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Dirac–Kähler Fields and the Lattice Shape Dependence of Fermion Flavour

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Abstract. We investigate the Dirac–Kähler operator on a triangular lattice in two dimensions and show that the number of degrees of freedom which survive in the continuum limit is the same as in the case of a square lattice.

As is well known, the naive discretization of the Dirac action leads to a theory which describes more than one species of fermions in the continuum limit [1]. To eliminate this unwanted degeneracy several methods have been proposed [2–4]. Yet none of them seems to be completely satisfactory. Instead of trying to get rid of the species doubling one might attempt a physical interpretation of the additional degrees of freedom in terms of some kind of flavour. Such a procedure was suggested by Susskind in his approach to the problem [3]. Of course, this only makes sense, if the degree of degeneracy does not depend on the structure of the lattice. This requirement is not fulfilled for the naive discretization, as was shown by Chodos and Healy [5].

On the other hand, in the geometric treatment of fermions, which starts from the Dirac–Kähler equation and is equivalent to Susskind's method for a free field on a cubic lattice, the degeneracy is not produced by the discretization but is already present in the continuum [6, 7]. In addition, the geometric character of this approach makes the discretization essentially unique even for an arbitrary lattice. So one might have a chance to associate a physical meaning with the different fermion species.

In this note, we first collect some formulae for the analogues of various continuum concepts on an arbitrary simplicial lattice. We then study the Dirac–Kähler operator in the Euclidean formulation on a two-dimensional triangular lattice. In the Dirac–Kähler approach, one degree of freedom is associated with each cell (point, link, plaquette, ...) of the lattice. But although the number of cells per lattice point is four in the case of the square lattice and six for the triangular lattice (in two dimensions), we show that the number of degrees of freedom which survive the continuum limit is four also in the case of a triangular lattice. Therefore, in the Dirac–Kähler formalism the degeneracy on the lattice seems to coincide with the continuum degeneracy, independent of the lattice shape, as was to be expected from the geometric content of the method.

We start with giving the lattice analogues of some continuum concepts, which we need in our analysis [8, 9]. For a more detailed discussion of the mathematical background the reader should consult [6, 7]. Let C_p be an oriented p-cell of an n-dimensional simplicial lattice. A 0-cell is a lattice point, a 1-cell a link, etc. We describe the geometry of the lattice by the incidence function $I(C_p, C_{p+1})$, which is $+1$, if C_p is contained in C_{p+1} with the right orientation, -1, if C_p is contained in C_{p+1} with the opposite orientation, and 0 otherwise. Let $*C_p$ denote the $(n-p)$-cell of the dual lattice which is dual to C_p (see [8]). We define the incidence function of the dual lattice by

$$I(*C_{p+1}, *C_p) = I(C_p, C_{p+1}). \tag{1}$$

A p-cochain f_p is a real- or complex-valued function of p-cells, which is linearly extended to arbitrary linear combinations of p-cells (p-chains). Moreover, we set $f_p(C_q) = 0$ for $p \neq q$. To each p-cochain f_p on the lattice there corresponds a dual $(n-p)$-cochain $*f_p$ on the dual lattice defined by

$$(*f_p)(*C_p) = f_p(C_p)\frac{V(*C_p)}{V(C_p)}, \tag{2}$$

where $V(C_p)$ is the (p-dimensional) volume of C_p. Analogues d_L and δ_L of the continuum operators d (exterior derivative) and δ (coderivative) are given by the formulae

$$(d_L f_p)(C_{p+1}) = \sum_{C_p} I(C_p, C_{p+1}) f_p(C_p), \tag{3}$$

$$(\delta_L f_p)(C_{p-1}) = (*^{-1} d_L * f_p)(C_{p-1})$$
$$= \sum_{C_p} I(C_{p-1}, C_p) f_p(C_p) \frac{V(*C_p)V(C_{p-1})}{V(C_p)V(*C_{p-1})}. \quad (4)$$

In the continuum, we have a symmetric bilinear form $(,)$ for real valued p-forms:

$$(\omega, \omega') = \int \omega \wedge *\omega'. \quad (5)$$

With respect to this bilinear form, δ is the adjoint of d:

$$(d\omega, \omega') = (\omega, \delta\omega'). \quad (6)$$

On the lattice we define:

$$(f_p, g_p) = \sum_{C_p} \frac{V(*C_p)}{V(C_p)} f_p(C_p) g_p(C_p). \quad (7)$$

Then we have:

$$(d_L f_p, g_{p+1}) = (f_p, \delta_L g_{p+1}). \quad (8)$$

Furthermore, we set $(f_p, g_q) = 0$ if $p \neq q$. For complex-valued cochains we take the complex conjugate in the first argument.

If our lattice admits a symmetry operation S, we let it act on cochains according to the equation

$$(S^* f_p)(C_p) = f_p(S(C_p)). \quad (9)$$

S^* commutes with d_L, δ_L and consequently with the lattice Laplacian

$$\Delta_L = (d_L - \delta_L)^2 = -d_L\delta_L - \delta_L d_L. \quad (10)$$

Now we consider a two-dimensional triangular lattice with lattice constant a (see Fig. 1). The 0-cells are the lattice points denoted by $(\mathbf{x}, \phi)$, where

$$\mathbf{x} = a \sum_{j=1}^{3} n_j \mathbf{e}_j, \quad n_j \in \mathbb{Z}. \quad (11)$$

The unit vectors $\mathbf{e}_i$ are given by

$$\mathbf{e}_1 = (\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}), \quad \mathbf{e}_2 = (0, -1), \quad \mathbf{e}_3 = (-\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}). \quad (12)$$

They satisfy

$$\sum_{j=1}^{3} \mathbf{e}_j = 0. \quad (13)$$

The 1-cells are the links $(\mathbf{x}, j)$ connecting the lattice points $\mathbf{x}$ and $\mathbf{x} + a\mathbf{e}_j$ $(j = 1, 2, 3)$. Finally we have two types of 2-cells: the triangles $(\mathbf{x}, 12)$ to the right of $\mathbf{x}$ and the triangles $(\mathbf{x}, 32)$ to the left of $\mathbf{x}$ (see Fig. 2). Hence, to each lattice point there correspond six cells: one 0-cell, three 1-cells and two 2-cells, two cells more than in a square lattice.

The incidence function is non-zero only for the following arguments:

$$I[(\mathbf{x}, \phi), (\mathbf{x}, j)] = -1, \quad I[(\mathbf{x} + a\mathbf{e}_j, \phi), (\mathbf{x}, j)] = 1,$$
$$j = 1, 2, 3,$$
$$I[(\mathbf{x}, 1), (\mathbf{x}, 12)] = -1, \; I[(\mathbf{x} - a\mathbf{e}_3, 3), (\mathbf{x}, 12)] = -1,$$
$$I[(\mathbf{x} + a\mathbf{e}_1, 2), (\mathbf{x}, 12)] = -1, \quad (14)$$
$$I[(\mathbf{x} + a\mathbf{e}_3, 2), (\mathbf{x}, 32)] = 1,$$
$$I[(\mathbf{x}, 3), (\mathbf{x}, 32)] = 1,$$
$$I[(\mathbf{x} - a\mathbf{e}_1, 1), (\mathbf{x}, 32)] = 1.$$

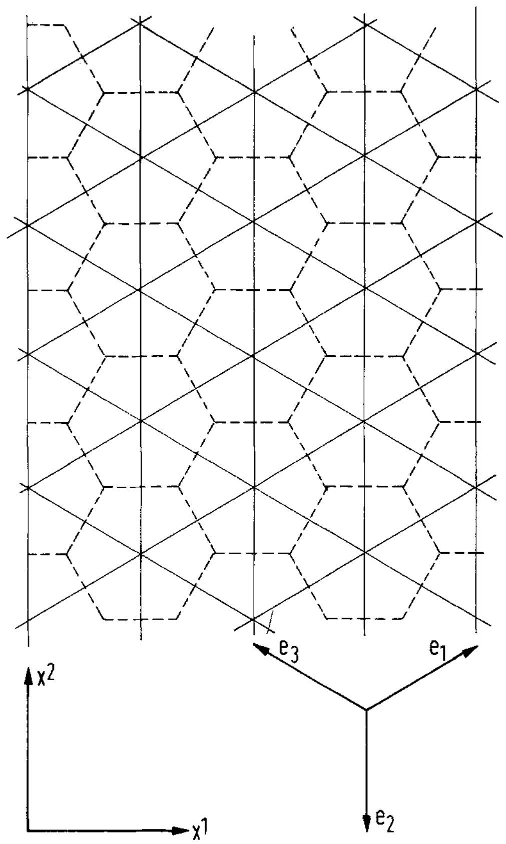

Fig. 1. The triangular lattice (solid lines) and the corresponding dual lattice (dashed lines). In addition, the coordinate axes and the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are shown

Fig. 2. The two types of 2-cells in the triangular lattice

The dual lattice consists of regular hexagons with edge length $a/\sqrt{3}$ (see Fig. 1). For the volumes of the cells we get:

$$V(\mathbf{x}, \phi) = 1, \qquad V(*(\mathbf{x}, \phi)) = \tfrac{1}{2}a^2\sqrt{3},$$
$$V(\mathbf{x}, j) = a, \qquad V(*(\mathbf{x}, j)) = a/\sqrt{3}, \quad (15)$$
$$V(\mathbf{x}, 12) = V(\mathbf{x}, 32) = \tfrac{1}{4}a^2\sqrt{3},$$
$$V(*(\mathbf{x}, 12)) = V(*(\mathbf{x}, 32)) = 1.$$

If we define elementary cochains $d^{\mathbf{x},H}$ according to

$$d^{\mathbf{x},H}(\mathbf{x}',H') = \delta_{\mathbf{x},\mathbf{x}'}\delta_{H,H'}V(\mathbf{x},H),$$
$$H,H' = \phi, 1, 2, 3, 12, 32, \tag{16}$$

we can write a general cochain as

$$\Phi = \sum_{\mathbf{x},H} \varphi(\mathbf{x},H) d^{\mathbf{x},H} \tag{17}$$

and find

$$d_L\Phi = \frac{1}{a}\sum_{\mathbf{x}}\left\{\sum_{j=1}^{3}[\varphi(\mathbf{x}+a\mathbf{e}_j,\phi)-\varphi(\mathbf{x},\phi)]d^{\mathbf{x},j} + \frac{4}{\sqrt{3}}\sum_{j=1}^{3}\varphi\left(\mathbf{x}-a\sum_{k=1}^{j}\mathbf{e}_k,j\right)d^{\mathbf{x},32} - \frac{4}{\sqrt{3}}\sum_{j=1}^{3}\varphi\left(\mathbf{x}+a\sum_{k=1}^{j-1}\mathbf{e}_k,j\right)d^{\mathbf{x},12}\right\}, \tag{18}$$

$$\delta_L\Phi = \frac{1}{a}\sum_{\mathbf{x}}\left\{\tfrac{2}{3}\sum_{j=1}^{3}[\varphi(\mathbf{x}-a\mathbf{e}_j,j)-\varphi(\mathbf{x},j)]d^{\mathbf{x},\phi} + \sqrt{3}\sum_{j=1}^{3}\left[\varphi\left(\mathbf{x}+a\sum_{k=1}^{j}\mathbf{e}_k,32\right) - \varphi\left(\mathbf{x}-a\sum_{k=1}^{j-1}\mathbf{e}_k,12\right)\right]d^{\mathbf{x},j}\right\}. \tag{19}$$

We now look for eigenvectors of the Dirac–Kähler operator $d_L-\delta_L$ in order to see, which of them belong to eigenvalues that remain finite in the continuum limit. It turns out to be advantageous to consider mainly the square of the Dirac–Kähler operator, the Laplacian. For the eigenvectors we make the ansatz

$$\varphi(\mathbf{x},H) = u(\mathbf{p},H)e^{i\mathbf{p}\cdot\mathbf{x}} \tag{20}$$

with the momentum $\mathbf{p}$ in the first Brillouin zone of our lattice. For two such plane waves

$$\Phi_j = \sum_{\mathbf{x},H} u_j(\mathbf{p}_j,H)e^{i\mathbf{p}_j\cdot\mathbf{x}}\, d^{\mathbf{x},H}, \quad j=1,2, \tag{21}$$

we have the scalar product

$$(\Phi_1,\Phi_2) = [u_1(\mathbf{p}_1,\phi)^* u_2(\mathbf{p}_2,\phi) + \tfrac{2}{3}\sum_{j=1}^{3} u_1(\mathbf{p}_1,j)^* u_2(\mathbf{p}_2,j) + \tfrac{1}{2}u_1(\mathbf{p}_1,12)^* u_2(\mathbf{p}_2,12) + \tfrac{1}{2}u_1(\mathbf{p}_1,32)^* u_2(\mathbf{p}_2,32)](2\pi)^2\delta(\mathbf{p}_1-\mathbf{p}_2). \tag{22}$$

With respect to this scalar product, the Dirac–Kähler operator is anti-hermitian, and consequently Δ_L is hermitian. So we use as normalization condition for our eigenvectors:

$$|u(\mathbf{p},\phi)|^2 + \tfrac{2}{3}\sum_{j=1}^{3}|u(\mathbf{p},j)|^2 + \tfrac{1}{2}|u(\mathbf{p},12)|^2 + \tfrac{1}{2}|u(\mathbf{p},32)|^2 = 1. \tag{23}$$

It will be interesting to study the transformation of the eigenvectors under the lattice point group, which is generated by R (rotation about $\pi/3$ around a lattice point) and S (reflection with respect to an axis in x^1-direction through a lattice point). We shall compare their properties with the behaviour of the eigenfunctions of the continuum Laplacian Δ under the same transformations. In the space of differential forms Δ has the eigenfunctions

$$\omega_0(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \omega_\mu(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{x}}\,dx^\mu, \quad \mu=1,2,$$
$$\omega(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{x}}\,dx^1\wedge dx^2. \tag{24}$$

They all belong to the eigenvalue $-\mathbf{p}^2$ and transform under R and S as follows:

$$\begin{aligned}
&R^*\omega_0(\mathbf{p}) = \omega_0(R^{-1}\mathbf{p}), \qquad S^*\omega_0(\mathbf{p}) = \omega_0(S^{-1}\mathbf{p}),\\
&R^*\omega_1(\mathbf{p}) = \tfrac{1}{2}\omega_1(R^{-1}\mathbf{p}) - \tfrac{1}{2}\sqrt{3}\omega_2(R^{-1}\mathbf{p}),\\
&S^*\omega_1(\mathbf{p}) = \omega_1(S^{-1}\mathbf{p}),\\
&R^*\omega_2(\mathbf{p}) = \tfrac{1}{2}\sqrt{3}\omega_1(R^{-1}\mathbf{p}) + \tfrac{1}{2}\omega_2(R^{-1}\mathbf{p}),\\
&S^*\omega_2(\mathbf{p}) = -\omega_2(S^{-1}\mathbf{p}),\\
&R^*\omega(\mathbf{p}) = \omega(R^{-1}\mathbf{p}), \qquad S^*\omega(\mathbf{p}) = -\omega(S^{-1}\mathbf{p}).
\end{aligned} \tag{25}$$

Turning now to the lattice Laplacian we have to distinguish the cases $\mathbf{p}=0$ and $\mathbf{p}\neq 0$. For $\mathbf{p}=0$ we get the eigenvectors

$$\begin{aligned}
&\hat{\Phi}_0 = \sum_{\mathbf{x}} d^{\mathbf{x},\phi} && \text{with eigenvalue } 0,\\
&\hat{\Phi}_1 = \tfrac{1}{2}\sqrt{3}\sum_{\mathbf{x}}(d^{\mathbf{x},1}-d^{\mathbf{x},3}) && 0,\\
&\hat{\Phi}_2 = \tfrac{1}{2}\sum_{\mathbf{x}}(d^{\mathbf{x},1}-2d^{\mathbf{x},2}+d^{\mathbf{x},3}) && 0,\\
&\hat{\Phi}_3 = \frac{1}{\sqrt{2}}\sum_{\mathbf{x}}(d^{\mathbf{x},1}+d^{\mathbf{x},2}+d^{\mathbf{x},3}) && -24a^{-2},\\
&\hat{\Phi}_+ = \sum_{\mathbf{x}}(d^{\mathbf{x},12}+d^{\mathbf{x},32}) && 0,\\
&\hat{\Phi}_- = \sum_{\mathbf{x}}(d^{\mathbf{x},12}-d^{\mathbf{x},32}) && -24a^{-2}.
\end{aligned} \tag{26}$$

Their transformation properties with respect to R and S are:

$$\begin{aligned}
&R^*\hat{\Phi}_0 = \hat{\Phi}_0, && S^*\hat{\Phi}_0 = \hat{\Phi}_0,\\
&R^*\hat{\Phi}_1 = \tfrac{1}{2}\hat{\Phi}_1 - \tfrac{1}{2}\sqrt{3}\hat{\Phi}_2, && S^*\hat{\Phi}_1 = \hat{\Phi}_1,\\
&R^*\hat{\Phi}_2 = \tfrac{1}{2}\sqrt{3}\hat{\Phi}_1 + \tfrac{1}{2}\hat{\Phi}_2, && S^*\hat{\Phi}_2 = -\hat{\Phi}_2,\\
&R^*\hat{\Phi}_3 = -\hat{\Phi}_3, && S^*\hat{\Phi}_3 = -\hat{\Phi}_3,\\
&R^*\hat{\Phi}_+ = \hat{\Phi}_+, && S^*\hat{\Phi}_+ = -\hat{\Phi}_+,\\
&R^*\hat{\Phi}_- = -\hat{\Phi}_-, && S^*\hat{\Phi}_- = -\hat{\Phi}_-.
\end{aligned} \tag{27}$$

So we see that, although there are six eigenvectors of Δ_L with $\mathbf{p}=0$, only four of them belong to eigenvalues which remain finite for $a\to 0$, and these are exactly those which transform analogously to the continuum eigenfunctions. The case $\mathbf{p}\neq 0$ is more complicated. It is convenient to introduce the following abbreviations:

$$\begin{aligned}
&\rho(\mathbf{p}) = \sum_{j=1}^{3} (\cos(a\mathbf{p}\cdot\mathbf{e}_j) - 1),\\
&\tau_j(\mathbf{p}) = (2\rho(\mathbf{p}) + 9)^{-1/4}\\
&\cdot[e^{ia\mathbf{p}\cdot\mathbf{e}_j} + e^{ia\mathbf{p}\cdot(\mathbf{e}_j-\mathbf{e}_l)} + e^{-ia\mathbf{p}\cdot\mathbf{e}_l}]^{1/2},\\
&\sigma_j(\mathbf{p}) = (2\rho(\mathbf{p}) + 9)^{-1/4}\\
&\cdot[e^{ia\mathbf{p}\cdot\mathbf{e}_j} + e^{ia\mathbf{p}\cdot(\mathbf{e}_j-\mathbf{e}_k)} + e^{-ia\mathbf{p}\cdot\mathbf{e}_k}]^{1/2}
\end{aligned} \tag{28}$$

with $(j, k, l) = (1, 2, 3)$ cyclic. The eigenvectors of Δ_L with $\mathbf{p} \neq 0$ are:

$$\begin{aligned}
&\Phi_0(\mathbf{p}) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} d^{\mathbf{x},\phi},\\
&\Phi_1(\mathbf{p}) = (-\tfrac{4}{3}\rho(\mathbf{p}))^{-1/2} \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \sum_{j=1}^{3} (1 - e^{ia\mathbf{p}\cdot\mathbf{e}_j}) d^{\mathbf{x},j},\\
&\Phi_2(\mathbf{p}) = (4 - \tfrac{4}{3}\sqrt{2\rho(\mathbf{p}) + 9})^{-1/2} \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\\
&\cdot \sum_{j=1}^{3} (\tau_j(\mathbf{p}) - \sigma_j(\mathbf{p})) d^{\mathbf{x},j},\\
&\Phi_3(\mathbf{p}) = (4 + \tfrac{4}{3}\sqrt{2\rho(\mathbf{p}) + 9})^{-1/2} \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\\
&\cdot \sum_{j=1}^{3} (\tau_j(\mathbf{p}) + \sigma_j(\mathbf{p})) d^{\mathbf{x},j},\\
&\Phi_\pm(\mathbf{p}) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} (\tau_1(\mathbf{p}) d^{\mathbf{x},12} \pm \tau_1(\mathbf{p})^* d^{\mathbf{x},32}).
\end{aligned} \tag{29}$$

For the corresponding eigenvalues we find:

$$\begin{aligned}
&\lambda_1(\mathbf{p}) = \lambda_0(\mathbf{p}) = \tfrac{4}{3}\rho(\mathbf{p}) a^{-2},\\
&\lambda_2(\mathbf{p}) = \lambda_+(\mathbf{p}) = (-12 + 4\sqrt{2\rho(\mathbf{p}) + 9}) a^{-2},\\
&\lambda_3(\mathbf{p}) = \lambda_-(\mathbf{p}) = (-12 - 4\sqrt{2\rho(\mathbf{p}) + 9}) a^{-2}.
\end{aligned} \tag{30}$$

One sees immediately that for $a \to 0$ with $\mathbf{p}$ fixed

$$\lambda_1(\mathbf{p}) \to -\mathbf{p}^2, \quad \lambda_2(\mathbf{p}) \to -\mathbf{p}^2, \quad \lambda_3(\mathbf{p}) \to -\infty. \tag{31}$$

So only four out of the six eigenvalues remain finite in the continuum limit, exactly as in the case $\mathbf{p} = 0$. But whereas

$$\begin{aligned}
&\lim_{\mathbf{p}\to 0} \Phi_0(\mathbf{p}) = \hat{\Phi}_0, \quad \lim_{\mathbf{p}\to 0} \Phi_3(\mathbf{p}) = \hat{\Phi}_3,\\
&\lim_{\mathbf{p}\to 0} \Phi_\pm(\mathbf{p}) = \hat{\Phi}_\pm,
\end{aligned} \tag{32}$$

the limit of $\Phi_\alpha(s\mathbf{p})$ as $s \to 0$ depends on $\mathbf{p}$ for $\alpha = 1, 2$. For $\mathbf{p} \neq 0$ all the eigenvectors are continuous.

Under the transformations R and S they behave as follows:

$$\begin{aligned}
&R^*\Phi_0(\mathbf{p}) = \Phi_0(R^{-1}\mathbf{p}), && S^*\Phi_0(\mathbf{p}) = \Phi_0(S^{-1}\mathbf{p}),\\
&R^*\Phi_1(\mathbf{p}) = \Phi_1(R^{-1}\mathbf{p}), && S^*\Phi_1(\mathbf{p}) = \Phi_1(S^{-1}\mathbf{p}),\\
&R^*\Phi_2(\mathbf{p}) = \Phi_2(R^{-1}\mathbf{p}), && S^*\Phi_2(\mathbf{p}) = -\Phi_2(S^{-1}\mathbf{p}),\\
&R^*\Phi_3(\mathbf{p}) = -\Phi_3(R^{-1}\mathbf{p}), && S^*\Phi_3(\mathbf{p}) = -\Phi_3(S^{-1}\mathbf{p}),\\
&R^*\Phi_\pm(\mathbf{p}) = \pm\Phi_\pm(R^{-1}\mathbf{p}), && S^*\Phi_\pm(\mathbf{p}) = -\Phi_\pm(S^{-1}\mathbf{p}).
\end{aligned} \tag{33}$$

Hence Φ_0 transforms like the continuum eigenfunction ω_0, and λ_0 tends to $-\mathbf{p}^2$ as $a \to 0$. Φ_+ transforms like ω, and the eigenvalue λ_+ remains finite for $a \to 0$. On the other hand, no eigenfunction of the continuum Laplacian Δ in the space of 2-forms, which is continuous as a function of $\mathbf{p}$, behaves like Φ_-, and $\lambda_- \to -\infty$ for $a \to 0$. So far everything is the same as in the case $\mathbf{p} = 0$. But in the space of 1-cochains the situation is different. The transformation behaviour of all of the eigenvectors Φ_1, Φ_2, Φ_3 has no counterpart among the continuous eigenfunctions of Δ in the space of 1-forms. Yet λ_1 and λ_2 remain finite, whereas $\lambda_3 \to -\infty$ for $a \to 0$. Moreover, Φ_1 and Φ_2 are not continuous at $\mathbf{p} = 0$. Nevertheless, one can easily find two orthonormal linear combinations of Φ_1 and Φ_2 which are continuous for all $\mathbf{p}$ and transform like the ω_μ. Of course, they are eigenvectors of the Laplacian only as $a \to 0$.

Finally we list the eigenvectors and eigenvalues of the Dirac–Kähler operator. For $\mathbf{p} = 0$, the eigenspace belonging to the fourfold eigenvalue 0 is spanned by $\hat{\Phi}_0$, $\hat{\Phi}_1$, $\hat{\Phi}_2$, $\hat{\Phi}_+$, and $2^{-1/2}(\hat{\Phi}_3 \pm i\hat{\Phi}_-)$ are eigenvectors with eigenvalues $\pm 2i\sqrt{6}/a$. In the case $\mathbf{p} \neq 0$ we get

$$\begin{aligned}
&2^{-1/2}(\Phi_0(\mathbf{p}) \pm i\Phi_1(\mathbf{p})) \text{ with eigenvalue } \pm i\sqrt{-\lambda_1(\mathbf{p})},\\
&2^{-1/2}(\Phi_2(\mathbf{p}) \pm i\Phi_+(\mathbf{p})) \qquad \pm i\sqrt{-\lambda_2(\mathbf{p})},\\
&2^{-1/2}(\Phi_3(\mathbf{p}) \pm i\Phi_-(\mathbf{p})) \qquad \pm i\sqrt{-\lambda_3(\mathbf{p})}.
\end{aligned} \tag{34}$$

These eigenvectors can be discussed in the same way as those of the Laplacian treated above.

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