THE GLUON PROPAGATOR IN TEMPORAL GAUGE

H.D. DAHMEN, B. SCHOLZ
Fachbereich Physik, Universität Siegen, Siegen, Germany

and

F. STEINER
II. Institut für Theoretische Physik, Universität Hamburg, Hamburg, Germany

Received 19 July 1982

The implementation of Gauss's law in perturbative calculations in temporal gauge is achieved through an explicit construction of the vacuum state. In this scheme the free gluon propagator is calculated. Terms in addition to the principal value part are found.

It is well known that perturbative calculations in non-abelian gauge theories profit from the use of the temporal gauge since there are no ghosts and the hamiltonian has a simple polynomial form. One of the problems discussed in the recent literature is the regularization of the propagator of the gauge field [1–3]. General methods as the path integral or canonical quantization as well as the principle of unitarity have led to a principal value prescription. A recent calculation by Müller and Rühl [4] of small coupling expansions on a lattice has shed doubts on this regulatization. These authors propose an improved propagator. Caracciolo et al. [5] come to a similar conclusion by studying the Wilson loop in temporal gauge.

In this note we propose a construction of a physical vacuum explicitly satisfying Gauss' law. This leads to a limiting procedure that has to be applied to every order of perturbation theory. For the free propagator we arrive at the usual principal value prescription plus terms constant respectively bilinear in time. In fourth order it is exactly these latter terms leading to additional contributions which can be simulated by a change in the free propagator of the kind proposed by refs. [4,5].

In temporal gauge the time components of the gauge fields A^a_μ (a refers to the colour of the field) are set equal to zero, $A^a_0(t,\boldsymbol{x}) = 0$. The hamiltonian of a pure non-abelian gauge theory without fermions takes the simple form

$$H = \frac{1}{2}\int (E^a_i E^a_i + B^a_i B^a_i)\, \mathrm{d}^3\boldsymbol{x} \tag{1}$$

in the chromoelectric and chromomagnetic fields

$$E^a_i = -\partial_0 A^a_i, \quad B^a_i = -\tfrac{1}{2}\epsilon_{ijk} F^a_{jk}. \tag{2}$$

The space components of the field strength tensor are as usual

$$F^a_{jk} = \partial_j A^a_k - \partial_k A^a_j + g f^{abc} A^b_j A^c_k. \tag{3}$$

Here g is the coupling constant, f^{abc} are the structure constants of the non-abelian group. The canonical quantization scheme starts from the canonical commutators at equal times for the gauge fields and their time derivatives

$$[A^a_i(t,\boldsymbol{x}), A^b_j(t,\boldsymbol{x}')] = 0 = [E^a_i(t,\boldsymbol{x}), E^b_j(t,\boldsymbol{x}')], \tag{4a}$$

and

$$[E^a_i(t,\boldsymbol{x}), A^b_j(t,\boldsymbol{x}')] = \mathrm{i}\delta^{ab}\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{4b}$$

Using the covariant derivative

$$D^{ab}_j = \delta^{ab}\partial_j - g f^{abc} A^c_j, \tag{5a}$$

Gauss' law has the form

$$D^{ab}_j E^b_j = 0 \tag{5b}$$

for the classical chromoelectric fields. Since it does not contain time derivatives it does not occur among the

Hamilton equations in temporal gauge and has to be required as a condition on the physical states of the theory

$$D_j^{ab}E_j^b\,|\mathrm{phys}\rangle = 0. \tag{6}$$

In the interaction picture the Maxwell equations in temporal gauge read

$$(\Box\delta_{ij} + \partial_i\partial_j)A_j^a = 0. \tag{7}$$

The gauge field A_i^a can be decomposed into a transverse field $A_i^{a\mathrm{T}}$ free of sources and a longitudinal field $A_i^{a\mathrm{L}}$ free of curls

$$A_i^a(t,\boldsymbol{x}) = A_i^{a\mathrm{T}}(t,\boldsymbol{x}) + A_i^{a\mathrm{L}}(t,\boldsymbol{x}). \tag{8}$$

The Maxwell equations for the two fields read

$$\Box A_i^{a\mathrm{T}} = 0, \quad (\Box\delta_{ij} + \partial_i\partial_j)A_j^{a\mathrm{L}} = 0. \tag{9}$$

The canonical commutation relations that do not vanish among the components read

$$[A_i^{a\mathrm{T}}(t,\boldsymbol{x}), \partial_0 A_j^{b\mathrm{T}}(t,\boldsymbol{x}')]$$
$$= \mathrm{i}\delta^{ab}\,[\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}') - d_{ij}(\boldsymbol{x}-\boldsymbol{x}')], \tag{10a}$$

$$[A_i^{a\mathrm{L}}(t,\boldsymbol{x}), \partial_0 A_j^{b\mathrm{L}}(t,\boldsymbol{x}')] = \mathrm{i}\delta^{ab}d_{ij}(\boldsymbol{x}-\boldsymbol{x}'), \tag{10b}$$

$$d_{ij}(\boldsymbol{x}-\boldsymbol{x}') = \partial_i\partial_j\,\Delta^{-1}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{10c}$$

According to Frenkel [3] the longitudinal field as determined by the Maxwell equation (9) is at most linear in time so that we may write

$$A_i^{a\mathrm{L}}(t,\boldsymbol{x}) = \partial_i[(-\Delta)^{-3/4}\chi_+^a(\boldsymbol{x}) + t(-\Delta)^{-1/4}\chi_-^a(\boldsymbol{x})]. \tag{11}$$

This takes the explicit representation of the operator $(-\Delta)^\alpha$ into account:

$$(-\Delta)^\alpha f(\boldsymbol{x}) = \int \mathrm{d}^3x'\,K_\alpha(\boldsymbol{x}-\boldsymbol{x}')\,f(\boldsymbol{x}'), \tag{12a}$$

with

$$K_\alpha(\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,(k^2)^\alpha\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$$
$$= \frac{2^{2\alpha+3/2}}{(2\pi)^{3/2}}\,\frac{\Gamma(\alpha+3/2)}{\Gamma(-\alpha)}\,\frac{1}{|\boldsymbol{x}|^{2\alpha+3}}. \tag{12b}$$

The particular powers of Δ are chosen such that the hermitian scalar fields χ_+, χ_- are of the same dimension. As a consequence of (10b) they fulfil the commutation relation

$$[\chi_+^a(\boldsymbol{x}), \chi_-^b(\boldsymbol{x}')] = \mathrm{i}\delta^{ab}\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{13}$$

with all other combinations vanishing. In order to satisfy Gauss' law for physical states perturbatively we impose on the unperturbed vacuum the constraint, see also refs. [3,6]

$$\partial_i E_i^a(t,\boldsymbol{x})|\Omega\rangle = 0, \tag{14a}$$

which implies

$$\chi_-^a(\boldsymbol{x})|\Omega\rangle = 0. \tag{14b}$$

In order to present an explicit construction of $|\Omega\rangle$ we decompose the χ's into

$$\chi_-^b(\boldsymbol{x}) = 2^{-1/2}[a^b(\boldsymbol{x}) + a^{b+}(\boldsymbol{x})],$$
$$\chi_+^b(\boldsymbol{x}) = 2^{-1/2}\mathrm{i}[a^b(\boldsymbol{x}) - a^{b+}(\boldsymbol{x})], \tag{15a}$$

with

$$[a^b(\boldsymbol{x}), a^{c+}(\boldsymbol{x}')] = \delta^{bc}\delta(\boldsymbol{x}-\boldsymbol{x}'),$$
$$[a^b(\boldsymbol{x}), a^c(\boldsymbol{x}')] = 0 = [a^{b+}(\boldsymbol{x}), a^{c+}(\boldsymbol{x}')]. \tag{15b}$$

We introduce the Fock-vacuum $|0\rangle$ by

$$a^b(\boldsymbol{x})|0\rangle = 0. \tag{16}$$

The construction of $|\Omega\rangle$ is analogous to a limiting process given in ref. [7]

$$|\Omega\rangle = \lim_{\lambda\to 1}|\Omega_\lambda\rangle, \tag{17a}$$

with

$$|\Omega_\lambda\rangle = N_\lambda\exp\left(-\frac{\lambda}{2}\int a^{c+}(\boldsymbol{x})\,a^{c+}(\boldsymbol{x})\,\mathrm{d}^3x\right)|0\rangle, \tag{17b}$$

where N_λ is a normalization ensuring $\langle\Omega_\lambda|\Omega_\lambda\rangle = 1$. It tends to zero as the number of degrees of freedom assumes infinity. Matrix elements have to be calculated with $|\Omega_\lambda\rangle$, the limit $\lambda\to 1$ has to be carried out from below at the very end.

The state $|\Omega_\lambda\rangle$ guarantees Gauss' law (14b) in the limit $\lambda\to 1$

$$\chi_-^c(\boldsymbol{x})|\Omega_\lambda\rangle = 2^{-1/2}(1-\lambda)\,a^{c+}(\boldsymbol{x})|\Omega_\lambda\rangle. \tag{18}$$

In contrast to the canonical quantization of ref. [3] the vacuum state $|\Omega\rangle$ secures the implementation of the commutation relations (13).

The calculation of the free longitudinal propagator yields

$$\langle\Omega_\lambda|TA_i^{aL}(t_1,\boldsymbol{x}_1)A_j^{bL}(t_2,\boldsymbol{x}_2)|\Omega_\lambda\rangle = -\tfrac{1}{2}\mathrm{i}\,\delta^{ab}$$
$$\times\{|t_1-t_2| - [(1-\lambda)/(1+\lambda)]\,t_1t_2\,\Delta_1^{1/2}$$
$$+[(1+\lambda)/(1-\lambda)]\,\Delta_1^{-1/2}\}\,d_{ij}(\boldsymbol{x}_1-\boldsymbol{x}_2). \qquad (19)$$

The first term is independent of λ and is the usual longitudinal propagator with principal value regularization. The second term vanishes with $\lambda \to 1$. The third term is time-independent, diverges as $\lambda \to 1$ but does not contribute to gauge invariant quantities. Obviously, the product of the second and the third term may lead to λ-independent, however time-dependent contributions in higher orders. As an example we look at the four point function of the longitudinal gauge fields. We find

$$\langle\Omega_\lambda|TA_i^{aL}(t_1,\boldsymbol{x}_1)A_j^{bL}(t_2,\boldsymbol{x}_2)A_k^{cL}(t_3,\boldsymbol{x}_3)A_l^{dL}(t_4,\boldsymbol{x}_4)|\Omega_\lambda\rangle$$
$$= -\tfrac{1}{4}\delta^{ab}\delta^{cd}(|t_1-t_2|\,|t_3-t_4|$$
$$-t_1t_2\Delta_1^{+1/2}\Delta_3^{-1/2} - t_3t_4\Delta_1^{-1/2}\Delta_3^{+1/2})$$
$$\times d_{ij}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,d_{kl}(\boldsymbol{x}_3-\boldsymbol{x}_4) \qquad (20)$$
$$+\text{ other contractions} + \mathrm{O}(1-\lambda) + \mathrm{O}((1-\lambda)^{-1}).$$

As expected, there are finite terms bilinear in time in addition to the product of the usual propagators. If one looks for instance into a one-loop propagator insertion one finds that the finite terms of all contractions are reproduced by the real part of the product of effective propagators of the form [‡1]

[‡1] The imaginary part of the product of two effective propagators does not contribute to a gauge invariant quantity as calculated in ref. [5].

$$\mathrm{i}D_{ij,\mathrm{eff}}^{abL} = -\tfrac{1}{2}\mathrm{i}\,\delta^{ab}$$
$$\times[|t_1-t_2| + \mathrm{i}\sigma(t_1+t_2)]\,d_{ij}(\boldsymbol{x}_1-\boldsymbol{x}_2), \qquad (21)$$

where σ is one of the values $+1, -1$.

The arguments presented here show how the improved propagator of refs. [4] and [5] can be understood on the basis of a canonical quantization as an effective propagator that reproduces the fourth order result. The i in front of σ in eq. (21) guarantees the correct hermiticity properties of the T-product of the hermitian longitudinal fields [‡2].

Of course, the λ-limiting procedure can be applied to any order of perturbation theory. Only higher order calculations can decide whether the effective propagator reproduces the results of the λ-limit also there.

[‡2] It is consistent with the findings of Müller and Rühl [4] since they are working in euclidean space–time.

References

[1] W. Kummer, Acta Phys. Austriaca 41 (1975) 315; W. Konetschny and W. Kummer, Nucl. Phys. B100 (1975) 106.
[2] J. Frenkel and J.C. Taylor, Nucl. Phys. B109 (1976) 439.
[3] J. Frenkel, Phys. Lett. 85B (1979) 63.
[4] V.F. Müller and W. Rühl, Ann. Phys. (NY) 133 (1981) 240.
[5] S. Caracciolo, G. Curci and P. Menotti, Pisa-Preprint IFUP-TH 4/82.
[6] L.D. Faddeev and A.A. Slavnov, Gauge fields: introduction to quantum theory (Benjamin/Cummings, New York, 1980) Ch. 3.
[7] R. Utiyama, T. Imamura, S. Sumakawa and T. Dodo, Prog. Theor. Phys. 6 (1951) 587; see also: J.M. Jauch and F. Rohrlich, The theory of photons and electrons, 2nd Ed. (Springer, Berlin, 1976) Ch. 6.