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CONTINUUM LIMIT IMPROVED LATTICE ACTION FOR PURE YANG–MILLS THEORY (I)*

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Symanzik's programme for constructing a lattice action with improved continuum limit behaviour is considered for the case of pure Yang–Mills theory. The structure of the action is proposed and discussed in detail to lowest order in perturbation theory.

1. Introduction

Despite the fact that the phenomenon of quark confinement and the existence of a mass gap in the theory of QCD have not yet been properly explained, it is widely accepted that QCD is a good candidate for the (or at least an effective) theory of strong interactions. As recognized for some time, there is qualitative and quantitative agreement with experiment in the asymptotic domains for a wide variety of processes where conventional perturbation theory is applicable [1]. Recently numerical studies, in the form of Monte Carlo (MC) calculations, yield similar agreement for non-perturbative quantities such as mass spectra [2, 3].

However, just as the perturbation theory applications have their unsatisfactory aspects [4], there are many questions concerning the MC results. The MC experiments are performed in the approximation where space–time is replaced by a discrete lattice with spacing a. The only other parameter (in the absence of bare quark masses) is the bare coupling constant g. Renormalization group arguments conclude that in four dimensions there is a function $\Lambda_{\rm L}$

$$\Lambda_{\rm L} = a^{-1}\, {\rm e}^{-1/2\beta_0 g^2}(\beta_0 g^2)^{-\beta_1/2\beta_0^2}(1+{\rm O}(g^2)) \qquad (1.1)$$

(where β_0, β_1 are the universal first two coefficients of the Callan–Symanzik β-function, $\beta_0 > 0$) such that physical masses for small g are of the form

$$m_i = k_i \Lambda_{\rm L}[1+{\rm O}(a^2\Lambda_{\rm L}^2 \ln a\Lambda_{\rm L})]\,, \qquad (1.2)$$

where the constants k_i and the ${\rm O}(a^2\Lambda_{\rm L}^2 \ln a\Lambda_{\rm L})$ part in (1.2) depend on the specific

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quantity under consideration. Stated in another way, the physical continuum theory is attained in the limit

$$a \to 0 , \qquad g \to 0 , \qquad \Lambda_{\rm L} \text{ fixed} ,$$

such that ratios of masses are constant up to exponentially small corrections

$$\frac{m_1}{m_2} = \frac{k_1}{k_2}[1 + {\rm O}(a^2 \Lambda_{\rm L}^2 \ln a\Lambda_{\rm L})] . \tag{1.3}$$

There are two types of problem which have to be considered, (i) finite size effects, and (ii) finite bare coupling effects. The two types of effects can in practice often not be clearly separated, although it is generally believed that finite size effects are under control for certain quantities not significantly probing the topological structure of the theory.

The finite bare coupling effects are, however, potentially disturbing, since in practice attempts are made to extract continuum results from domains where [although effects (i) are estimated small] the bare coupling constant is still of order unity. Indeed initially it was rather surprising that Creutz [2] observed the asymptotic freedom prediction (1.2) setting in at such large g^2. Having accepted this fact, it is, however, still rather optimistic to claim extraction of the constants k_i in (1.2) to within an accuracy of better than 100% without knowledge of the $O(g^2)$ behaviour in (1.1). In fact Bhanot and Dashen [5] observed large discrepancies for Λ/m ratios extracted using the same procedure for different lattice actions, and the origin for this is probably accounted for by finite coupling corrections [6]. Ratios of masses presumably fare better with respect to (ii) and it would be desirable to have available more detailed data on the bare coupling dependence of mass ratios. Certain ratios may be smooth even through the "cross-over region" and explain the surprising approximate validity of strong coupling calculations [7].

Assuming that QCD is a well-defined theory (in the sense that different lattice regularizations have the same continuum limit), there are two approaches to the question of whether the theory is really physically relevant. The first is to accumulate a sufficient amount of "circumstantial evidence" and to establish the absence of contradictions. The second is to try to investigate whether theoretically motivated improvements also actually lead to improvement between theory and experiment.

In the latter spirit, a suggestion of Symanzik [8] is to attempt to systematically construct the lattice action so that the cutoff dependence is reduced and the "continuum limit more rapidly approached". For example as a first step, one would like to reduce the $O(a^2)$ dependence on the r.h.s. of (1.3) to an $O(a^4)$ dependence. It is to be stressed that this improvement is one primarily associated with only the continuum limit behaviour and is not done with a view to eliminating other lattice artifacts such as monopoles in intermediate coupling domains. Actually what is accomplished is slightly more modest since the $O(a^2)$ in (1.3) have a variety of origins, e.g. (1) irrelevant terms in the effective continuum action (LEL) [9], (2) non-perturbative. Only the former effects are treated. With an improved action motivated in this way, computer experiments should be repeated and the order of magnitude of changes in physical predictions studied. If improved smoothness for mass ratios, for example, is noted, one could be satisfied; drastic irregularities, on the other hand, would be a cause for concern and point to a need for a revaluation.

The improvement programme was first discussed thoroughly in the framework of ϕ_4^4 theory by Symanzik [8], and recently studied in the non-linear sigma model in two dimensions by Symanzik [10] and by Martinelli , Parisi and Petronzio [11]. This paper deals with the application of the programme to pure Yang–Mills theory. The programme can be extended to full QCD; however, a proper inclusion of fermions on the lattice has additional problems associated with chiral symmetry, which should be satisfactorily tackled first [12].

The plan of the paper is as follows. In the subsequent section the ideas are outlined in more detail and the proposed structure of the improved action motivated (2.12). It includes, in addition to the usual one-plaquette terms, terms involving paths of length $6a$. The resulting action is similar in structure to that considered by Wilson [13] in his real space renormalization studies. Sect. 3 discusses the determination of the relative strengths of the contributions in the action (2.12), to lowest order in perturbation theory. Sect. 4 records the consequence of improvement for the static potential. The paper concludes with a short discussion of the practical applicability. The constraints obtained from considerations of next order in perturbation theory is the topic of a subsequent paper.

2. Structure of the improved action

We consider pure SU(N) gauge theory on an infinite hypercubical lattice with spacing a in d euclidean dimensions*. The dynamical variables are specified as elements $U_\mu(x)$ of SU(N) associated with directed links from x to $x+\hat{\mu}$, where $\hat{\mu}$ is a vector in the direction $\mu = 1, \dots, d$ with length a.

The choice of the lattice action is highly arbitrary. The only *a priori* restriction is that in the limit $a \to 0$ the lattice action S_L tends to the classical expression S_cl

$$S_\text{L} \xrightarrow[a\to 0]{} S_\text{cl} = -\sum_{i=1}^{N^2-1} \int \mathrm{d}^d x \tfrac{1}{4} F^{i2}_{\mu\nu} . \tag{2.1}$$

The ultimate aim of Symanzik's improvement is to systematically construct an action so that physical quantities, i.e. gauge-invariant quantities having a finite limit without multiplicative renormalization, have weaker cutoff dependence. This is rather difficult to establish for non-perturbative quantities and the best one can do at present (analytically) is to improve physical quantities which are non-trivial in perturbation theory (or perhaps in a $1/N$ expansion).

* Many considerations will be specifically relevant only for the case $d=4$; however, various computations will be made keeping d arbitrary.

Consider, for example, the Wilson loop expectation in $d=4$ Yang–Mills theory with an ultraviolet regularization (cutoff a^{-1}) respecting gauge invariance,

$$W_{\mathrm{R}}(\mathscr{C}, g, a)=\frac{1}{d_{\mathrm{R}}}\left\langle \operatorname{tr} \mathrm{P} \exp\left(ig\int_{\mathscr{C}} \mathrm{d}x_\mu A^i_\mu(x)\mathrm{R}^i\right)\right\rangle, \tag{2.2}$$

where $\mathscr{C}$ is some closed curve, R some irreducible representation of SU(N) of dimension d_{R} and R^i the corresponding representation of infinitesimal generators; and g the bare coupling. Then the important result [14] is that there exists, to all orders of perturbation theory, a renormalization constant $Z_{\mathrm{R}}(\mathscr{C}, g, a, M)$ depending on $\mathscr{C}$ only through the perimeter $P(\mathscr{C})$ of $\mathscr{C}$ and the number and angles of kinks and self-intersections such that the limit

$$\lim_{\substack{a\to 0\\ g(M)\ \text{fixed}}} [Z_{\mathrm{R}}(\mathscr{C}, g, a, M)W_{\mathrm{R}}(\mathscr{C}, g, a)]$$

exists and is non-trivial, where $g(M)$ (a function of g and Ma) is a suitably defined renormalized coupling. In particular, for curves with no kinks or self-intersections

$$Z_{\mathrm{R}}=\mathrm{e}^{-P(\mathscr{C})a^{-1}F_{\mathrm{R}}(g, Ma)}. \tag{2.3}$$

Many physical quantities in the above sense can now be identified. These include derivatives of the static potential $V_{\mathrm{R}}(L)$ defined by

$$V_{\mathrm{R}}(L)=-\lim_{T\to\infty}\frac{1}{T}\ln W_{\mathrm{R}}(\mathscr{C}_{LT}), \tag{2.4}$$

where $\mathscr{C}_{LT}$ is a rectangular $L\times T$ loop. The static potential has a small a, g expansion

$$V_{\mathrm{R}}(L)=a^{-1}f_{\mathrm{R}}(g)-\frac{g^2}{4\pi L}\sum_{j,k=0}^{\infty}\left(\frac{a}{L}\right)^{2j}\left(\ln\frac{a}{L}\right)^k f_{jk,\mathrm{R}}(g). \tag{2.5}$$

The terms in (2.5) with $j=0$ are treated by renormalization of the coupling constant. It is the object of the first step of the improvement programme to remove all terms with $j=1$.

For the ϕ^4_4 theory [8] the steps are as follows. (1) Start with a lattice action with nearest neighbour couplings. (2) Calculate the small a dependence of Green functions in perturbation theory. This can be summarized concisely by a local effective lagrangian (LEL) with specified calculational rules. The existence of such a LEL is non-trivial [9]. (3) Add terms to the original action so that the small a dependence is improved, in the sense explained, order by order in perturbation theory. This amounts here to a (in perturbation theory, up to finite renormalizations, unique) choice of dimension 6 irrelevant terms in the lattice action.

It is probable that a programme, analogous to the one outlined above, (which does work for the ϕ^4_4 theory [8] and $d=2$ non-linear σ-model) can also be applied to Yang–Mills theory. The starting point (1) is an action involving just a sum over plaquettes p,

$$S_{\text{start}} = -\frac{1}{g^2} \sum_{\mathrm{R}} \sum_{\mathrm{p}} k_{\mathrm{R}} \,\mathrm{tr}\,(1 - U_{\mathrm{R}}(\mathrm{p}))\,, \tag{2.6}$$

of the form originally proposed by Wilson. One could then proceed with step 2, restricting attention to gauge-invariant quantities, and one expects the small a dependence to be summarized by a gauge-invariant LEL. To incorporate the correct small a behaviour to order a^2, we require for $d=4$, a list of all gauge-invariant operators of dimension 6, invariant under parity and $\frac{1}{2}\pi$ rotations. We find that there are only 3 such independent operators, and a particular basis is given by

$$\begin{aligned} S_1 &= \tfrac{1}{2} \sum_{\mu,\nu,\rho} \mathrm{tr}\, J_{\mu\nu\rho} J_{\mu\nu\rho}\,, \\ S_2 &= \sum_{\mu,\nu,\rho} \mathrm{tr}\, J_{\mu\mu\rho} J_{\nu\nu\rho}\,, \\ S_3 &= \sum_{\mu,\nu} \mathrm{tr}\, J_{\mu\mu\nu} J_{\mu\mu\nu}\,, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} J_{\mu\nu\rho} &= [D_\mu, F_{\nu\rho}]\,, \\ F_{\mu\nu} &= \frac{1}{g}[D_\mu, D_\nu]\,, \end{aligned} \tag{2.8}$$

and D_μ is a covariant derivative

$$D_\mu = \partial_\mu + gA_\mu\,. \tag{2.9}$$

Any other gauge-invariant operator of dimension 6 can be written as a linear combination of these plus a total derivative, e.g.

$$\sum_{\mu,\nu,\rho} g\, \mathrm{tr}\, F_{\mu\nu} F_{\nu\rho} F_{\rho\mu} = \tfrac{1}{2}(S_2 - S_1) + \text{total derivative}\,. \tag{2.10}$$

The LEL then assumes the form

$$\mathscr{L}_{\text{eff}} = Z_0(g^2)\, \mathrm{tr} \sum_{\mu,\nu} F_{\mu\nu}^2 + a^2 \sum_{i=1}^{3} Z_i(g^2) S_i\,. \tag{2.11}$$

The small a dependence cannot be systematically improved merely by adjusting the coefficients k_{R} in (2.6). Analogously to the ϕ_4^4 case, where next nearest neighbours are also added in step 3, one must in this case add to the action terms involving longer paths. To the order in a we are considering we propose that it suffices to take into account (in a specified way) only paths of length $6a$. The

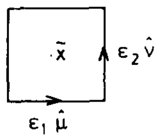

Fig. 1. Curve $C_{\tilde{x},\epsilon,\hat{\mu},\epsilon_2\hat{\nu}}$ enclosing a single plaquette with centre $\tilde{x}$ and lying in the μ, ν plane.

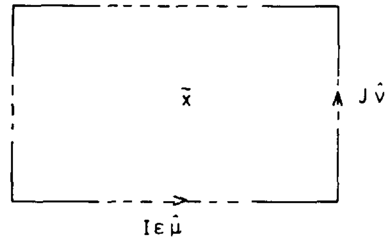

Fig. 2. Rectangular $Ia \times Ja$ loop in the μ, ν plane.

proposed improved lattice action then takes the form

$$S = -\frac{a^{d-4}}{g^2} \sum_{i=0}^{3} \sum_{\mathscr{C} \in \mathrm{T}_i} \sum_{R} c_{\mathrm{R}i}(g^2)\, \mathrm{tr}\,(1 - U_{\mathrm{R}}(\mathscr{C}))\,, \tag{2.12}$$

where $\mathscr{C}$ are oriented paths belonging to sets T_i, $i = 1, 2, 3$ of structurally equivalent curves,

T_0 = set of curves enclosing one plaquette (fig. 1);

T_1 = set of planar curves with perimeter 6a enclosing two plaquettes (fig. 2 with $I + J = 3$);

T_2, T_3 = set of non-planar curves depicted in figs. 3, 4 respectively.

Note that the number of different classes matches the number of independent operators in the LEL (2.11) as expected. The sum over all irreducible representations R is included for possible later applications.

Already in lowest order perturbation theory one finds contributions in (2.5) with $j = 1$, unless the coefficients in (2.12) are suitably adjusted. For the ϕ_4^4 theory the elimination of the lowest order $j = 1$ contributions is a purely kinematical problem, and involves [8] replacing the difference operator by an amputated SLAC derivative [15]. In gauge theory the procedure is analogous although in this case, of course, gauge invariance enters as an additional constraint. Introducing the gauge potential $A_\mu(x)$ through

$$U_{\mu \mathrm{R}}(x) = \mathrm{e}^{iagA_\mu^i R^i} \tag{2.13}$$

one develops Feynman perturbation theory. For this one requires a suitable gauge

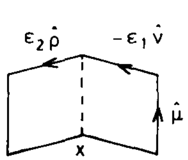

Fig. 3. Non-planar "L-shaped" curve with perimeter $6a$.

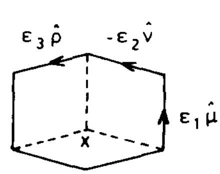

Fig. 4. Non-planar parallelogram with perimeter $6a$.

fixing, and a covariant α-gauge will be employed in our calculations. To lowest order, with which we are concerned in this paper, full details of the gauge fixing are not necessary, since only the transverse part is required.

At first sight one might think that improvement at lowest order means chosing the coefficients in (2.12) such that the inverse A-propagator, with A_μ defined via (2.13), takes the form (before gauge fixing)

$$g_{\mu\nu}k^2 - k_\mu k_\nu + \mathrm{O}(a^4k^6)\,.$$

However this is not quite correct (and incidentally cannot be achieved). Comparison of the continuum and lattice expressions for Wilson loop expectations shows that improvement to lowest order means improvement of the A-propagator multiplied by a definite factor. This is dealt with in the next section.

3. Gaussian terms in the gauge field action

Let $\mathscr{C}$ be an oriented closed curve and $U(\mathscr{C})$ the associated phase factor

$$U(\mathscr{C}) = \prod_{\text{link } \mathscr{L}\in\mathscr{C}} \mathrm{e}^{iagA_{\mathscr{L}}}\,, \qquad A_{\mathscr{L}} = A_{\mathscr{L}}^+\,. \tag{3.1}$$

Then

$$\operatorname{tr}\,(2 - U(\mathscr{C}) - U(\mathscr{C})^+) = a^2g^2 \operatorname{tr} A(\mathscr{C})^2 + \mathrm{O}(A^3)\,, \tag{3.2}$$

where

$$A(\mathscr{C}) = \sum_{\mathscr{L}\in\mathscr{C}} A_{\mathscr{L}}\,. \tag{3.3}$$

Let $\mathscr{L}$ be the directed link from x to $x+\epsilon\hat{\mu}$, $\epsilon = \pm 1$, and introduce the following Fourier transform:

$$A_{\mathscr{L}} = \epsilon \int_k \mathrm{e}^{ikx + i\epsilon a k_\mu/2} \tilde{A}_\mu(k)\,, \tag{3.4}$$

where $\int_k$ denotes $\prod_{\mu=1}^{d} \int_{-\pi/a}^{\pi/a} \mathrm{d}k_\mu/2\pi$.

Evidently any $A(\mathscr{C})$ can be determined from the knowledge of that for a curve $\mathscr{C}_{\tilde{x},\epsilon_1\mu,\epsilon_2\nu}$ enclosing a single plaquette in the μ, ν plane (fig. 1). One has

$$A(C_{\tilde{x},\epsilon_1\mu,\epsilon_2\nu}) = a\epsilon_1\epsilon_2 \int_k \mathrm{e}^{ik\tilde{x}} \tilde{f}_{\mu\nu}(k)\,, \tag{3.5}$$

with

$$\tilde{f}_{\mu\nu}(k) = i(\hat{k}_\mu \tilde{A}_\nu(k) - \hat{k}_\nu \tilde{A}_\mu(k))\,, \tag{3.6}$$

where

$$\hat{k}_\mu = \frac{2}{a} \sin \tfrac{1}{2} k_\mu a\,. \tag{3.7}$$

We require for our considerations the three types of curves in figs. 2–4. For the planar $Ia \times Ja$ loop (fig. 2) one finds

$$A(\text{fig. 2}) = a\epsilon \int_k e^{ik\bar{x}} \frac{\sin \frac{1}{2}Ik_\mu a}{\sin \frac{1}{2}k_\mu a} \cdot \frac{\sin \frac{1}{2}Jk_\nu a}{\sin \frac{1}{2}k_\nu a} \tilde{f}_{\mu\nu}(k)\,, \tag{3.8}$$

and for the non-planar loops of perimeter $6a$ depicted in figs. 3, 4 one finds respectively

$$A\ (\text{fig. 3}) = a \int_k e^{ik(x+\hat{\mu}/2)}(-\epsilon_1\, e^{i\epsilon_1 k_\nu a/2}\tilde{f}_{\mu\nu}(k) + \epsilon_2\, e^{i\epsilon_2 k_\rho a/2}\tilde{f}_{\mu\rho}(k))\,, \tag{3.9}$$

$$A\ (\text{fig. 4}) = a\epsilon_1\epsilon_2\epsilon_3 \int_k e^{ik(x+\epsilon_1\hat{\mu}/2+\epsilon_2\hat{\nu}/2+\epsilon_3\hat{\rho}/2)} \tag{3.10}$$
$$\times (\epsilon_1\, e^{-i\epsilon_1 k_\mu a/2}\tilde{f}_{\rho\nu}(k) + \epsilon_2\, e^{-i\epsilon_2 k_\nu a/2}\tilde{f}_{\mu\rho}(k) + \varepsilon_3\, e^{-\epsilon_3 k_\rho a/2}\tilde{f}_{\nu\mu}(k))\,.$$

For the quadratic contributions to the action one uses (3.8)–(3.10) to obtain

$$\sum_{I,J} C_{IJ} \sum_{\mu>\nu} \sum_x \operatorname{tr} A\ (\text{fig. 2})^2 = a^{2-d} \sum_{I,J} C_{IJ} \sum_{\mu>\nu} \int_k \left(\frac{\sin \frac{1}{2}Ik_\mu a}{\sin \frac{1}{2}k_\mu a}\right)^2$$
$$\times \left(\frac{\sin \frac{1}{2}Jk_\nu a}{\sin \frac{1}{2}k_\nu a}\right)^2 \operatorname{tr} \tilde{f}_{\mu\nu}(k)\tilde{f}_{\mu\nu}(-k)\,, \tag{3.11}$$

and, defining $\hat{k}^2 = \sum_\mu \hat{k}_\mu^2$,

$$\sum_x \sum_{\mu\neq\nu>\rho\neq\mu} \sum_{\epsilon_1,\epsilon_2} \operatorname{tr} A\ (\text{fig. 3})^2 = 4a^{2-d} \sum_{\mu>\nu} \int_k [2(d-2) - \tfrac{1}{4}a^2(\hat{k}^2 - \hat{k}_\mu^2 - \hat{k}_\nu^2)]$$
$$\times \operatorname{tr} \tilde{f}_{\mu\nu}(k)\tilde{f}_{\mu\nu}(-k)\,, \tag{3.12}$$

$$\tfrac{1}{2} \sum_{\mu>\nu>\rho} \sum_{\epsilon_1,\epsilon_2,\epsilon_3} \sum_x \operatorname{tr} A\ (\text{fig. 4})^3 = 4a^{2-d} \sum_{\mu>\nu} \int_k [(d-2) - \tfrac{1}{4}a^2(\hat{k}^2 - \hat{k}_\mu^2 - \hat{k}_\nu^2)]$$
$$\times \operatorname{tr} \tilde{f}_{\mu\nu}(k)\tilde{f}_{\mu\nu}(-k)\,. \tag{3.13}$$

The sums in (3.11)–(3.13), (with $c_{IJ} = 1$), correspond to each unoriented curve taken once.

The quadratic part of the improved action (2.12) then assumes the form

$$S^{(2)} = -\tfrac{1}{2} \sum_{\mu>\nu} \sum_i \int_k [c_0(g^2) + 8c_1(g^2) + 4(d-2)(2c_2(g^2) + c_3(g^2))$$
$$-(c_1(g^2) - c_2(g^2) - c_3(g^2))a^2(\hat{k}_\mu^2 + \hat{k}_\nu^2)$$
$$-(c_2(g^2) + c_3(g^2))a^2\hat{k}^2]\tilde{f}^1_{\mu\nu}(k)\tilde{f}^2_{\mu\nu}(-k)\,, \tag{3.14}$$

where

$$c_i(g^2) = \sum_{\mathrm{R}} 2t_{\mathrm{R}} c_{\mathrm{R}i}(g^2)\,, \tag{3.15}$$

with

$$\delta^{ij}t_{\mathrm{R}} = \mathrm{tr}\, R^i R^j \,. \tag{3.16}$$

As overall normalization we set

$$c_0(g^2) + 8c_1(g^2) + 4(d-2)[2c_2(g^2) + c_3(g^2)] = 1 \,. \tag{3.17}$$

To see precisely what has to be done to achieve improvement we consider the expectation value of the Wilson loop for an arbitrary loop $\mathscr{C}$ on the lattice. To lowest order, using (3.2) and (3.3),

$$\langle \mathrm{tr}\,(1 - U(\mathscr{C}))\rangle \propto g^2 \langle A(\mathscr{C}) A(\mathscr{C})\rangle_0 \,. \tag{3.18}$$

Let $\mathscr{S}$ be a surface spanned by $\mathscr{C}$. Then, as used previously, $A(\mathscr{C})$ is given as a sum of contributions coming from the plaquettes on $\mathscr{S}$, and it follows that

$$\langle \mathrm{tr}\,(1 - U(\mathscr{C}))\rangle = \mathrm{const} \cdot g^2 \sum_{\mathrm{p}\in\mathscr{S}} \sum_{\mathrm{p}'\in\mathscr{S}} \int_k \mathrm{e}^{ik(\tilde{x}_{\mathrm{p}} - \tilde{x}_{\mathrm{p}'})} \times \sum_{\substack{\mu>\nu \\ \rho>\lambda}} D_{\mu\nu,\rho\lambda}(k) \epsilon_{\mathrm{p},\mu\nu} \epsilon_{p',\rho\lambda} \tag{3.19}$$

where

$$\begin{aligned} \epsilon_{\mathrm{p},\mu\nu} &= \pm 1 \,, && \text{if p is in the } \theta, \nu \text{ plane} \,, \\ &= 0 \,, && \text{otherwise} \,. \end{aligned} \tag{3.20}$$

$\tilde{x}_{\mathrm{p}}$ is the midpoint of the plaquette p; and $D_{\mu\nu,\rho\lambda}(k)$ is the free $\tilde{f}$ propagator

$$\langle \tilde{f}^i_{\mu\nu}(k) \tilde{f}^j_{\rho\lambda}(k')\rangle_0 = \delta^{ij}(2\pi)^d \delta(k + k') D_{\mu\nu,\rho\lambda}(k) \,, \tag{3.21}$$

the functional form of which is determined by the quadratic part of the action (3.14), which is considered in detail in the appendix.

For the same curve in the continuum formulation one has

$$\langle \mathrm{tr}\,(1 - U(\mathscr{C}))\rangle_{\mathrm{cont}} = \mathrm{const} \cdot g^2 \sum_{\mathrm{p}\in\mathscr{S}} \sum_{\mathrm{p}'\in\mathscr{S}} \int_{k,r} \mathrm{e}^{ik(\tilde{x}_{\mathrm{p}} - \tilde{x}_{\mathrm{p}'})} \times \sum_{\substack{\mu>\nu \\ \rho>\lambda}} \frac{k_\mu k_\nu k_\rho k_\lambda}{k_\mu k_\nu k_\rho k_\lambda} D^{\mathrm{cont}}_{\mu\nu,\rho\lambda} \epsilon_{p,\mu\nu} \epsilon_{p',\rho\lambda} \,, \tag{3.22}$$

where

$$D^{\mathrm{cont}}_{\mu\nu,\rho\lambda}(k) = k^{-2}(\delta_{\lambda\nu} k_\mu k_\rho - \delta_{\lambda\mu} k_\nu k_\rho + \delta_{\rho\mu} k_\nu k_\lambda - \delta_{\rho\nu} k_\mu k_\lambda) \tag{3.23}$$

and $\int_{k,r}$ denotes some ultravioletly regularized integral. Comparison of the lattice and continuum expressions (3.19) and (3.22), leads to the conclusion that, to lowest order, the small a behaviour is improved by ensuring that*

$$\frac{k_\mu k_\nu k_\rho k_\lambda}{\hat{k}_\mu \hat{k}_\nu \hat{k}_\rho \hat{k}_\lambda} D_{\mu\nu,\rho\lambda}(k) = D^{\text{cont}}_{\mu\nu,\rho\lambda}(k) + \mathrm{O}(a^4) . \tag{3.24}$$

This is achieved by consideration of the inverse propagator, and choosing the coefficients c_i in such a way that the expression in square brackets in (3.14) multiplied by the factor

$$\left(\frac{\hat{k}_\mu \hat{k}_\nu}{k_\mu k_\nu}\right)^2 = 1 - \tfrac{1}{12}a^2(k_\mu^2 + k_\nu^2) + \cdots$$

has no term order a^2 for $g=0$ (consult the appendix for details). This occurs when

$$c_2(0) + c_3(0) = 0 , \tag{3.25}$$

$$c_0(0) + 20c_1(0) + 4c_2(0)(2d-7) + 4c_3(0)(d-5) = 0 . \tag{3.26}$$

Hence, combining (3.17), (3.26) and (3.27) we have

$$c_1(0) = -\tfrac{1}{12} , \tag{3.27}$$

$$c_0(0) - 4(d-2)c_3(0) = \tfrac{5}{3} . \tag{3.28}$$

Note that a further relation is required to fix all the coefficients $c_i(0)$ completely**. In the formulation of the programme described above, we cannot set $c_3(0)=0$ at this stage. To see this, note that the relation between the $Z_i(0)$ occurring in the LEL (2, 11) and the $c_i(0)$ are non-linear. In particular, starting with $c_2(0)=c_3(0)=0$ implies only that $Z_2(0)+Z_3(0)=0$, since S_2 and S_3 are the same (up to total derivatives) to lowest order but differ in order g (2.10).

Finally we remark that no criteria for special choices of the representation coefficients $c_{\mathrm{R}i}(0)$ making up the $c_i(0)$, (3.15), emerge from our weak coupling considerations so far.

4. The Wilson loop in lowest order

Define the coefficients $w_{n\mathrm{R}}(\mathscr{C})$ by (d_R = dim. rep. R)

$$\ln \frac{1}{d_\mathrm{R}} \langle \mathrm{tr}\, U_\mathrm{R}(\mathscr{C}) \rangle = -\sum_{n=1}^{\infty} \frac{(a^{4-d}g^2)^n}{(2n)!} w_{n\mathrm{R}}(\mathscr{C}) . \tag{4.1}$$

Then, in lowest order

$$w_{1\mathrm{R}}(\mathscr{C}) = a^{d-2} \frac{t_\mathrm{R}}{d_\mathrm{R}} \sum_{i=1}^{N^2-1} \langle A^i(\mathscr{C}) A^i(\mathscr{C}) \rangle_0 . \tag{4.2}$$

* The origin factor in (3.24) can be understood as arising from the fact that the lattice link potential, defined in (2.13), is to be set in correspondence with a line integral in the continuum theory.

** The relations (3.27) and (3.28) for $c_3(0)=0$ were known to G. 't Hooft and M. Lüscher. (Private communication from K. Symanzik.)

In particular, for a planar $Ia \times Ja$ loop we have

$$w_{1\mathrm{R}}(I,J) = a^d C_\mathrm{R} \int_k \left(\frac{\sin \frac{1}{2}k_1 Ia}{\sin \frac{1}{2}k_1 a}\right)^2 \left(\frac{\sin \frac{1}{2}k_d Ja}{\sin \frac{1}{2}k_d a}\right)^2 D_{1d,1d}(k)\,, \tag{4.3}$$

where $D_{1d,1d}(k)$ is the transverse free propagator (3.21) and

$$\sum_i \mathrm{tr}\, R^i R^i = C_\mathrm{R} = \frac{(N^2-1)t_\mathrm{R}}{d_\mathrm{R}}\,. \tag{4.4}$$

For the simplest Wilson action, $c_i = 0$, $i > 0$ and $c_{\mathrm{R}0} = 0$ for $R \neq$ the fundamental representation, one has

$$w_{1\mathrm{R}}(I,J)_\mathrm{Wilson} = C_\mathrm{R} I_1(I,J,d)\,, \tag{4.5}$$

where

$$I_1(I,J,d) = a^d \int_k \left(\frac{\sin \frac{1}{2}k_1 Ia}{\sin \frac{1}{2}k_1 a}\right)^2 \left(\frac{\sin \frac{1}{2}k_d Ja}{\sin \frac{1}{2}k_d a}\right)^2 \frac{\hat{k}_1^2 + \hat{k}_d^2}{\hat{k}^2}\,, \tag{4.6}$$

with exact known results [16],

$$\begin{aligned}
&I_1(I,J,2) = IJ\,,\\
&I_1(1,1,d) = 2/d\,,\\
&I_1(I,J,3) \underset{I,J\to\infty}{\sim} \frac{1}{\pi}(I \ln J + J \ln I)\,,\\
&I_1(I,J,d) \underset{I,J\to\infty}{\sim} (I+J)\int_0^\infty \mathrm{d}\beta\, I_0^{d-1}(\beta)\, \mathrm{e}^{-(d-1)\beta}\,, \qquad (d \geq 4)\,.
\end{aligned} \tag{4.7}$$

Now consider the static potential (2.4) in four dimensions ($d = 4$). For the modified actions only the terms in $D_{1d,1d}$ with no $\hat{k}_d^2$ factor in the numerator [see in particular (A.8)] contribute to the leading $J \to \infty$ limit behaviour. Let $L = aI$, $T = aJ$, then one sees for the special choice of coefficients determined in the last section [see (A.12)]

$$\begin{aligned}
\lim_{\substack{T\to\infty\\ a,L \text{ fixed}}} \frac{1}{T} w_{1\mathrm{R}}(I,J) &= C_\mathrm{R} a^{-1} \int_{-\pi}^{\pi} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{(1-\cos(Lk_1/a))}{2\sum_{i=1}^3 (\sin^2(k_i/2) + \frac{1}{3}\sin^4(k_i/2))}\\
&\underset{a\to 0}{\longrightarrow} C_\mathrm{R} a^{-1} \int_{-\pi}^{\pi} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\sum_{i=1}^3 (\sin^2(k_i/2) + \frac{1}{3}\sin^4(k_i/2))}\\
&\quad - \frac{C_\mathrm{R}}{2L} + \mathrm{O}(a^4)\,.
\end{aligned} \tag{4.8}$$

The important point is that (by construction) the static potential has an improved small a behaviour in lowest order; after subtraction of the linear divergence, the corrections are of order a^4.

Finally, we record the $\chi_R(I, J)$, often used in MC analyses since (just like the static potential) they are free from the kink divergences,

$$\chi_R(I,J) = -\ln\left(\frac{\langle \operatorname{tr} U_R(I,J)\rangle\langle \operatorname{tr} U_R(I-1,J-1)\rangle}{\langle \operatorname{tr} U_R(I,J-1)\rangle\langle \operatorname{tr} U_R(I-1,J)\rangle}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(a^{4-d}g^2)^n}{(2n)!}\chi_{nR}(I,J)\,. \tag{4.9}$$

One has

$$a^{-2}\chi_{1R}(I,J) = C_R a^{d-2}\int_k \frac{\sin k_1 a(I-\frac{1}{2})}{\sin\frac{1}{2}k_1 a}\,\frac{\sin k_d a(J-\frac{1}{2})}{\sin\frac{1}{2}k_d a}D_{1d,1d}(k)\,. \tag{4.10}$$

The calculation of Wilson loop expectations in next order involves perturbation theory with Feynman rules derived from the action (2.12). The vertices are algebraically complicated, and the result of the lengthy computation will be presented in a subsequent paper.

5. Discussion

In this paper we have discussed the improvement of the Wilson action on the lines suggested by Symanzik [8]. The lowest order in perturbation theory has been treated in detail, and the next order calculation is in progress. The hope is that for a systematic improvement programme for the gauge-invariant observables, it will be sufficient to use the proposed action (2.12) and not to complicate the description further*. We stress that the conjecture above has not yet been proven; the self-consistency, or otherwise, should show up in the next order.

The qualitative nature of the improvement is presently not known. For example one can raise the question as to how delicate is the balance between the improvement, in the sense of Symanzik, and the finite size effects or more elusive non-perturbative effects. Note for example that, in general, the improved action violates Osterwalder–Schrader positivity (a property holding for the Wilson action [17]). However, it is sufficient that this property is restored in the continuum limit, just as ordinary rotation invariance should be, although this may be difficult to establish analytically. Strong coupling expansions with the improved action also become more technically involved and have not been investigated yet. However improvement of rotational invariance of the strong coupling expansion is not the immediate aim of the programme at this stage; (for such attempts see e.g. ref. [18]).

* For example one could imagine changing the group structure SU(N) to a larger group and regaining the SU(N) theory only in the continuum limit. Possibilities of this type have been stressed to me by Symanzik. In any case an improved definition of certain quantities will also be necessary. For example the x's are not suitably improved as they appear in (4.9).

The philosophy, at present, is merely to do ones best within the framework of perturbation theory; to investigate the order of magnitude of corrections to quantitative predictions and to see whether they go in the correct direction. The hope is that ratios of masses become smoother functions of g in the neighbourhood of the continuum limit. From a practical MC point of view the addition of paths of length $6a$ to the usual Wilson action, requires more computer time in order to obtain comparable statistics. This can be appreciated by considering the number of curves n_i, belonging to the various classes T_i (defined in sect. 2), passing through a given link $\mathscr{L}$, which are needed in one step of the MC updating procedure. They are given by

$$\begin{aligned} n_0 &= 2(d-1)\,, \\ n_1 &= 6(d-1)\,, \\ n_2 &= 12(d-1)(d-2)\,, \\ n_3 &= 4(d-1)(d-2)\,. \end{aligned} \tag{5.1}$$

Thus for $d=4$ one has, for example, $n_2=72$ compared to $n_0=6$. Montvay has suggested that the curves in classes $T_{1,2,3}$ could also be treated statistically. Cleverly constructed programmes could also reduce the, at first sight, large effective factor. In either case, working efficiently with more complicated actions, as the ones discussed above, seems to be one of the applications of parallel processors.

Indeed a programme using an action with paths belonging to classes T_0, T_1 and T_3 has been used by Wilson [13] in his real space renormalization group studies. Wilson's theoretical ideas are similar spirit to those of Symanzik, but the latter seems to be more systematically implementable. Working on an 8^4 lattice, Wilson found that the convergence of effective actions was more rapid, than the simplest action, for a choice of coefficients

$$c_0=4.376\,,\qquad c_1=-0.252\,,\qquad c_2=0\,,\qquad c_3=-0.17\,. \tag{5.2}$$

Whether these numbers are optimal in some respect is not made clear in the paper [13]. The numbers differ somewhat from those given in (3.25), (3.27), (3.28), but it must be recalled that the fit is made at finite g. Note also, that Wilson's coefficients (5.2) still obey the constraint (3.26) accurately, a relation which ensures some extent of rotational symmetry.

One could imagine running MC programmes in various regions of c_i space and determining some optimum set of coefficients experimentally. This would however require an enormous amount of computer time and in addition no estimates of $\Lambda_{\mathrm{L}}/\Lambda_{\mathrm{cont}}$ could be explicitly made. Preliminary MC experiments will first be made for the non-linear σ-model in two dimensions. Success for Symanzik's programme in this case would encourage application to the theory of QCD.

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Appendix

Consider lattice actions which to lowest order in g have the form, after covariant gauge fixing.

$$(S+S_{\mathrm{gf}})^{(2)} = -\tfrac{1}{2}\sum_{\mu,\nu}\int_k \mathrm{tr}\left[q_{\mu\nu}(k)\tilde{f}_{\mu\nu}(k)\tilde{f}_{\mu\nu}(-k)+\frac{2}{\alpha}\hat{k}_\mu\hat{k}_\nu\tilde{A}_\mu(k)\tilde{A}_\nu(-k)\right], \tag{A.1}$$

with $q_{\mu\nu}$ satisfying:

(i) $q_{\mu\mu}=0$ for all μ;
(ii) $q_{\mu\nu}=q_{\nu\mu}$;
(iii) $q_{\mu\nu}(k)=q_{\mu\nu}(-k)$;
(iv) $q_{\mu\nu}(0)=1, \qquad \mu\neq\nu$.

The free propagator defined in

$$\langle A^i_\mu(x)A^j_\nu(y)\rangle_0=\delta^{ij}\int_k \mathrm{e}^{ik(x-y)}\,\mathrm{e}^{ia(k_\mu-k_\nu)/2}D_{\mu\nu}(k) \tag{A.2}$$

is the solution to the equation

$$\sum_\nu\left[\frac{1}{\alpha}\hat{k}_\mu\hat{k}_\nu+\sum_\rho(\hat{k}_\rho\delta_{\mu\nu}-\hat{k}_\mu\delta_{\rho\nu})q_{\mu\rho}\hat{k}_\rho\right]D_{\nu\tau}=\delta_{\mu\tau}. \tag{A.3}$$

It can be written in the form

$$D_{\nu\tau}(k)=(\hat{k}^2)^{-2}\left[\alpha\hat{k}_\nu\hat{k}_\tau+\sum_\sigma(\hat{k}_\sigma\delta_{\tau\nu}-\hat{k}_\tau\delta_{\sigma\nu})A_{\tau\sigma}\hat{k}_\sigma\right] \tag{A.4}$$

with $A_{\mu\nu}$ independent of α and satisfying the properties (i)–(iv) above. For the Wilson action

$$q^{\mathrm{Wilson}}_{\mu\nu}=(1-\delta_{\mu\nu})=A^{\mathrm{Wilson}}_{\mu\nu}. \tag{A.5}$$

For a general $q_{\mu\nu}$ the functional dependence of A on q is more complicated for $d>2$.

Defining, for dimension d

$$\Delta_d=\frac{\alpha}{(\hat{k}^2)^2}\det D^{-1} \tag{A.6}$$

one finds for the cases $d \leqslant 4$,

$$\Delta_2 = q_{12}\,,$$

$$\Delta_3 = \sum_{\mu} \hat{k}_{\mu}^2 \prod_{\nu \neq \mu} q_{\nu\mu}\,, \tag{A.7}$$

$$\Delta_4 = \sum_{\mu} \hat{k}_{\mu}^4 \prod_{\nu \neq \mu} q_{\nu\mu} + \sum_{\substack{\mu > \nu \\ \rho > \tau \\ \{\rho,\tau\} \cap \{\mu,\nu\} = \varnothing}} \hat{k}_{\mu}^2 \hat{k}_{\nu}^2 q_{\mu\nu} (q_{\mu\rho} q_{\nu\tau} + q_{\mu\tau} q_{\nu\rho})\,,$$

and the matrix element A_{12} (and the other elements by appropriate replacement of indices) given by

$$d = 2: \quad A_{12} = \frac{1}{\Delta_2}\,,$$

$$d = 3: \quad A_{12} = \frac{1}{\Delta_3}[q_{13}(\hat{k}^2 - \hat{k}_2^2) + q_{23}(\hat{k}^2 - \hat{k}_1^2) - q_{12}\hat{k}_3^2]\,,$$

$$\begin{aligned} d = 4: \quad A_{12} = \frac{1}{\Delta_4}[&(\hat{k}^2 - \hat{k}_2^2)(q_{13}q_{14}\hat{k}_1^2 + q_{13}q_{34} + \hat{k}_3^2 + q_{14}q_{34}\hat{k}_4^2) \\ &+ (\hat{k}^2 - \hat{k}_1^2)(q_{23}q_{24}\hat{k}_2^2 + q_{23}q_{34}\hat{k}_3^2 + q_{24}q_{34}\hat{k}_4^2) \\ &+ q_{13}q_{24}(\hat{k}_1^2 + \hat{k}_3^2)(\hat{k}_2^2 + \hat{k}_4^2) + q_{14}q_{23}(\hat{k}_1^2 + \hat{k}_4^2)(\hat{k}_2^2 + \hat{k}_3^2) \\ &- q_{12}q_{34}(\hat{k}_3^2 + \hat{k}_4^2)^2 - (q_{13}q_{23} + q_{14}q_{24})\hat{k}_3^2\hat{k}_4^2 \\ &- q_{12}(q_{13}\hat{k}_1^2\hat{k}_4^2 + q_{14}\hat{k}_1^2\hat{k}_3^2 + q_{23}\hat{k}_2^2\hat{k}_4^2 + q_{24}\hat{k}_2^2\hat{k}_3^2)]\,. \end{aligned} \tag{A.8}$$

The free propagator of the transverse $\tilde{f}$'s is given by

$$\langle \tilde{f}_{\mu\nu}^i(k) \tilde{f}_{\rho\lambda}^j(k') \rangle_0 = \delta^{ij} (2\pi)^d \delta(k + k') D_{\mu\nu,\rho\lambda}(k)\,, \tag{A.9}$$

with

$$\begin{aligned} D_{\mu\nu,\rho\lambda}(k) = (\hat{k}^2)^{-2} \Big\{ &\sum_{\sigma} \hat{k}_{\sigma}^2 [\hat{k}_{\rho} A_{\lambda\sigma} (\delta_{\lambda\nu} \hat{k}_{\mu} - \delta_{\lambda\mu} \hat{k}_{\nu}) - \hat{k}_{\lambda} A_{\rho\sigma} (\delta_{\rho\nu} \hat{k}_{\mu} - \delta_{\rho\mu} \hat{k}_{\nu})] \\ &- \hat{k}_{\mu} \hat{k}_{\rho} \hat{k}_{\nu} \hat{k}_{\lambda} [A_{\lambda\nu} - A_{\lambda\mu} + A_{\rho\mu} + A_{\rho\mu} - A_{\rho\nu}] \Big\}\,. \end{aligned} \tag{A.10}$$

For the special case $\mu = \rho$, $\nu = \lambda$ we have

$$\begin{aligned} D_{12,12}(k) &= (\hat{k}^2)^{-2} \Big\{ \sum_{\mu} \hat{k}_{\mu}^2 (\hat{k}_1^2 A_{2\mu} + \hat{k}_2^2 A_{1\mu}) + 2\hat{k}_1^2 \hat{k}_2^2 A_{12} \Big\} \\ &= \begin{cases} \dfrac{1}{\Delta_2}\,, & \text{for } d = 2\,, \\[2ex] \dfrac{1}{\Delta_3}(q_{13}\hat{k}_1^2 + q_{23}\hat{k}_2^2)\,, & \text{for } d = 3\,, \end{cases} \end{aligned}$$

$$= \frac{1}{\Delta_4}[\hat{k}_1^2(q_{13}q_{14}\hat{k}_1^2 + q_{13}q_{34}\hat{k}_3^2 + q_{14}q_{34}\hat{k}_4^2) + \hat{k}_2^2(q_{23}q_{24}\hat{k}_2^2 + q_{23}q_{34}\hat{k}_3^2 + q_{24}q_{34}\hat{k}_4^2) + \hat{k}_1^2\hat{k}_2^2(q_{13}q_{24} + q_{14}q_{23})]\,, \qquad \text{for } d = 4\,. \tag{A.11}$$

Remark, in particular, a relation which we use in sect. 4,

$$D_{1d,1d}\Big|_{k_d=0} = \hat{k}_1^2\Big(\sum_\mu q_{\mu d}\hat{k}_\mu^2\Big)^{-1}\Big|_{k_d=0}\,. \tag{A.12}$$

Note also, the identity

$$\sum_{\mu>\nu} q_{\mu\nu}D_{\mu\nu,\mu\nu} = d - 1 \tag{A.13}$$

and the special Wilson case

$$D^{\text{Wilson}}_{\mu\nu,\mu\nu} = \frac{\hat{k}_u^2 + \hat{k}_\nu^2}{\hat{k}^2}\,. \tag{A.14}$$

For later numerical calculations working with the improved action, we require the special case

$$q_{\mu\nu} = (1 - \delta_{\mu\nu})(1 + ca^2[\hat{k}_\mu^2 + \hat{k}_\nu^2])\,, \qquad c = \tfrac{1}{12}\,. \tag{A.15}$$

Then

$$\begin{aligned} \Delta_2 &= 1 - ca^2\hat{k}^2\,,\\ \Delta_3 &= (1 + ca^2\hat{k}^2)\Big(\hat{k}^2 + ca^2\sum_\mu \hat{k}_\mu^4\Big) + 3c^2a^4\prod_\nu \hat{k}_\nu^2\,,\\ \Delta_4 &= \Big(\hat{k}^2 + ca^2\sum_\mu \hat{k}_\mu^4\Big)\Big[\hat{k}^2 + ca^2\Big((\hat{k}^2)^2 + \sum_\mu \hat{k}_\mu^4\Big)\\ &\quad + \tfrac{1}{2}c^2a^4\Big((\hat{k}^2)^3 + 2\sum_\mu \hat{k}_\mu^6 - \hat{k}^2\sum_\mu \hat{k}_\mu^4\Big)\Big] + 8c^3a^6\sum_\mu \hat{k}_\mu^4 \prod_{\nu\neq\mu} \hat{k}_\nu^2\,, \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} \Delta_3 A_{12} &= \hat{k}^2 + ca^2\Big(\hat{k}^2\hat{k}_3^2 + \sum_\mu \hat{k}_\mu^4\Big)\,, \qquad \text{for } d = 3\,,\\ \Delta_4 A_{12} &= (\hat{k}^2)^2 + ca^2\hat{k}^2\Big(2\sum_\mu \hat{k}_\mu^4 + \hat{k}^2[\hat{k}_3^2 + \hat{k}_4^2]\Big)\\ &\quad + c^2a^2\Big(\Big(\sum_\mu \hat{k}_\mu^4\Big)^2 + \hat{k}^2\sum \hat{k}_\mu^4(\hat{k}_3^2 + \hat{k}_4^2) + (\hat{k}^2)^2\hat{k}_3^2\hat{k}_4^2\Big)\,. \end{aligned} \tag{A.17}$$

One immediately checks that in each case

$$A_{\mu\nu} = (1-\delta_{\mu\nu})(1-ca^2[\hat{k}_\mu^2+\hat{k}_\nu^2]+\mathrm{O}(a^4)). \tag{A.18}$$

Thus

$$D_{\mu\nu,\rho\lambda} = (\hat{k}^2)^{-2}\left\{\hat{k}_\rho\hat{k}_\mu\delta_{\lambda\nu}\left(\hat{k}^2-ca^2\left[\hat{k}^2\hat{k}_\nu^2+\sum_\tau\hat{k}_\tau^4\right]\right)\pm 3\text{ perms}+\mathrm{O}(a^4)\right\}, \tag{A.19}$$

and it follows that (3.24) is satisfied.

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