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3-Dimensional $SU(2)$ Lattice Gauge Theory in Terms of Gauge Invariant Variables*

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Abstract. As a first step towards a duality transformation for the $SU(2)$ lattice gauge theory in 3 dimensions, the integration over all gauge variant variables is performed explicitly after introducing gauge invariant auxiliary variables. The resulting new Hamiltonian is complex and involves a sum over closed loops. Each of these loops is confined to an elementary cube of a dual lattice. Like in a previous investigation for the $O(4)$ symmetric Heisenberg ferromagnet Rühl's boson representation is used to derive the result.

1. Introduction

It is commonly believed that nonabelian gauge theories are the most promising candidate for a correct description of hadronic matter. However, the existence of the essential property of confinement has up to now only been conjectured.

Some time ago, 't Hooft and Mandelstam [1, 2] suggested a dual Meissner effect as a possible quark confining mechanism. It thus seems desirable to develop a concept of duality transformations for nonabelian gauge theories.

Up to now, duality transformations have only been worked out for abelian systems [3]. There it turned out to be important to formulate the theory in terms of gauge or rotation invariant variables. In a previous paper [4], the analogue of this step has been carried out for the simple case of the 2-dimensional Heisenberg ferromagnet with global $O(4)$ symmetry. Our main concern, however, are nonabelian gauge theories. In the present paper, we apply our method to the $SU(2)$ lattice gauge theory in 3 dimensions.

To avoid the appearance of vector coupling coefficients, use is made of a formalism introduced by Rühl in his investigation of $SU(N)$ lattice gauge theories [5, 6]. Its main ingredient is the Bargmann space realization of group representations of $SU(N)$ [7].

After a character expansion of the partition function and introduction of auxiliary gauge variant $\mathbb{C}^2$ variables the integration over the original $SU(2)$ variables can be carried out. Rühl's formalism allows to sum explicitly over the irreducible unitary representations of $SU(2)$. The quartic interaction of the auxiliary $\mathbb{C}^2$ variables can be rewritten in quadratic form with the help of new gauge invariant complex variables. The gaussian integrals over the $\mathbb{C}^2$ variables factorize and result in a product of determinants each of which is expanded into a system of closed loops.

In contrast to the situation for the $O(4)$ symmetric Heisenberg ferromagnet [4], these loops are localized, i.e. they are each confined to elementary cubes of the dual lattice.

As intended, the new system of closed loops is formulated entirely in gauge invariant variables.

Unfortunately, our result has the undesirable, but possibly inevitable feature that the new Hamiltonian is complex and thus does not allow an interpretation as a system of statistical mechanics.

2. The Model

We consider Wilson's action for an $SU(2)$ gauge theory on a 3-dimensional cubic lattice $\Lambda \subset \mathbb{Z}^3$

$$L(u) = \frac{\beta}{2} \sum_p \operatorname{tr} u_p \tag{2.1}$$

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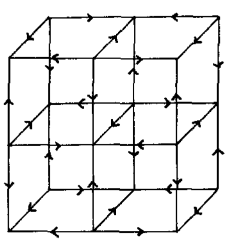

Fig. 1. Orientation of links on the lattice

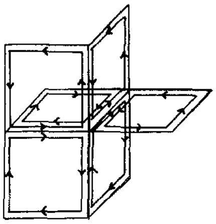

Fig. 2. Orientation of plaquettes on the lattice

where u_p is the product of four $SU(2)$ variables u_b attached to the links of the plaquette p.

The partition function of the system is

$$z=\int \prod_{b\in\Lambda} du_b\, e^{L(u)} \tag{2.2}$$

where du_b is the normalized Haar measure on $SU(2)$.

We assume periodic boundary conditions. If we further assume that the lattice contains an even number of links in every direction it is possible to orient the links in alternating order (see Fig. 1) so that the lattice consists of a set Λ_i of starting points of links and a set Λ_f of end points. Then each link is denoted by $b=\langle x\,y\rangle$, $x\in\Lambda_i$, $y\in\Lambda_f$.

On this lattice, each plaquette variable u_p takes the form

$$u_p=u_{b_1}u_{b_2}^{-1}u_{b_3}u_{b_4}^{-1}. \tag{2.3}$$

We have an additional freedom of choice for the orientation of each plaquette. We will use the notation $b\in\partial p$ resp. $b^{-1}\in\partial p$ if a given link b belonging to the boundary of the plaquette p has an orientation parallel resp. antiparallel to the orientation of the plaquette p. If $b^{-1}\in\partial p$ then u_p will contain the inverse u_b^{-1} rather than u_b. In the present case, it is convenient to choose alternating orientation for the plaquettes, as shown in Fig. 2.

We will further use the notation $(p,p')\wedge b$ for an unordered pair of plaquettes touching at the link b.

3. Integration of the Group Variables

The partition function is expanded into characters of irreducible unitary representations of $SU(2)$

$$z=\int \prod_b du_b \prod_p \sum_{j_p} c_{j_p}\chi_{j_p}(u_p) \tag{3.1}$$

with the expansion coefficients

$$c_j=\frac{2}{\beta}(2j+1)\,I_{2j+1}(\beta) \tag{3.2}$$

where the I_n are modified Bessel functions.

Each plaquette carries a representation $j=0$, $1/2, 1, \ldots$ of $SU(2)$. The characters are decomposed into

$$\chi_j(u_p)=\sum_{mm'} D^j_{m_1m_1'}(u_{b_1})\,D^j_{m_1'm_2}(u_{b_2}^{-1})\cdot D^j_{m_2m_2'}(u_{b_3})\,D^j_{m_2'm_1}(u_{b_4}^{-1}). \tag{3.3}$$

In complete analogy to [4], the integration over the field variables is performed in the Bargmann space formalism [5-7] where a Hilbert space of entire analytic functions over $\mathbb{C}^2$ is introduced with gaussian measure

$$d\mu(z)=\frac{1}{\pi^2}\prod_{i=1,2} dx_i\,dy_i\,e^{-x_i^2-y_i^2}$$
$$\equiv dz\,dz^+\,e^{-z^+z},$$
$$z=\begin{pmatrix} z_1\\ z_2\end{pmatrix}\in\mathbb{C}^2; \qquad z_i=x_i+i\,y_i.$$

The analytic homogeneous polynomials of degree $2j$ form a subspace carrying the irreducible representation D^j. A basis of this subspace is given by the polynomials

$$v^j_m(z)=\frac{z_1^{j+m}\,z_2^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}, \qquad -j\leqq m\leqq j$$

so that the representation matrices D^j take the form

$$D^j_{mm'}(u)=\int d\mu(\rho)\,d\mu(z')\,\bar{v}^j_m(\rho)\,K(u;\rho,z')\,v^j_{m'}(z') \tag{3.4}$$

with the kernel

$$K(u;\rho,z)=e^{z^+\cdot u^T\rho}.$$

Using

$$D^j_{m'm}(u^{-1})=(-1)^{2j-m-m'}D^j_{-m,-m'}(u)$$

we get

$$D^j_{m'm}(u^{-1})=(-1)^{2j-m-m'}\int d\mu(\rho')\,d\mu(z)\cdot\bar{v}^j_{-m'}(z)\,K(u;\rho'^+,z^+)\,v^j_{-m}(\rho'). \tag{3.5}$$

For each link $b=\langle x\,y\rangle$ of each plaquette p we have introduced two $\mathbb{C}^2$ vectors $\rho^{(\prime)}_{p,x}$ and $z^{(\prime)}_{p,y}$, where the ρ resp. z variables are associated with the m resp. m' variables and therefore with Λ_i resp. Λ_f.

The partition function now reads

$$z=\int\prod_b du_b \prod_p \{\sum_{j_p} c_{j_p} \prod_{\substack{x\in\partial p\\ x\in A_i}} \sum_{m_{p,x}} (-1)^{j_p-m_{p,x}}$$

$$\cdot \prod_{\substack{y\in\partial p\\ y\in A_i}} \sum_{m_{p,y}} (-1)^{j_p-m_{p,y}} \prod_{\substack{b\in\partial p\\ b=\langle xy\rangle}} [\int d\mu(\rho_{p,x})\, d\mu(z'_{p,y})$$

$$\cdot \bar{v}^{j_p}_{m_{p,x}}(\rho_{p,x})\, K(u_b;\rho_{p,x}, z'_{p,y})\, v^{j_p}_{m_{p,y}}(z'_{p,y})]$$

$$\cdot \prod_{\substack{b^{-1}\in\partial p\\ b=\langle xy\rangle}} [\int d\mu(\rho'_{p,x})\, d\mu(z_{p,y})\, \bar{v}^{j_p}_{-m_{p,y}}(z_{p,y})$$

$$\cdot K(u_b;\rho'^{+}_{p,x}, z^{+}_{p,y})\, v^{j_p}_{-m_{p,x}}(\rho'_{p,x})]\}. \tag{3.6}$$

Doing the summation over m

$$\sum_m (-1)^{j-m}\, \bar{v}^j_m(\rho)\, v^j_{-m}(\rho') = \frac{(\rho^+\varepsilon\rho')^{2j}}{(2j)!} \tag{3.7a}$$

and over m'

$$\sum_{m'} (-1)^{j-m'}\, v^j_{m'}(z')\, \bar{v}^j_{-m'}(z) = \frac{(z'\varepsilon z^+)^{2j}}{(2j)!} \tag{3.7b}$$

we get

$$z=\int\prod_b du_b\, D\mu(\rho,z)\, D\mu(\rho',z')$$

$$\cdot\prod_p\left\{\sum_{j_p} c_{j_p}\prod_{\substack{x\in\partial p\\ x\in A_i}}\frac{(\rho'_{p,x}\varepsilon\rho^+_{p,x})^{2j_p}}{(2j_p)!}\prod_{\substack{y\in\partial p\\ y\in A_f}}\frac{(z'_{p,y}\varepsilon z^+_{p,y})^{2j_p}}{(2j_p)!}\right.$$

$$\left.\cdot\prod_{b\in\partial p} K(u_b;\rho_{p,x}, z'_{p,y})\prod_{b^{-1}\in\partial p} K(u_b;\rho'^{+}_{p,x}, z^{+}_{p,y})\right\} \tag{3.8}$$

where

$$D\mu(\rho,z)\equiv\prod_p\prod_{\substack{x\in\partial p\\ x\in A_i}} d\mu(\rho_{p,x})\prod_{\substack{y\in\partial p\\ y\in A_f}} d\mu(z_{p,y}).$$

Setting

$$\gamma_{p,x}(b)=\begin{cases}\rho_{p,x} & \text{if } b\in\partial p\\ \rho'^{+}_{p,x} & \text{if } b^{-1}\in\partial p\end{cases}$$

$$\alpha_{p,y}(b)=\begin{cases} z'_{p,y} & \text{if } b\in\partial p\\ z^{+}_{p,y} & \text{if } b^{-1}\in\partial p\end{cases} \tag{3.9}$$

the integrals over the field variables u_b can be done by means of the formula

$$\int du\, e^{\sum_i \alpha_i^+(u^T\gamma_i)} = \frac{i}{2\pi}\oint dv\, e^{-\frac{1}{v}+\sum_{(ij)}(\gamma_i\varepsilon\gamma_j)(\alpha_i^+\varepsilon^{-1}\alpha_j^+)}. \tag{3.10}$$

The summation is over unordered pairs (ij).

The partition function is now

$$z=\int\prod_b\left(\frac{i}{2\pi}dv_b\, e^{-\frac{1}{v_b}}\right) D\mu(\rho,z)\, D\mu(\rho',z')$$

$$\cdot\prod_p\left\{\sum_{j_p} c_{j_p}\prod_{\substack{x\in\partial p\\ x\in A_i}}\frac{(\rho'_{p,x}\varepsilon\rho^+_{p,x})^{2j_p}}{(2j_p)!}\prod_{\substack{y\in\partial p\\ y\in A_f}}\frac{(z'_{p,y}\varepsilon z^+_{p,y})^{2j_p}}{(2j_p)!}\right\}$$

$$\cdot\prod_{b=\langle xy\rangle}\exp[v_b\sum_{(p,p')\wedge b}(\gamma_{p,x}\varepsilon\gamma_{p',x})(\alpha^+_{p,y}\varepsilon^{-1}\alpha^+_{p',y})] \tag{3.11}$$

$(p,p')\wedge b$ denotes an unordered pair of plaquettes with common link b. The ε products in the exponential relate variables belonging to the same site but to different plaquettes.

With each corner of each plaquette we associate a complex variable $\tau_{p,x}$ by means of the complex contour integral

$$\frac{(z'\varepsilon z^+)^{2j}}{(2j)!}=\frac{1}{2\pi i}\oint\frac{d\tau}{\tau^{2j+1}}e^{\tau z'\varepsilon z^+}. \tag{3.12}$$

The sum over j yields a factor

$$B(\tau_p)=(\tau_p^{-1}-\tau_p)\,e^{\frac{\beta}{2}(\tau_p+\tau_p^{-1})}$$

$$\tau_p\equiv\prod_{x\in\partial p}\tau_{p,x} \tag{3.13}$$

for each plaquette (see [4]).

Thus we arrive at the partition function

$$z=\int Dv\, D\mu(\rho,z)\, D\mu(\rho',z')$$

$$\cdot\prod_p\left\{\prod_{\substack{x\in\partial p\\ x\in A_i}}\oint\frac{d\tau_{p,x}}{2\pi i}e^{\tau_{p,x}\rho'_{p,x}\varepsilon\rho^+_{p,x}}\right.$$

$$\left.\cdot\prod_{\substack{y\in\partial p\\ y\in A_f}}\oint\frac{d\tau_{p,y}}{2\pi i}e^{\tau_{p,y}z'_{p,y}\varepsilon z^+_{p,y}}\right\}\cdot B(\tau_p)$$

$$\cdot\prod_{b=\langle xy\rangle}\exp[v_b\sum_{(p,p')\wedge b}(\gamma_{p,x}\varepsilon\gamma_{p',x})(\alpha^+_{p,y}\varepsilon^{-1}\alpha^+_{p',y})] \tag{3.14}$$

with the abbreviation

$$Dv\equiv\prod_b\frac{i}{2\pi}dv_b\, e^{-\frac{1}{v_b}}.$$

4. Elimination of All Gauge Variant Variables – Loop Expansion

The quartic terms in the exponential are eliminated by introducing one gauge invariant complex variable

$$\eta_{pp'}=-\eta'_{p\,p} \tag{4.1}$$

for each pair of plaquettes (p,p') with a common link.

$$e^{v(\gamma\varepsilon\gamma')(\alpha^+\varepsilon^{-1}\alpha'^+)}=\frac{1}{\pi}\int_{\mathbb{C}} d\eta\, d\bar{\eta}\, e^{-\eta\bar{\eta}}\, e^{v^{1/2}(\gamma\varepsilon\gamma'\eta+\alpha^+\varepsilon^{-1}\alpha'^+\bar{\eta})}. \tag{4.2}$$

The partition function becomes

$$z=\int Dv\, D\mu(\eta)\, D\mu(\rho,z)\, D\mu(\rho',z')$$

$$\cdot\prod_p\left\{\prod_{\substack{x\in\partial p\\ x\in A_i}}\oint\frac{d\tau_{p,x}}{2\pi i}e^{\tau_{p,x}\rho'_{p,x}\varepsilon\rho^+_{p,x}}\right.$$

$$\left.\cdot\prod_{\substack{y\in\partial p\\ y\in A_f}}\oint\frac{d\tau_{p,y}}{2\pi i}e^{\tau_{p,y}z'_{p,y}\varepsilon z^+_{p,y}}\right\}\cdot B(\tau_p)$$

$$\cdot\prod_{b=\langle xy\rangle}\exp[v_b^{1/2}\sum_{(p,p')\wedge b}(\gamma_{p,x}\varepsilon\gamma_{p',x}\eta_{pp'}+\alpha^+_{p,y}\varepsilon^{-1}\alpha^+_{p',y}\bar{\eta}_{pp'})] \tag{4.3}$$

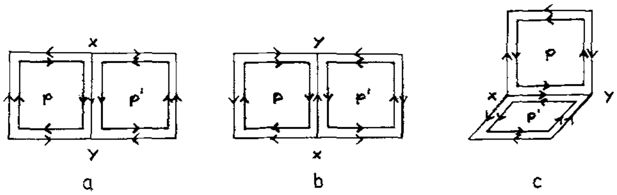

Fig. 3a–c. Possible relative orientations for a given link and two neighboring plaquettes

where

$$D\mu(\eta)=\prod_{(p,p')}\frac{1}{\pi}\,d\eta_{pp'}\,d\bar{\eta}_{pp'}\,e^{-\eta_{pp'}\bar{\eta}_{pp'}}.$$

The integration over ρ', z' amounts to the substitution [4]

$$\begin{aligned}&\bar{z}'^{i}_{p,y}\to\varepsilon^{ij}\tau_{p,y}\bar{z}^{j}_{p,y}\\&\bar{\rho}'^{i}_{p,x}\to\varepsilon^{ij}\tau_{p,x}\bar{\rho}^{j}_{p,x}\end{aligned}\tag{4.4}$$

within the exponent of the last line of (4.3).

Due to our choices for the orientation of links and plaquettes we have to distinguish three cases which are illustrated in Fig. 3.

The substitution (4.4) yields

$$\gamma_{p,x}\,\varepsilon\,\gamma_{p',x}\to\begin{cases}\rho_{p,x}\,\varepsilon\,\rho_{p',x} & \text{if } b\in\partial p,\partial p'\\ \tau_{p,x}\tau_{p',x}\rho^{+}_{p,x}\,\varepsilon\,\rho^{+}_{p',x} & \text{if } b^{-1}\in\partial p,\partial p'\\ \tau_{p',x}\rho_{p,x}\cdot\rho^{+}_{p',x} & \text{if } b\in\partial p, b^{-1}\in\partial p'\end{cases}$$

$$\alpha^{+}_{p,y}\,\varepsilon^{-1}\,\alpha^{+}_{p',y}\to\begin{cases}\tau_{p,y}\tau_{p',y}z^{+}_{p,y}\,\varepsilon^{-1}\,z^{+}_{p',y} & \text{if } b\in\partial p,\partial p'\\ z_{p,y}\,\varepsilon^{-1}\,z_{p',y} & \text{if } b^{-1}\in\partial p,\partial p'\\ \tau_{p,y}z^{+}_{p,y}\cdot z_{p',y} & \text{if } b\in\partial p, b^{-1}\in\partial p'.\end{cases}$$

Evidently, only variables belonging to the same site interact. Consequently, the ρ and z integrals factorize.

For each site $x\in\Lambda_i$ we introduce two antisymmetric matrices λ and κ

$$\lambda_{pp'}(x)=\begin{cases}v_b^{1/2}\,\eta_{pp'} & \text{if } x\in\partial p,\partial p'\\ & \text{and } b\in\partial p,\partial p' \text{ for any } b\\ 0 & \text{otherwise}\end{cases}$$

$$\kappa_{pp'}(x)=\begin{cases}v_b^{1/2}\,\tau_{p,x}\tau_{p',x}\eta_{pp'} & \text{if } x\in\partial p,\partial p'\\ & \text{and } b^{-1}\in\partial p,\partial p' \text{ for any } b\\ 0 & \text{otherwise}\end{cases}$$

and a matrix ρ

$$\rho_{pp'}(x)=\begin{cases}-v_b^{1/2}\,\tau_{p',x}\eta_{pp'} & \text{if } x\in\partial p,\partial p'\\ & \text{and } b\in\partial p, b^{-1}\in\partial p' \text{ for any } b\\ 0 & \text{otherwise.}\end{cases}$$

Analogously for each $y\in\Lambda_f$:

$$\lambda'_{pp'}(y)=\begin{cases}v_b^{1/2}\,\bar{\eta}_{pp'} & \text{if } y\in\partial p,\partial p'\\ & \text{and } b^{-1}\in\partial p,\partial p' \text{ for any } b\\ 0 & \text{otherwise}\end{cases}$$

$$\kappa'_{pp'}(y)=\begin{cases}v_b^{1/2}\,\tau_{p,y}\tau_{p',y}\bar{\eta}_{pp'} & \text{if } y\in\partial p,\partial p'\\ & \text{and } b\in\partial p,\partial p' \text{ for any } b\\ 0 & \text{otherwise}\end{cases}$$

$$\rho'_{pp'}(y)=\begin{cases}v_b^{1/2}\,\tau_{p',y}\bar{\eta}_{pp'} & \text{if } y\in\partial p,\partial p'\\ & \text{and } b^{-1}\in\partial p, b\in\partial p' \text{ for any } b\\ 0 & \text{otherwise.}\end{cases}$$

Thus the partition function takes the form

$$\begin{aligned}z=&\int Dv\,D\mu(\eta)\,D\tau\,D\mu(\rho,z)\\&\cdot\prod_{x\in\Lambda_i}\exp\sum_{pp'}\left(\tfrac{1}{2}\lambda_{pp'}\rho_p\,\varepsilon\,\rho_{p'}+\tfrac{1}{2}\kappa_{pp'}\rho_p^{+}\,\varepsilon^{-1}\rho_{p'}^{+}+\rho_{pp'}\rho_p\cdot\rho_{p'}^{+}\right)\\&\cdot\prod_{y\in\Lambda_f}\exp\sum_{pp'}\left(\tfrac{1}{2}\lambda'_{pp'}z_p\,\varepsilon\,z_{p'}+\tfrac{1}{2}\kappa'_{pp'}z_p^{+}\,\varepsilon^{-1}z_{p'}^{+}+\rho'_{pp'}z_p\cdot z_{p'}^{+}\right)\end{aligned}\tag{4.5}$$

where we have introduced the abbreviation

$$D\tau\equiv\prod_p\left\{\prod_{z\in\partial p}\frac{d\tau_{p,z}}{2\pi i}\right\}B(\tau_p).$$

The gaussian integrals over z and ρ are evaluated in the appendix. The result is the loop expansion

$$\begin{aligned}z=&\int Dv\,D\mu(\eta)\,D\tau\\&\cdot\prod_{x\in\Lambda_i}\exp\sum_C\frac{(-1)^{s_C}}{n_C}v(C)\,\tau(C)\,\eta(C)\\&\cdot\prod_{y\in\Lambda_f}\exp\sum_C\frac{(-1)^{s_C}}{n_C}v(C)\,\tau(C)\,\bar{\eta}(C)\end{aligned}\tag{4.6}$$

where

$$v(C)\equiv\prod_{\substack{b\wedge(p,p')\\(p,p')\in C}}v_b^{1/2},$$

$$\tau(C)\equiv\prod_{p\in C}\tau_{p,z},$$

$$\eta(C)\equiv\prod_{(p,p')\in C}\eta_{pp'}$$

s_C is the number of ordered pairs $(p,p')\in C$ with $b\in\partial p$, $b^{-1}\in\partial p'$, and n_C is the length of the path C in units of the lattice constant. For a given site $z\in\Lambda$, the sum in the exponent extends over all closed paths C consisting of ordered pairs (p_1,p_2), $(p_2,p_3)\ldots(p_n,p_1)$ with $(p_i,p_{i+1})\wedge b$ for any $b\ni z$ and obeying the restriction that p_i, p_{i+1}, and p_{i+2} do not share a common link. Paths which are cyclic permutations of one another are not identified.

The sums over paths C in the partition function (4.6) can be replaced by a sum over equivalence

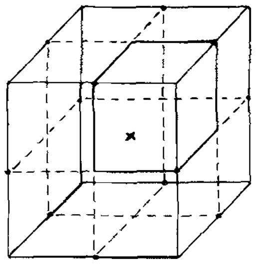

Fig. 4. Simplest allowed path on a cube of the dual lattice. × Original site; ● dual sites

classes $\tilde{C}$ of paths. We define two closed paths to be equivalent if they differ only in initial point or direction. Every path C can be represented uniquely in the form $C=C_0^{p_c}$ where C_0 is a simple closed path, i.e. cannot be expressed as a power of a path of lower order. This defines an integer p_c for every closed path C. The corresponding equivalence class contains $2n_c/p_c$ paths. Then the partition function can be rewritten as

$$z=\int Dv\, D\mu(\eta)\, D\tau$$
$$\cdot \prod_{x\in\Lambda_i} \exp 2\sum_{\tilde{C}} \frac{(-1)^{s_C}}{p_c} v(C)\,\tau(C)\,\eta(C)$$
$$\cdot \prod_{y\in\Lambda_f} \exp 2\sum_{\tilde{C}} \frac{(-1)^{s_C}}{p_C} v(C)\,\tau(C)\,\bar{\eta}(C). \tag{4.7}$$

Finally, we may replace the sum over all equivalence classes $\tilde{C}$ by a sum over all equivalence classes $\tilde{C}_0$ of simple paths only. The partition function eventually becomes

$$z=\int Dv\, D\mu(\eta)\, D\tau$$
$$\cdot \prod_{x\in\Lambda_i} \{\prod_{\tilde{C}_0} [1-(-1)^{s_C}\tau(C_0)\,v(C_0)\,\eta(C_0)]^{-2}\}$$
$$\cdot \prod_{y\in\Lambda_f} \{\prod_{\tilde{C}_0} [1-(-1)^{s_C}\tau(C_0)\,v(C_0)\,\bar{\eta}(C_0)]^{-2}\}. \tag{4.8}$$

In contrast to the $O(4)$ symmetric Heisenberg model [4], the paths are localized, i.e. cannot extend over the whole lattice. In fact, the allowed paths may be visualized on a "dual" lattice whose sites lie in the centers of the old plaquettes. If we draw a cube around each original site the new sites lie in the middle of the edges as shown in Fig. 4. An allowed path is confined to the surface of a single such "dual" cube. It connects "dual" sites in such a way that three subsequent sites never lie in a plane. If the paths are drawn along the dashed lines in Fig. 4 we have the equivalent condition that no backtracking paths are allowed.

The path displayed by a solid line in Fig. 4 is of lowest possible, i.e. third, order.

Appendix

Consider one gaussian integral of the partition function (4.5) belonging to a given site $x\in\Lambda_i$.

$$I\equiv\int\prod_{\substack{p\\ x\in\partial p}} d\mu(\rho_p)\exp\sum_{pp'}(\tfrac{1}{2}\lambda_{pp'}\,\rho_p\,\varepsilon\,\rho_{p'}$$
$$+\tfrac{1}{2}\kappa_{pp'}\,\rho_p^+\,\varepsilon^{-1}\,\rho_{p'}^+ +\rho_{pp'}\,\rho_p\cdot\rho_{p'}^+).$$

It is transformed into an integral over real variables by introducing vectors $r_p\in\mathbb{R}^4$ via

$$\rho_p=\begin{pmatrix} r_1+i\,r_2\\ r_3+i\,r_4\end{pmatrix}.$$

Furthermore, we introduce an antisymmetric block matrix C consisting of 4×4 blocks

$$C_{pp'}=\lambda_{pp'}\begin{pmatrix}0 & F\\ -F & 0\end{pmatrix}+\kappa_{pp'}\begin{pmatrix}0 & -F^*\\ F^* & 0\end{pmatrix}=-C_{p'p}$$

where

$$F=\frac{1}{2}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}$$

and a block matrix D consisting of antisymmetric 4×4 blocks

$$D_{pp'}=\rho_{pp'}\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}+\rho_{p'p}\begin{pmatrix}E^* & 0\\ 0 & E^*\end{pmatrix}$$

where

$$E=\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}.$$

The result of the gaussian integration is

$$I=\int\prod_p d^4r_p\, e^{-\sum_p (r_p, r_p)+\sum_{pp'}(r_p,[C+D]_{pp'}r_{p'})}$$
$$=[\det(1-C-D)]^{-1/2}$$
$$=\exp\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{tr}(C+D)^n \tag{A.1}$$

with

$$(r_p, r_q)=\sum_{i=1}^{4} r_p^i\, r_q^i.$$

The powers of D are given by

$$(D^n)_{pp'}=(\rho^n)_{pp'}\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}+(\rho^n)_{p'p}\begin{pmatrix}E^* & 0\\ 0 & E^*\end{pmatrix}. \tag{A.2}$$

We have to distinguish even and odd powers of C

$$(C^{2n})_{pp'}=(\lambda\,\kappa)^n_{pp'}\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}+(\kappa\,\lambda)^n_{pp'}\begin{pmatrix}E^* & 0\\ 0 & E^*\end{pmatrix},$$

$$(C^{2n+1})_{pp'}=[(\lambda\,\kappa)^n\,\lambda]_{pp'}\begin{pmatrix}0 & F\\ -F & 0\end{pmatrix}$$
$$+[(\kappa\,\lambda)^n\,\kappa]_{pp'}\begin{pmatrix}0 & -F^*\\ F^* & 0\end{pmatrix}. \tag{A.3}$$

For the trace in (A.1) not to vanish C has to appear an even number of times.

From the relations (A.3) and

$$[CD^n C]_{pp'} = (\lambda\, \rho^{T^n} \kappa)_{pp'} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + (\kappa\, \rho^n \lambda) \begin{pmatrix} E^* & 0 \\ 0 & E^* \end{pmatrix} \tag{A.4}$$

we see that λ and κ have to appear alternatingly where λ may be followed by powers of ρ^T and κ by powers of ρ. Thus, the nonvanishing terms in the exponential of (A.1) take the form

$$\mathrm{tr} \ldots \kappa\, \rho^{n_1} \lambda (\rho^T)^{n_2} \kappa\, \rho^{n_3} \lambda \ldots$$

These contributions may be represented by closed paths connecting the centers of neighboring plaquettes which have the point x in common.

With each oriented closed path C consisting of n_C ordered pairs of plaquettes $(p_1, p_2), (p_2, p_3) \ldots (p_{n_C}, p_1)$ we associate the algebraic expressions

$$v(C) = \prod_{\substack{b \wedge (p, p') \\ (p, p') \in C}} v_b^{1/2}$$

$$\tau(C) = \prod_{p \in C} \tau_{p,x}$$

$$\eta(C) = \prod_{(p, p') \in C} \eta_{pp'}. \tag{A.5}$$

Moreover, we define an integer s_C which is the number of pairs $(p, p') \in C$ with $b \in \partial p$, $b^{-1} \in \partial p'$.

Then (A.1) can be written as

$$I = \exp \sum_C \frac{(-1)^{s_C}}{n_C} v(C)\, \tau(C)\, \eta(C) \tag{A.6}$$

where the sum is over all closed paths C

$$\{C\} = \{(p_1, p_2) \ldots (p_n, p_1) \mid \text{for all } i,\ (p_i, p_{i+1}) \wedge b,\ b = \langle x\, y \rangle \text{ with } x \in \Lambda_i \text{ fixed, and } p_i, p_{i+1}, p_{i+2} \text{ do not share a common link}\}.$$

The integrals belonging to sites $y \in \Lambda_f$ are being treated in a similar way.

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