Table 2). Our previous result [1] on smaller lattices was

$$ma=2.21\begin{array}{l}+0.60\\-0.38\end{array}. \qquad (20)$$

The results again agree within errors.

In Fig. 5 we plot our new and old mass estimates. The vertical scale is in GeV in the same sense that the scale in Fig. 1 was in fermi; more significant are the mass ratios. It is interesting to note that the ratio of 0^+ to 2^+ masses as calculated here is very close to the same ratio as calculated in the $SU(3)$ theory [3].

The current calculation involved measurements on about 9000 lattice configurations in contrast to the ~30000 configurations in our previous calculation [1]. The relative error bars indicate the efficiency of using momentum smeared wave functions, particularly so for higher mass glueballs where the momentum smearing is less important.

Table 2. Measured values of Γ_a/Γ_0 for momentum smeared wave functions at $\beta=2.2, 2.3, 2.4$ and $\mathbf{p}=0$ wave functions at $\beta=2.3$ (as indicated) on the 8^4 lattice

β	Γ_a/Γ_0 0^+ (1×1)	0^+ (2×2)	2^+ (1×1)	2^+ (2×2)
2.2	0.0857 $\pm$0.0014	0.0604 $\pm$0.0021	0.0328 $\pm$0.0009	0.0288 $\pm$0.0014
2.3 (cube)	0.0821 $\pm$0.0009	0.0732 $\pm$0.0011	0.0301 $\pm$0.0006	0.0342 $\pm$0.0007
2.3 ($\mathbf{p}=0$)	0.105 $\pm$0.007	0.122 $\pm$0.007	0.028 $\pm$0.005	0.046 $\pm$0.005
2.4	0.0662 $\pm$0.0023	0.0662 $\pm$0.0023	0.0286 $\pm$0.0015	0.0399 $\pm$0.0015

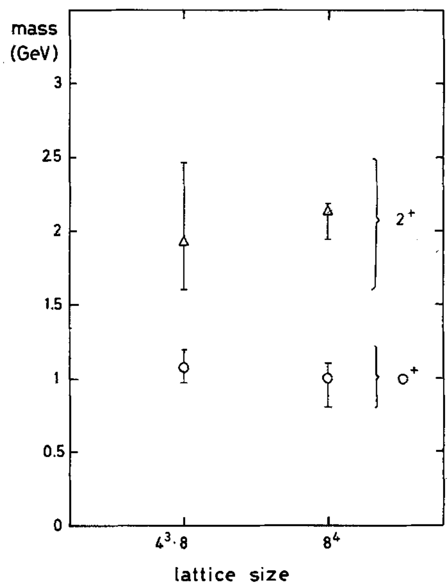

Fig. 5. 0^+ and 2^+ glueball masses as measured on the 8^4 lattice and on the $4^3\cdot8$ lattice [1]

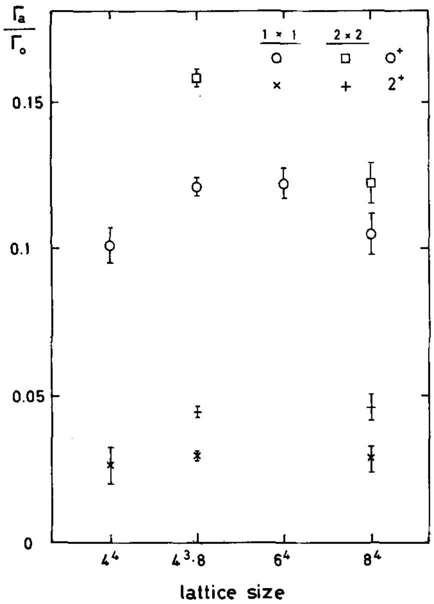

Fig. 6. Γ_a/Γ_0 as a function of lattice size for a variety of zero momentum wave functions

Finite Size Effects

In our calculation we also have measured Γ_a/Γ_0 for $\mathbf{p}=0$ wave functions at $\beta=2.3$. Together with our previous calculations we thus have $\Gamma_a/\Gamma_0|_{\mathbf{p}=0}$ for 1×1 plaquettes on 4^4, $4^3\cdot 8$, 6^4 and 8^4 lattices, and for 2×2 plaquettes we have data on $4^3\cdot 8$ and 8^4 lattices. It is interesting to search for finite size effects by comparing our various results. In Fig. 6 we plot $\Gamma_a/\Gamma_0|_{\mathbf{p}=0}$ separately for 1×1 and 2×2 plaquettes, and for 0^+ and 2^+ wave functions. We observe that the only appreciable variation with lattice size occurs for the 0^+ wave function consisting of 2×2 plaquettes, and even here the change in going from $4^3\cdot 8$ to an 8^4 lattice is only about 20%. In view of our results of the previous section, this is presumably mainly due to changes in wave function overlaps rather than in masses. In any case this comparison again confirms that a $4^3\cdot 8$ lattice at $\beta=2.3$ is adequately large for a mass gap calculation at the current level of statistical accuracy.

Renormalisation Group Behaviour

Up to now we have assumed that for $\beta \geqq 2.2$ we are indeed in the continuum limit, so that the glueball masses are independent of β. We shall now analyse our data to see to what extent we can substantiate this assumption.

We shall calculate the 0^+ and 2^+ glueball masses as functions of β and verify to what extent we have the desired renormalisation group relation

$$m(\beta)=\text{constant},\quad \beta\geqq 2.2, \tag{21}$$

where we relate β and the lattice spacing, a, by (2) (the validity of (2) for $\beta\geqq 2.2$ follows from Wilson loop [5] and finite size scaling [8] studies). We shall combine our present data with our previously obtained data [1]. As we have seen at the level of mass estimates these measurements are consistent with each other.

We estimate $m(\beta)$ from the relation

$$[m(\beta)a(\beta)]^2+\delta^2=\left(\ln\frac{\Gamma_{2a}}{\Gamma_a}\right)^2. \tag{22}$$

To begin with we must calculate δ^2. Working at $\beta=2.3$ (and dropping the argument β where its value is clear) we have from our old calculation [1] of the 0^+ mass

$$(ma)^2=1.53\begin{array}{l}+0.28\\-0.23\end{array}, \tag{23}$$

while our present momentum smeared measurements at $\beta=2.3$ give us

$$(ma)^2+\delta^2=2.88\begin{array}{l}+0.18\\-0.16\end{array}. \tag{24}$$

From (23) and (24) we find

$$\delta^2=1.35\begin{array}{l}+0.29\\-0.32\end{array}. \tag{25}$$

Using Γ_{2a}/Γ_a as in Table 1 and this value of δ^2 enables us to calculate $m(\beta)$ for 0^+ and 2^+ glueballs. We find that the β dependence is comparable for all δ^2 allowed by (25), and so we show in Fig. 7 $m(\beta)$ only for the most probable value, $\delta^2=1.35$ (varying δ^2 shifts the overall normalisation slightly).

Within statistical errors we see that we have the desired renormalisation group behaviour for both 0^+ and 2^+ glueball masses. For the 0^+ any β variation is certainly small compared to the variation in the lattice spacing as β changes from 2.2 to 2.4 (a factor of 3/5). For the 2^+ the errors are larger, but a change

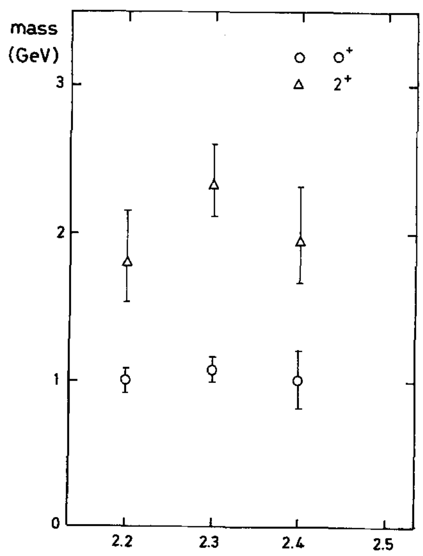

Fig. 7. 0^+ and 2^+ glueball masses, extracted separately at each β, expressed in physical units

of more than 40% looks ruled out. We especially note that there is no evidence for anomalously low glueball masses at $\beta=2.2$. This possibility was a serious one in view of the specific heat peak [8,9] located near $\beta=2.2$.

Wave Functions and Mass Corrections

Our calculations so far have depended on the assumption that, for the wave function we have used, both Γ_{2a} and Γ_a are dominated by the lowest glueball contribution (when we expand as in (3)):

$$\frac{\Gamma_a}{\Gamma_0}=\alpha(\beta)e^{-(m^2a^2(\beta)+\delta^2)^{1/2}}, \tag{26}$$

$$\frac{\Gamma_{2a}}{\Gamma_0}=\alpha(\beta)e^{-2(m^2a^2(\beta)+\delta^2)^{1/2}}, \tag{27}$$

so that Γ_{2a}/Γ_a gives us m as in (12). In the same spirit, having determined m and δ^2, we can use our data on Γ_a/Γ_0 to determine the wave function overlaps, $\alpha(\beta)$. This will enable us to make some estimates of the errors in the approximation (26) and (27) and to indicate how this will affect our previous mass estimates.

For the masses and δ^2 we use our previous results in (15), (16) and (19). Using the Γ_a/Γ_0 data in Table 2, we then obtain four $\alpha(\beta)$; two each for 0^+ and 2^+ corresponding to the two basic operators in Fig. 2. These are plotted in Fig. 8. Errors are shown only for the points at $\beta=2.2$. The errors are comparable at other values of β and are mainly systematic in the sense of being strongly correlated amongst differing β. The $\alpha(\beta)$ decrease rapidly with increasing β as expected: for increasing β the glueball becomes rapidly larger in units of the lattice spacing, and the simplest and smallest loops will rapidly come to have little overlap with the true glueball wave function. Already at $\beta=2.4$ we are perhaps optimistic in regarding our wave function as "reasonably good".

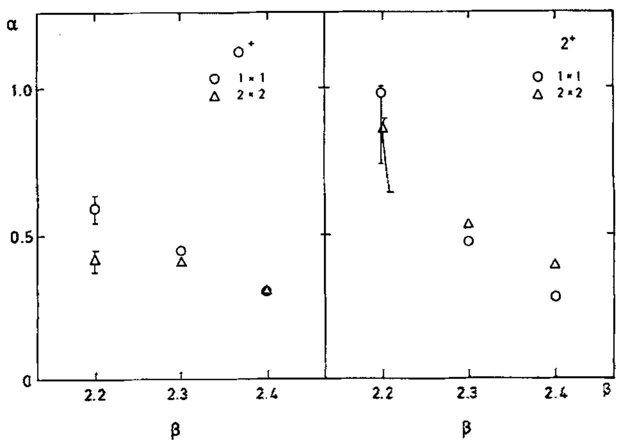

Fig. 8. The wave function overlaps, $\alpha(\beta)$, extracted as functions of β for our various momentum smeared wave functions

A similar analysis for the 0^+ glueball of our $\mathbf{p}=0$ data at $\beta=2.3$ on the $4^3\cdot 8$ lattice yields $\alpha\approx 0.41$ for the 1×1 plaquette and $\alpha\approx 0.55$ for the 2×2 plaquette.

We now wish to estimate the sizes of the terms we have dropped in (26) and (27) and hence to calculate the corresponding corrections in our previous mass estimates. We begin with the example of the 2×2 plaquette, $\mathbf{p}=0$ wave function for the 0^+ glueball that was taken at $\beta=2.3$ on a $4^3\cdot 8$ lattice in our previous work [1]. We parametrise the corrections to the $\mathbf{p}=0$ versions of (26) and (27) as

$$\alpha e^{-2ma}=\frac{\Gamma_{2a}}{\Gamma_0}=0.046, \tag{28}$$

$$\alpha e^{-ma}+(1-\alpha)e^{-cma}=\frac{\Gamma_a}{\Gamma_0}=0.16. \tag{29}$$

For the moment we neglect the much smaller corrections in (28). We have represented the correction as being located at one average mass $cm>m$. We do not know what c is, but it is clear that c should increase with β (decreasing a) and that the correction increases the smaller is c. Since the 0^+ glueball has a next higher spin admixture of 4^+, a plausibly conservative choice for c might be $c=3$. With this choice of c we can solve (28) and (29) and we find

$$\alpha\approx 0.42,\ ma\approx 1.11 \tag{30}$$

in contrast to our previous results, without the corrections, of

$$\alpha\approx 0.55,\ ma=1.24\begin{array}{l}+0.11\\-0.10\end{array}. \tag{31}$$

The mass is decreased (as it must be) but only by about 10%, which is almost covered by the statistical errors. The correction term neglected in (28) is only a negligible 2% or so.

We now repeat these calculations for our results on the 8^4 lattice. At $\beta=2.2$ using $c=3$ leads to negligible corrections. At $\beta=2.3$ the mass is reduced by about 5% which is within statistical errors. At $\beta=2.4$ the solution becomes sensitive to the choice of c. However, if c has increased to $c\approx 5$ we again find very small corrections. In every case the corrections to Γ_{2a}/Γ_a are negligible. For $\beta=2.5$ the α is now so small that we cannot claim Γ_{2a}/Γ_a to give anything more than a mass upper bound. Moreover, the mass is now much smaller than the momentum smearing, so that for our momentum smeared wave functions the values of Γ_a and Γ_{2a} are little affected by the actual masses. For these reasons we do not use our $\beta=2.5$ data in this paper.

We now turn to the 2^+ glueball mass. If we use (19) to estimate $\Gamma_{2a}/\Gamma_a|_{\mathbf{p}=0}$ for our 8^4 lattice, plus $\Gamma_a/\Gamma_0|_{\mathbf{p}=0}\approx 0.46$ as measured for the 2×2 plaquette wave function and choose $c\approx 2$, we find $\alpha\approx 0.42$ and $ma\approx 2.35$, that is a reduction in the mass that is well within the statistical errors of our "naive" mass estimate in (19).

Conclusions

The results of this paper provide evidence that in the region of coupling $2.2 \leqq \beta \leqq 2.4$ the 0^+ and 2^+ glueball masses possess the asymptotic freedom renormalisation group behaviour characteristic of the continuum theory. This supports our expectation that the glueball masses we have previously calculated [1] at $\beta = 2.3$ are indeed good approximations to the masses of the continuum theory.

To be able to vary β over a non-trivial range, we were forced to work on a large 8^4 lattice. To reduce the computing time on such a large lattice to a reasonable level, we modified the straightforward approach of using zero momentum wave functions. Our technique of using momentum smeared wave functions is particularly powerful for higher mass states, and has allowed us to obtain a much tighter value for the 2^+ glueball mass than previously.

Comparing our previous results [1] with those obtained on the 8^4 lattice confirms that a $4^3 \cdot 8$ lattice is already large enough for quite accurate mass gap calculations.

We were also able to estimate how good our wave functions were, and it is apparent that calculations for $\beta \gtrsim 2.5$ will probably require harder work on the wave functions in order to be useful. We estimated the corrections to our mass estimates due to our approximations and find that these will typically be 0(5%) and hence will be covered by our statistical errors. However, the lesson is that any further reduction of statistical errors needs to be also accompanied by improvements in technique (in obvious ways) if it is to be useful.

We finally point out that if we take all our $SU(2)$ data together we find a mass ratio

$$SU(2): \frac{m(2^+)}{m(0^+)} = 2.05 \begin{array}{l} +0.18 \\ -0.22 \end{array} \tag{32}$$

that is very similar to the corresponding ratio in the $SU(3)$ theory [3]:

$$SU(3): \frac{m(2^+)}{m(0^+)} = 2.25 \pm 0.33. \tag{33}$$

This supports speculations that even for 2 or 3 colours the theory is already in some sense in the large number of colour limit.

Note Added

After completing this work we received a paper by Berg, Billoire and Rebbi [10] in which the $SU(2)$ 0^+ glueball mass is estimated by a pure variational calculation on a (smaller) $4^3 \cdot 16$ lattice at various values of β. There is some evidence for the desired renormalisation group behaviour, but only for $\beta \lesssim 2.25$. This is presumably due to the small spatial extent of their lattice as can be inferred from Fig. 1. We show the renormalisation group dependence of their 0^+ glueball

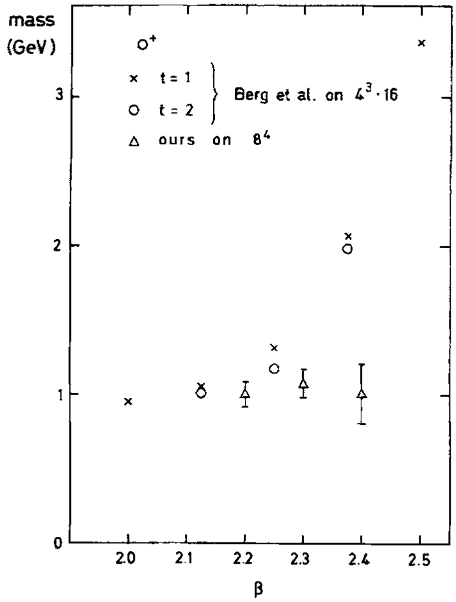

Fig. 9. Comparison of the renormalisation group dependence of the 0^+ glueball mass of [10] to ours

mass (final maximised best values) and compare it to ours in Fig. 9. Their errors are not shown but for $t = 1$ are presumably small and do not overlap. The estimate of the 0^+ glueball mass of these authors is similar to ours. The recent paper by Mütter and Schilling [11] also addresses some of these questions but from a different approach.

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Appendix: Error Analysis

Consider a wave function $\phi(t)$ with $\langle \phi \rangle = 0$. We wish to estimate the correlation functions

$$\Gamma_t = \langle \phi(t) \phi(0) \rangle \tag{A.1}$$

for $t = 0$, a, $2a$ from n independent measurements of the product $\phi(t)\phi(0)$ from n different field configurations

$$\Gamma_t^{\exp} = \frac{1}{n} \sum_{i=1}^{n} \phi_i(t) \phi_i(0). \tag{A.2}$$

What is the error in such an estimate, $\sigma(\Gamma_t)$?

Because Γ_0 is a large signal, the error on $\Gamma_0^{\exp}$ is negligible, so we may assume

$$\Gamma_0^{\exp} = \Gamma_0. \tag{A.3}$$

Furthermore, as long as $\Gamma_a, \Gamma_{2a} \ll \Gamma_0$, which is always the case in our work, we may assume to a good approximation (for the error analysis) that $\phi(a)$ and $\phi(2a)$ are independent of $\phi(0)$. Then

$$\sigma(\langle\phi(a)\phi(0)\rangle) = \frac{1}{\sqrt{n}}\langle(yy')^2\rangle^{1/2}, \tag{A.4}$$

where y and y' are independent random variables with the distribution of $\phi(0)$. One may easily show that in this case

$$\langle(yy')^2\rangle = \langle y^2\rangle^2 \equiv \Gamma_0^2. \tag{A.5}$$

The same argument holds for $\sigma(\langle\phi(2a)\phi(0)\rangle)$ and so we have

$$\left(\frac{\Gamma_a}{\Gamma_0}\right)^{\exp} = \frac{\Gamma_a}{\Gamma_0} \pm \frac{1}{\sqrt{n}}, \tag{A.6}$$

$$\left(\frac{\Gamma_{2a}}{\Gamma_a}\right)^{\exp} = \frac{\Gamma_{2a}}{\Gamma_a} + \frac{(\Gamma_a/\Gamma_0)^{-1}}{\sqrt{n}}. \tag{A.7}$$

In (A.7) we neglect the error from $\Gamma_a^{\exp}$ as compared to that on $\Gamma_{2a}^{\exp}$.

Having obtained (A.6) and (A.7) we must determine n. In practice we obtain on each configuration of our $L_s^3 L_t$ size lattice m separate measurements at each time. If we have N_{IT} such configurations we may write

$$n = mL_t N_{IT}/(\zeta_m \zeta_t \zeta_{IT}), \tag{A.8}$$

where ζ_m is a measure of the correlation amongst the m wave functions at a given time, ζ_t a measure of the correlation between neighbouring times and ζ_{IT} between neighbouring configurations.

For $\mathbf{p}=0$ wave functions $m=1$ for 0^+ and $m=3$ for 2^+. For the cube wave functions on the 8^4 lattice m is increased by a factor of $3^3=27$. On the basis of our measurements of ma we expect $\zeta_t \approx 1$. Our cubed wave functions are separated enough that we expect $\zeta_m \approx 1$. An 8^4 lattice is large enough that subsequent sweeps should not be highly correlated, so that $\zeta_{IT} = 0(1)$.

An analysis of our experimentally observed errors shows that a choice of

$$n(0^+) = mL_t N_{IT}, \tag{A.9}$$

$$n(2^+) = mL_t N_{IT}\left(\frac{2.3}{2}\right) \tag{A.10}$$

is appropriate. Our errors in Table 1 and Table 2 are then calculated from (A.6), (A.7), (A.9) and (A.10), after first having checked that the measured errors are distributed around these theoretically determined values.

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