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A Lattice Version of the Wess-Zumino Model[1]

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Abstract. A lattice Lagrangian of the Wess–Zumino model is constructed using perturbation theory up to two loops. It is shown that the renormalized vertex-functions have the correct continuum limit if nonsupersymmetric counterterms up to dimension four are added to the Lagrangian. The structure of these terms is analysed with the Wilson prescription for the fermions.

1. Introduction

Supersymmetric field theories [1] have become popular in connection with attempts to go beyond the standard $SU(3) \times SU(2) \times U(1)$-model of strong, weak, and electromagnetic interactions. In particular it is hoped that supersymmetry might be able to explain why the mass scale of weak interactions (i.e. the mass of Higg's scalars) is so small in comparison with the Planck scale [2]. It is, however, clear that at low energies supersymmetry cannot be exact, and some mechanism would have to be responsible for the breaking of supersymmetry. As a consequence of certain nonrenormalization theorems [3], breaking of supersymmetry either has to be put in by an appropriate choice of Higgs parameters or has to come from nonperturbative effects. The latter alternative seems to be realized in certain two-dimensional models [4]. Witten's theorem [5], on the other hand says that this cannot be realized in a large class of theories.

All this makes it highly desirable to know more about nonperturbative aspects of supersymmetric field theories. During the last years it has turned out to be extremely useful to formulate field theories on the lattice and then to use either analytical methods (e.g. strong coupling expansion, mean field theory) or numerical methods (Monte Carlo computer calculations). In the context of gauge field theories [6] these tools have given both qualitative insight into the phase structure and numerical results for low-lying states of the spectrum. It is therefore tempting to try similar calculations for supersymmetric field theories. Several attempts have been made in order to find a lattice formulation of supersymmetric field theories. They all start from the requirement that such a lattice version should still be manifestly invariant under the supersymmetry algebra (or at least a part of it). Since, however, the (continuum) algebra of supersymmetry contains the generators of translations and Lorentz rotations which on the lattice do not exist such a requirement could be expected to be too strong. Nicolai and Dondi [7] kept the superfield formalism, but they concluded that either one has to give up the supersymmetry algebra on the lattice or one is forced into a form of the derivative operator which contains long range correlations (SLAC derivative [8]). Banks and Windey [9] and, later on, Rittenberg and Yankielowicz [10] defined a modified version of the supersymmetry algebra ("lattice supersymmetry") which could be kept on the lattice. These attempts, however, lead to lattice models which in the continuum limit are not fully Lorentz invariant. Recently Elitzur, Rabinovici and Schwimmer [11], continuing along these lines, succeeded in finding a lattice version of $N=2$ extended supersymmetry (their method also works for the Wess–Zumino model in 2 dimensions, but in 4 dimensions not for $N=1$ supersymmetry): whether this model in the limit of vanishing lattice spacing agrees with the continuum theory, however, has not yet firmly been established.

In view of all these difficulties which seem to arise because some manifest symmetry properties on the lattice are imposed, it seems useful to approach the problem from a somewhat different angle. From Symanzik's [12] formulation of the "local effective Lagrangian" it is known that a lattice action must exist, such that its Green's functions and vertexfunctions, in the limit of zero lattice spacing, coincide with those of the continuum theory (in the sense of perturbation

[1] Dedicated to Rudolf Haag on the occasion of his 60th birthday

theory). The question then is: how much supersymmetry will this lattice action have? The results mentioned above make it probable that part of the symmetry requirements have to be given up, but a priori it is not clear, to what extent this has to be done and what form the lattice action will have.

It is the aim of this paper to find an answer to these questions. More precisely, we shall construct a lattice action of the Wess–Zumino model [13] in such a way that its (renormalized) vertexfunctions have the correct continuum limit. This then guarantees that they will satisfy the Ward identities which are a manifestation of the supersymmetry invariance. We work in perturbation theory, use Wilson's method for the fermions and our conclusions will be based on the two-loop approximation.

Our result for the lattice action confirms our expectation that on the lattice supersymmetry cannot be maintained, for several reasons. First, there is the effect which has been noticed already in [7] and which is not subject of this paper: since the nearest-neighbor derivative operator does not obey the product rule of differentation, already the tree approximation is not invariant under supersymmetry. The new effect we find is that *quantum corrections leed to finite* (in the limit of vanishing lattice spacing) *deviations from the continuum theory.* This must be compensated by a readjustment of the bare parameters in the action, and this correction does not respect the equality of bosonic and fermionic mass terms (and also of couplings between bosons and fermions). Furthermore we find that the correct continuum limit holds only if we add new operators of dimensions $\leqq 4$ to the Lagrangian which are not present in the continuum theory. Their coefficients, which we can calculate only in our framework of perturbation theory, are strongly dependent upon the Wilson parameter r. It therefore seems as if the presence of these new operators is closely related to the way in which we deal with the problem of additional fermionic degrees of freedom on the lattice.

As we have already said, our considerations are entirely restricted to perturbation theory: we calculate vertexfunctions in the two-loop approximation, and it is also only in this approximation that we can explicitly construct the coefficients of the operators in the lattice action. This will be of little use for practical purposes. Since the Wess–Zumino model is known [15] to be not asymptotically free, it is clear that the limit of zero lattice spacing is not going to drive us into the weak coupling regime where our results are quantitatively valid. This implies that our final result for the lattice action is only a qualitative one: couplings and masses have to be chosen in such a way that the vertexfunctions satisfy the requirements of supersymmetry (Ward identities). The way in which these functions depend upon the input parameters is, outside of perturbation theory, not known, but one might try to use other approximation methods. The question how our results can be used for practical calculations will be left to future work.

Our paper will be organized as follows. In Sect. II we briefly review the main features of the Wess–Zumino model in the continuum, and we then define our lattice action. In Sect. III we present one-loop calculations on the lattice and study the limit of vanishing lattice spacing. This already brings our main result: agreement of the renormalized vertexfunctions with the continuum theory is achieved only if bare masses and coupling constants are properly adjusted. This way we get counter terms in the Lagrangian which break supersymmetry explicitly. In sect. IV we describe two-loop calculations. They confirm the results of sect. III, but also lead to the new operators in the lattice Lagrangian. In sect. V we summarize our results and draw a few conclusions.

II. The Model

Before we define our model we first want to review a few features of the Wess–Zumino model [13] in the continuum. In particular we wish to emphasize those properties which will be affected by the lattice regularization. The Wess–Zumino model is defined through the following Lagrangian:

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_0 + \mathscr{L}_m + \mathscr{L}_g \\ &= -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 + \tfrac{1}{2}\bar{\psi}\gamma\cdot\partial\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \\ &+ m(AF + BG - \tfrac{1}{2}\bar{\psi}\psi) \\ &+ g(FA^2 - FB^2 + 2GAB - \bar{\psi}(A + i\gamma_5 B)\psi) \qquad (2.1)\end{aligned}$$

Here A, B are scalar and pseudoscalar fields, resp., and F, G are auxiliary fields which could be eliminated via the equations of motion. We prefer to keep the auxiliary fields. We work in Euklidean space with $g_{\mu\nu} = -\delta_{\mu\nu}$, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, $\gamma_5^2 = 1$, and ψ is a Majorana spinor analytically continued from Minkowsky to Euklidean space. The Lagrangian (2.1) is invariant (up to four-divergences) under supersymmetry transformations. This invariance can be used [3] to derive a set of Ward identities between unrenormalized (but suitably regularized) Greens functions or vertexfunctions. For example, if we denote the (amputated) two-point vertexfunctions by Γ_{AA}, Γ_{AF}, Γ_{FF}, $\Gamma_{\psi\psi} = \gamma\cdot p\Gamma^{(1)}_{\psi\psi} + \Gamma^{(2)}_{\psi\psi}$ the following equations hold:

$$\begin{aligned}\Gamma_{AA}(p^2) &= \Gamma_{BB}(p^2) = p^2\Gamma^{(1)}_{\psi\psi} = -p^2\Gamma_{FF}(p^2) \\ &= -p^2\Gamma_{GG}(p^2) \qquad (2.2)\end{aligned}$$

$$\Gamma_{AF}(p^2) = \Gamma_{BG}(p^2) = \Gamma^{(2)}_{\psi\psi}(p^2) \qquad (2.3)$$

From the first set of equations it follows that $\Gamma_{AA}(0) = \Gamma_{BB}(0) = 0$. Relations similar to (2.2) and (2.3) can be derived for 3-point and 4-point functions. For the vertices A^3 and A^4 they are of the form:

$$\Gamma_{AAA} \sim \gamma\cdot p\Gamma_{\psi\bar{\psi}A}, \Gamma_{AAAA} \sim \gamma\cdot p\Gamma_{\psi\bar{\psi}AA} \qquad (2.4)$$

which again implies that Γ_{AAA} and Γ_{AAAA} vanish at zero external momenta.

As Iliopoulos and Zumino [3] have shown, this model has very simple renormalization properties: first, quadratic divergences always cancel and only logarithmic divergences remain, secondly, provided a supersymmetric regularization scheme is used, there is only one common wave function renormalization constant:

$$Z^{-1} = \Gamma'_{AA}(0) = \Gamma'_{BB}(0) = \Gamma^{(1)}_{\psi\psi}(0) = -\Gamma_{FF}(0) = -\Gamma_{GG}(0) \tag{2.5}$$

As a result of this, the Ward identities (2.2)–(2.4) also hold for the renormalized vertexfunctions ($\Gamma^R_{AA} = Z\Gamma_{AA}$, $A_R = Z^{-1/2} \cdot A$ etc.)

$$\Gamma^R_{AA} = \Gamma^R_{BB} = p^2\Gamma^{(1)R}_{\psi\psi} = -p^2\Gamma^R_{FF} = -p^2\Gamma^R_{GG} \tag{2.6}$$

$$\Gamma^R_{AF} = \Gamma^R_{BG} = \Gamma^{(2)R}_{\psi\psi} \tag{2.7}$$

$$\Gamma^R_{AAA} \sim \gamma \cdot p\Gamma^R_{\psi\psi A}, \Gamma^R_{AAAA} \sim \gamma \cdot p\Gamma^R_{\psi\psi AA} \tag{2.8}$$

Furthermore, renormalization of masses and coupling constants only comes from the wave function renormalization:

$$\Gamma^R_{AF}(0) = \Gamma^R_{BG}(0) = \Gamma^{(2)R}_{\psi\psi}(0) = m_R \tag{2.9}$$

$$m_R = Z \cdot m \tag{2.10}$$

$$\Gamma^R_{FA^2}(0,0,0) = -\Gamma^R_{FB^2}(0,0,0) = \Gamma^R_{GAB}(0,0,0) = -\Gamma^R_{\bar\psi A\psi}(0,0,0) = -i\Gamma_{\bar\psi B\psi}(0,0,0) = g_R \tag{2.11}$$

$$g_R = Z^{3/2} g \tag{2.12}$$

The way in which (2.9)–(2.12) are realized in perturbation theory is very simple. In the tree approximation, (2.9) and (2.11) are trivially correct. Loop corrections to any of the 2-point or 3-point functions of (2.9) and (2.11) are ultraviolet finite and vanish, when the external momenta are taken to zero. The same also happens with those other vertexfunctions which, by dimensional arguments, could be logarithmic divergent: $\Gamma_{A^3}, \Gamma_{AB^2}, \Gamma_{A^4}, \Gamma_{B^4}$, and $\Gamma_{A^2B^2}$. This is the content of (2.4) and (2.8). That renormalization preserves sypersymmetry can also be seen by writing down the necessary counter term:

$$\mathscr{L}_R = \mathscr{L}_{0R} + \mathscr{L}_{mR} + \mathscr{L}_{gR} + \Delta\mathscr{L}_{0R} \tag{2.13}$$

$$\Delta\mathscr{L}_{0R} = (Z-1)\mathscr{L}_{0R} \tag{2.14}$$

Here $\mathscr{L}_{0R}$, $\mathscr{L}_{mR}$ and $\mathscr{L}_{gR}$ are the same as in (2.1), with the bare parameters g, m and the fields A, B, F, G, ψ being replaced by their renormalized partners g_R, m_R, A_R, B_R, F_R, G_R, ψ_R, respectively.

For the following it will be useful to compare the renormalization properties of (2.1) with a model which has the same terms as (2.1) but not necessarily equal masses and coupling constants. Such a theory has quadratic divergencies but is still renormalizable, and each field now has its own wave function renormalization constant Z_A, Z_F, and Z_ψ. Furthermore, renormalization would require counterterms for masses, coupling constant renormalization constants, as well as new operators A^2, A^3, A^4 which are not present in the original Lagrangian. Most important, in such a theory it would, in general, not be possible to satisfy the Ward identities (2.2)–(2.4) or (2.6)–(2.8).

As a result of the analysis given below we will show that a lattice-regularized version of (2.1) has a general form, which is quite similar, but where it is possible to adjust the parameters of all the interaction terms in such a way that the renormalized vertexfunctions in the limit $a \to 0$ agree with those of the continuum theory. All this will be done in perturbation theory: the coefficients must then be adjusted order by order of renormalized perturbation theory. As a starting point, we choose the following lattice analogue of (2.1):

$$S = a^4 \sum_x (\mathscr{L}_0 + \mathscr{L}_m + \mathscr{L}_g) \tag{2.15}$$

$$\begin{aligned}\mathscr{L}_0 = &-\frac{1}{2}\frac{1}{4a^2}\sum_\mu (A(x+ae_\mu) - A(x-ae_\mu))^2 \\ &-\frac{1}{2}\frac{1}{4a^2}\sum_\mu (B(x+ae_\mu) - B(x-ae_\mu))^2 \\ &-\frac{1}{2ia}\sum_\mu (\bar\psi(x)\gamma_\mu\psi(x+ae_\mu) - \bar\psi(x+ae_\mu)\gamma_\mu\psi(x)) \\ &+\tfrac{1}{2}F^2 + \tfrac{1}{2}G^2\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathscr{L}_m = &\, m(FA + GB - \tfrac{1}{2}\bar\psi\psi) \\ &+\frac{r}{2a}\Big[\tfrac{1}{2}\sum_\mu (\bar\psi(x)\psi(x+ae_\mu) + \bar\psi(x+ae_\mu)\psi(x) - 2\bar\psi(x)\psi(x)) \\ &-\sum_\mu (F(x)A(x+ae_\mu) + F(x+ae_\mu)A(x) - 2F(x)A(x)) \\ &-\sum_\mu (G(x)B(x+ae_\mu) + G(x+ae_\mu)B(x) \\ &-2G(x)B(x))\Big]\end{aligned} \tag{2.17}$$

$$\mathscr{L}_g = g[F(A^2 - B^2) + 2GAB - \bar\psi(A + i\gamma_5 B)\psi] \tag{2.18}$$

We have chosen to use the Wilson [14] method of dealing with the additional superficial fermionic degrees of freedom caused by the lattice. In order to preserve as much as possible of supersymmetric invariance, we then are forced to treat the other fields in the same way. For the same reason we use the symmetric form of the derivative operator for the scalar fields. In the following we shall see that this is necessary for cancelling divergencies between fermionic diagrams and diagrams with scalar fields. Finally we mention that, although (2.15)–(2.18) looks rather supersymmetric, the action (2.15) is not invariant under supersymmetry transformations: in the continuum version (2.1) supersymmetry transformations generate total derivatives which do not contribute to the action integral. On the lattice terms remain, since the derivative operators of (2.16) do not obey the usual product rules of differentiation.

We then conclude this section by listing the propagators of lattice perturbation theory:

$$G_{AA} = G_{BB} = [M^2(ak) + F^2(ak)]^{-1} \tag{2.19}$$

$$G_{FF} = G_{GG} = -F^2(ak)[M^2(ak) + F^2(ak)]^{-1} \tag{2.20}$$

$$G_{AF} = G_{BG} = -M(ak)[M^2(ak) + F^2(ak)]^{-1} \qquad ((2.21)$$

$$G_{\psi\psi} = [M(ak) + \gamma \cdot F(ak)]^{-1} = (M(ak) - \gamma \cdot F(ak)) \cdot [M^2(ak) + F^2(ak)]^{-1} \qquad (2.22)$$

where:

$$M(ak) = m + \frac{r}{a}\sum_\mu (1 - \cos ak_\mu) \qquad (2.23)$$

$$F^2(ak) = \sum_\mu \frac{\sin^2 ak_\mu}{a^2} \qquad (2.24)$$

$$\gamma \cdot F(ak) = \sum_\mu \gamma_\mu \cdot \frac{\sin ak_\mu}{a} \qquad (2.25)$$

We see that the usual doubling problem of the spectrum which would be present for $r = 0$ is cured in a symmetric way for bosons and fermions by additional Wilson terms in all propagators. It is clear that $r > 0$ is required.

In the following we shall calculate from (2.15)–(2.18) vertexfunctions, study their limit $a \to 0$ and then improve [16] the action by adding counterterms in such a way that the renormalized vertexfunctions have the correct continuum limit.

III. One-Loop Calculations

In this section we calculate all those vertexfunctions in the one loop approximation, which, because of possible divergencies in the limit $a \to 0$, have a chance to disagree with the continuum theory.

We begin with the two-point vertexfunctions. The most dangerous one is the AA-vertex (Fig. 1). In the continuum theory (2.1) some of the graphs of Fig. 1 diverge quadratically, but this divergence cancels in the sum of all diagrams. The same cancellation occurs in our lattice formulation because of our particular choice of lattice derivatives for scalar fields. We find for the sum of these graphs:

$$\Gamma_{AA(\text{one-loop})} = 4g^2 \sum_\mu \left(2\frac{\sin(ap_\mu/2)}{a}\right)^2 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \cdot \frac{\cos^2(a(2k_\mu - p_\mu)/2)}{D(ak)D(ap - ak)} \qquad (3.1)$$

where

$$D(ak) = M^2(ak) + F^2(ak) \qquad (3.2)$$

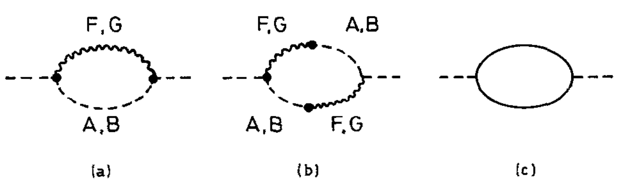

Fig. 1a–c. Diagrams contributing to the two-point function Γ_{AA}. The dotted line denotes scalar particle A or B, the wavy line denotes the auxiliary fields F or G, and the straight line denotes the fermion

For calculating the limit $a \to 0$ (p fixed) [17] we rescale the integration momentum: $k_\mu \to k'_\mu = ak_\mu$, and approximate the factor in front of the integral:

$$(3.1) \underset{a\to 0}{\sim} 4g^2 \sum_\mu p_\mu^2 \int_{-\pi}^{\pi} \frac{d^4k \cos^2((2k_\mu - ap_\mu)/2)}{(2\pi)^4 a^2 D(k) a^2 D(ap - k)} \qquad (3.3)$$

In the limit $a \to 0$ this integral digerges logarithmically near $k_\mu = 0$. After combining the denominators and shifting the integration variable $k_\mu \to l_\mu = k_\mu - (1-x)ap_\mu$, we divide the region of l-integration into two pieces: $|l| < \delta$ and $|l| > \delta$ (δ small but fixed). In the small-l region, we only keep leading terms near $l_\mu = 0$; in the large-l part, the limit $a \to 0$ can be taken immediately. All this leads to:

$$\int_{-\pi}^{\pi} \frac{d^4k \cos^2((2k_\mu - ap_\mu)/2)}{(2\pi)^4 a^2 D(k) a^2 D(ap - k)} = \int_0^1 dx \int_{|l|<\delta} \frac{d^4l}{(2\pi)^4} \cdot \frac{1}{[l^2 + a^2(m^2 + x(1-x)p^2) + O(l^3)]^2} + \int_{|l|>\delta} \frac{d^4l}{(2\pi)^4} \cdot \frac{\cos^2 l_\mu}{[r^2(1 - \sum_\mu \cos l_\mu) + a^2F^2(l)]^2} + O(a) \qquad (3.4)$$

Direct evaluation of the first integral on the right-hand side yields:

$$\Gamma_{AA(\text{one-loop})} = -4g^2p^2 \left\{\frac{\ln a^2}{16\pi^2} + \frac{1}{16\pi^2} \cdot \left[\int_0^1 dx \ln(m^2 + x(1-x)p^2) + 1\right] - \sigma_A\right\} \qquad (3.5)$$

where

$$\sigma_A = \frac{\ln\delta^2}{16\pi^2} + \frac{1}{4}\int_{|l|>\delta} \frac{d^4l}{(2\pi)^4} \frac{\sum_\mu \cos^2 l_\mu}{(D_0(l))^2} \qquad (3.6)$$

$$D_0(l) = r^2\Big(\sum_\mu (1 - \cos l_\mu)\Big)^2 + a^2F^2(l) \qquad (3.7)$$

Equation (3.5) is correct up to terms $\sim \delta$: since δ can be taken arbitrarily small and the result is finite at $\delta = 0$, we now can put $\delta \to 0$. With the renormalization condition (2.5) we find:

$$Z_A^{-1} - 1 = -4g^2\left\{\frac{\ln a^2}{16\pi^2} + \frac{1}{16\pi^2}(\ln m^2 + 1) - \sigma_A\right\} \qquad (3.8)$$

and

$$\Gamma_{AA}^R = p^2\left(1 - 4g^2 \frac{1}{16\pi^2}\int_0^1 dx \ln(1 + x(1-x)p^2/m^2)\right) \qquad (3.9)$$

This agrees with the continuum theory. For later purpose we mention that (3.4)–(3.9) are valid only in the limit $a \to 0$, $ap \to 0$. Had we chosen to take $a \to 0$ with ap fixed, (3.5) would have to be replaced by:

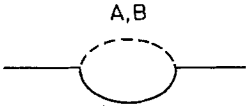

Fig. 2. Diagram which contributes to the two-point function $\Gamma_{\psi\psi}$

$$(3.1) \underset{\substack{a\to 0\\ ap\neq 0,\text{fixed}}}{\sim} 4g^2\frac{1}{a^2}\sum_\mu\left(2\sin\frac{ap_\mu}{2}\right)^2\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\cdot\frac{\cos^2((2k_\mu-ap_\mu)/2)}{D_0(k)D_0(ap-k)} \quad (3.10)$$

Next we study the ψ-field. The relevant graphs are shown in Fig. 2 and their formula is:

$$\Gamma_{\psi\psi\,(\text{one-loop})} = 4g^2\sum_\mu\gamma_\mu\frac{2\sin(ap_\mu/2)}{a}\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}\cdot\frac{\cos((2k_\mu-p_\mu)a/2)}{D(ak)D(ap-ak)} \quad (3.11)$$

The limit $a\to 0$ is obtained in the same way as for the A-field:

$$(3.11)\underset{a\to 0}{\sim} -4g^2\gamma p\left\{\frac{\ln a^2}{16\pi^2}+\frac{1}{16\pi^2}\cdot\left[\int_0^1 dx\ln(m^2+x(1-x)p^2)+1\right]-\sigma_\psi\right\} \quad (3.12)$$

where

$$\sigma_\psi=\frac{\ln\delta^2}{16\pi^2}+\frac{1}{4}\int_{|l|>\delta}\frac{d^4k}{(2\pi)^4}\frac{\sum_\mu\cos l_\mu}{(D_0(l))^2} \quad (3.13)$$

The renormalization condition (2.5) leads to:

$$Z_\psi^{-1}-1=-4g^2\left\{\frac{\ln a^2}{16\pi^2}+\frac{1}{16\pi^2}(\ln m^2+1)-\sigma_\psi\right\} \quad (3.14)$$

$$\Gamma_{\psi\psi}^{(1)R}=1-4g^2\frac{1}{16\pi^2}\int_0^1 dx\ln(1+x(1-x)p^2/m^2) \quad (3.15)$$

Equation (3.15) again agrees with the continuum theory, but it is important to note that Z_ψ (3.14) does not agree with Z_A (3.8). They differ by the constants σ_A and σ_ψ which depend upon the large-momentum behavior of lattice propagators. The implications of this will be discussed later. Since (3.12) is proportional to $\gamma\cdot p$, there is no contribution to $\Gamma_{\psi\psi}^{(2)}$ in this one-loop correction. We therefore have:

$$\Gamma_{\psi\psi}^{(2)}=m \quad (3.16)$$

and

$$\Gamma_{\psi\psi}^{(2)R}=Z_\psi m \quad (3.17)$$

For the F-field we proceed in the same way (Fig. 3). We find:

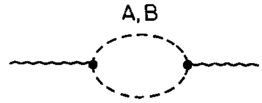

Fig. 3. The two-point function Γ_{FF}

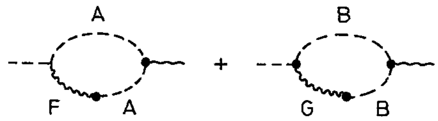

Fig. 4. The two-point function Γ_{AF}

$$\Gamma_{FF\,(\text{one-loop})} = -4g^2\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}\frac{1}{D(ak)D(ap-ak)} \quad (3.18)$$

$$\underset{a\to 0}{\sim} 4g^2\left\{\frac{\ln a^2}{16\pi^2}+\frac{1}{16\pi^2}\cdot\left[\int_0^1 dx\ln(m^2+x(1-x)p^2)+1\right]-\sigma_F\right\} \quad (3.19)$$

$$\sigma_F=\frac{\ln\delta^2}{16\pi^2}+\frac{1}{16\pi^2}\int_{|l|>\delta}\frac{d^4l}{(2\pi)^4}\frac{1}{D_0(l)} \quad (3.20)$$

$$Z_F^{-1}-1=-4g^2\left\{\frac{\ln a^2}{16\pi^2}+\frac{1}{16\pi^2}(\ln m^2+1)-\sigma_F\right\} \quad (3.21)$$

$$\Gamma_{FF}^R=-(1-4g^2\frac{1}{16\pi^2}\int_0^1 dx\ln(1+x(1-x)p^2/m^2)) \quad (3.22)$$

Equation (3.22) is in agreement with the continuum theory, but the wave function renormalization constant again differs from both Z_A and Z_ψ. Finally, all graphs which contribute to the transition vertex AF (Fig. 4), add up to zero. This leaves us with:

$$\Gamma_{AF}=m \quad (3.23)$$

and

$$\Gamma_{AF}^R=(Z_AZ_F)^{1/2}m \quad (3.24)$$

In the continuum theory the Ward identities require that the renormalized mass of the ψ-field and of the AF-transition vertex are identical (2.7) and (2.9). In our lattice version, the results (3.17) and (3.24) teach us that the renormalized masses can be made equal only by an appropriate change of the bare mass parameters: in the action (2.15) the (common) mass parameter m has to be replaced by:

$$m_\psi=m_RZ_\psi^{-1} \quad (3.25)$$

$$m_{AF}=m_R(Z_AZ_F)^{-1/2} \quad (3.26)$$

This is the main finding of this section: *in order that the renormalized vertexfunctions have the correct $a\to 0$ limit* (and, hence, satisfy the Ward identities), *we have to break the symmetry of the bare parameters.* In the

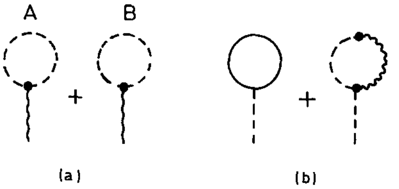

Fig. 5a and b. Tadpole graphs in the one-loop approximation

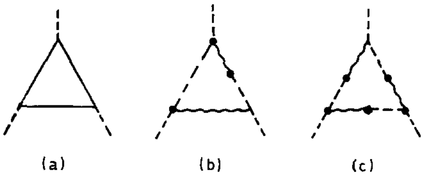

Fig. 6a–c. The three-point function Γ_{AAA} in the one-loop approximation

remainder of this section we shall show that, in the one-loop approximation, all symmetry breaking resides in the different wave function renormalization constants.

We conclude the discussion of the two-point functions by mentioning that the results for the pseudoscalar fields B and G are the same as those for the scalar fields A and F and that the tadpole graphs (Fig. 5) add up to zero. Our discussion so far then leads to the following counter terms in the Lagrangian (2.16)–(2.18):

$$\begin{aligned}\Delta\mathcal{L}_{0R} = & -\frac{1}{2}\frac{1}{4a^2}(Z_A-1)\sum_\mu(A_R(x+ae_\mu) \\ & - A_R(x-ae_\mu))^2 \\ & -\frac{1}{2}\frac{1}{4a^2}(Z_B-1)\sum_\mu(B_R(x+ae_\mu)-B_R(x-ae_\mu))^2 \\ & -\frac{1}{2ia}(Z_\psi-1)\sum_\mu(\bar\psi_R(x)\gamma_\mu\psi_R(x+ae_\mu)-\bar\psi_R(x+ae_\mu) \\ & \cdot\gamma_\mu\psi_R(x)) + \tfrac{1}{2}(Z_F-1)F_R^2 + \tfrac{1}{2}(Z_G-1)G_R^2 \end{aligned} \tag{3.27}$$

The renormalization constants Z_A, Z_F and Z_ψ are (for finite a):

$$Z_A^{-1} = Z_B^{-1} = 1 + 4g_R^2 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \frac{\frac{1}{4}\sum_\mu \cos^2 ak_\mu}{(D(ak))^2} \tag{3.28}$$

$$Z_F^{-1} = Z_G^{-1} = 1 + 4g_R^2 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \frac{1}{(D(ak))^2} \tag{3.29}$$

$$Z_\psi^{-1} = 1 + 4g_R^2 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \frac{\frac{1}{4}\sum_\mu \cos ak_\mu}{(D(ak))^2} \tag{3.30}$$

In the limit $a \to 0$, they reduce to (3.8), (3.21), and (3.14), respectively. Since these renormalization constants (which in the continuum theory would coincide) are different from each other, the counter term (3.27) clearly breaks supersymmetry. The origin of the difference between (3.28), (3.29), and (3.30) is the behavior of the lattice propagators for large values (of order $1/a$) of the momentum k: if we were allowed to take the limit $a \to 0$ inside the integral (which corresponds to the continuum theory), all Z's would become equal.

Next we come to the 3-point and four-point vertices. In the continuum theory only the vertices A^3, AB^2, A^4, A^2B^2, and B^4 contain logarithmic divergencies. In each case, however, the sum of all diagrams is finite, and the one-loop corrections vanish if the external momenta are taken to zero. On the lattice we start with the vertexfunction Γ_{AAA} (Fig. 6). The sum of all graphs leads to:

$$\begin{aligned}\Gamma_{AAA\,(\text{one-loop})} & = 16g^3\left\{\sum_\mu\left(2\frac{\sin(p_{1\mu}a/2)}{a}\right)^2 \int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}\right. \\ & \cdot\frac{M(ak)\cos^2((2k+p_2-p_3)_\mu a/2)}{D(ak)D(ak+ap_2)D(ak-ap_3)} \\ & + 2 \text{ other terms, cycl. permutation of } (p_1, p_2, p_3)\end{aligned} \tag{3.31}$$

We again rescale the integration variable $k_\mu \to k'_\mu = ak_\mu$ and approximate the factor in front of the integral:

$$\begin{aligned}\Gamma_{AAA(\text{one loop})} & \underset{a\to 0}{\sim} 16g^3\left\{\sum_\mu p_{1\mu}^2 a\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\right. \\ & \left.\cdot\frac{aM(k)\cos^2(k+(ap_2-ap_3)/2)_\mu}{a^2D(k)a^2D(k+ap_2)a^2D(k-ap_3)} + \text{two other terms}\right\}\end{aligned} \tag{3.32}$$

Because of $aM(k) = am + r\sum_\mu(1-\cos k_\mu)$ this integral splits into two terms. The first one is proportional to a^2, but the integral diverges like $1/a^2$ near $k=0$. After combining the denominators and shifting from k_μ to $l_\mu = k_\mu + xap_2 - yap_3$, we divide the integration into a small-l_μ ($|l|<\delta$) and a large-l_μ region ($|l|>\delta$). The latter one does not contribute since it is multiplied by a^2, whereas the first one goes like

$$\begin{aligned}& a^2\int_0^1 dx\int_0^1 dy \int_{|l|<\delta}\frac{d^4k}{(2\pi)^4}[l^2 + a^2(m^2+xy)p_1^2 \\ & + x(1-x-y)p_2^2 + y(1-x-y)p_3^2]^{-3}\end{aligned} \tag{3.33}$$

$$\underset{a\to 0}{\to} \int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k+p_2)^2+m^2}\frac{1}{(k-p_3)^2+m^2} = I_3$$

The second part of the integral (3.32) (which is proportional to r) diverges logarithmically near $k=0$. But since it has a factor a in front of it, this term does not contribute to the leading $a \to 0$ behavior.

Before we go on it is useful to make a comment on (3.31) and its limit $a \to 0$, contained in (3.33). We notice that the same result would have been obtained, if in (3.31) we had taken the limit $a \to 0$ *inside* the integral. Comparing this with our experience from the two-point functions where the limit $a \to 0$ under the integral would have given the wrong answer, we conclude that the limit $a \to 0$ commutes with the k-integral only in those cases where the continuum analogue of the lattice integral is UV finite (in case of the two-point functions, the continuum integrals diverge logarithmically). We have tested this "rule" [18] on all our one- and two-loop integrals and found no exception. In connection with the two-loop three-point vertices, where we shall make use of this rule, we shall give a more detailed discussion.

Returning to the leading term of (3.33) we have, as the limit $a \to 0$ of (3.31):

$$\Gamma_{AAA\,(\text{one-loop})} \underset{a \to 0}{\to} 16 g^3 (p_1^2 + p_2^2 + p_3^2) I_3 \tag{3.34}$$

This is in full agreement with the continuum theory. Similar calculations for the vertices AB^2, A^4, A^2B^2, and B^4 (the four-point vertices give somewhat complicated expressions which we do not want to write down) all lead to the same conclusion, namely that the limit $a \to 0$ agrees with the continuum theory. The same is true also for all the other 3-point and 4-point vertexfunctions which in the continuum theory are finite integrals.

So far we have been discussing *unrenormalized* 3-point and 4-point vertexfunctions. Our result that the limit $a \to 0$ directly leads us to the correct continuum limit means that they satisfy the Ward identities. But since the wave function renormalization constants for the A, ψ, and F-field are different from each other, the Ward identities for the *renormalized* vertexfunctions cannot be fulfilled simultaneously. This then forces us to adjust the bare couplings in the same way as we did with the mass parameters. This can be illustrated best if, instead of the renormalized quantities in the Lagrangian $\mathscr{L}_R = \mathscr{L}_{0R} + \mathscr{L}_{mR} + \mathscr{L}_{gR} + \Delta \mathscr{L}_{0R}$ we return to bare quantities. The part $\mathscr{L}_g$ and $\mathscr{L}_m$ then become:

$$\mathscr{L}_m = m_{AF} AF + m_{BG} BG - \tfrac{1}{2} m_\psi \bar{\psi} \psi \tag{3.35}$$

$$\mathscr{L}_g = g_{FAA} FA^2 - g_{FBB} FB^2$$
$$+ 2 g_{GAB} GAB - g_{\bar{\psi}A\psi} \bar{\psi} A \psi - g_{\bar{\psi}B\psi} \bar{\psi} i \gamma_5 B \psi \tag{3.36}$$

where the bare masses and couplings are given by (3.25), (3.26) and:

$$g_{FAA} = g_{FBB} = g_{GAB} = Z_A^{-1} Z_F^{-1/2} g_R$$
$$g_{\bar{\psi}A\psi} = g_{\bar{\psi}B\psi} = Z_\psi^{-1} Z_A^{-1/2} g_R$$

Equation (3.27) or, alternatively, (3.35) and (3.36) contain the result of our one-loop analysis.

IV. Two-Loop calculations

In this section we wish to extend our analysis to the two-loop approximation. The main result will be that, in addition to the counterterms of the previous section, further operators in the Lagrangian will be needed, in order to have the correct $a \to 0$ limit of the renormalized vertexfunctions. First we shall study the two-point functions, then the three- and four-point vertices. In both cases, we begin with the unrenormalized two-loop contributions and study their limit $a \to 0$. From this we then deduce the necessary steps of renormalization. A short summary will be given at the end of this section.

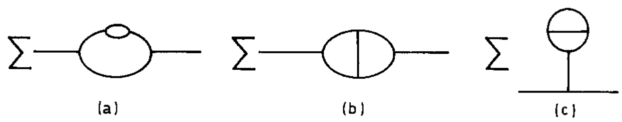

Fig. 7a–c. Classes of diagrams which contribute to the two-point function in the two-loop approximation. The $\sum$ indicates that one has to sum over all possible internal lines

A) Two-Point Functions

Our main emphasis will be on the two-point functions which in the continuum theory are the only ones that contain divergencies. In order to have agreement with the continuum theory, we will need the operator A^2 in the lattice Lagrangian, and its coefficient turns out to be proportional to the Wilson parameter r. We have three classes of diagrams that contribute to any of the two-point functions. They are shown in Fig. 7. After summing over all possible internal lines we find, in the limit $a \to 0$, the following types of contributions:

1) terms of the form a^{-2}·constant, where "constant" means: independent of p^2. They are reflections of quadratic divergencies in the continuum theory, and they appear in the tadpole graphs (Fig. 7c) as well as in the Γ_{AA} graphs of Fig. 7a. As it will be explained below, in case of Γ_{AF} and $\Gamma_{\psi\psi}$ these terms can be absorbed in a redefinition of the bare mass parameter, whereas in the case of Γ_{AA} the two contributions from Figs. 7a and c cancel;

2) terms which are proportional to the Wilson parameter r (or r^2) and independent of p^2. They go with inverse powers of a, and they require a counterterm A^2 in the Lagrangian;

3) functions of p^2. In case of Γ_{AA}, these pieces are of the form $p^2 \cdot f(p^2)$, where $f(p^2)$ is either a sum of powers of $\ln a^2$ a finite function of p^2 and constants or simply a finite function of p^2. In the first case these terms are quite analogous to the one-loop of (3.5): after multiplication with Z_A, the terms $\sim \ln a^2$ and the constants drop out, and the correct continuum theory result emerges:

$$\Gamma_{AA}^R \sim p^2 (f(p^2) - f(0)) \tag{4.1}$$

In case of $\Gamma_{\psi\psi}$, the corresponding pieces are of the form $\gamma \cdot p f(p^2) + p^2 \cdot g(p^2)$, where $g(p^2)$ is a finite function of p^2 and $f(p^2)$ has the same properties as in the case of Γ_{AA}. Finally, for Γ_{AF} and Γ_{FF} these terms have the form $f(p^2)$, with $f(p^2)$ having the features described before. In all cases, these terms lead to the behavior of the continuum theory.

After this general description of results let us now see in more detail how these terms arise and how they are affected through renormalization. We begin with the vertex Γ_{AA}. Diagrams of the type Fig. 7a sum up to:

$$8g^2 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \frac{1}{(D(ak))^2 D(ap-ak)} \Big[\Sigma^R_{AA}(ak) \cdot (-F^2(ak-ap) + M^2(ak) + 2M(ak)M(ak-ap)) + \Sigma^R_{FF}(ak)(F^2(ak)F^2(ak) - M^2(ak)F^2(ak-ap) - 2F^2(ak)M(ak)M(ak-ap)) + 4\sum_\mu F_\mu(ak)\Sigma^R_{\psi\mu}(ak) (F(ak)F(ak-ap) - M(ak)M(ak-ap)) - 2\sum_\mu F_\mu(ak-ap)\Sigma^R_{\psi\mu}(ak)(F^2(ak)+M^2(ak)) \Big] \tag{4.2}$$

Here Σ^R_{AA} stands for the renormalized one-loop contribution to Γ^R_{AA}, evaluated for *finite* a: equation (3.1), with the integral being subtracted at $p^2=0$. A similar definition holds for Σ^R_{FF} (3.22) and $\Sigma^R_{\psi\mu}$ (3.11). The limit $a\to 0$ of (4.2) is studied by multiplying both numerator and denominator by $(a^2)^4$ and then rescaling the integration variable: $k_\mu \to k'_\mu = ak_\mu$. The integrand then contains only combinations, such as $a^2D(k')$, $a^2\Sigma^R_{AA}(k')$, a^2M^2, $a^2F^2(k')$, which are finite in the limit $a\to 0$. Moreover, a factor a^{-2} stands in front of the integral. We now expand the integrand in powers of a and find the various terms 1)–3) described before.

We begin with the leading term which is obtained by simply putting $a=0$ inside the integral. The numerator can then be written as:

$$\text{numerator of (4.2)} = (a^2F^2 - 3a^2M^2)\Big(-a^2\Sigma^R_{AA} + a^2F^2\Sigma^R_{FF} + 2a^2\sum_\mu F_\mu \Sigma^R_{\psi\mu}\Big) = (a^2D^2 - 4a^2M^2) \cdot \Big(-a^2\Sigma^R_{AA} + a^2F^2\Sigma^R_{FF} + 2a^2\sum_\mu F_\mu\Sigma^R_{\psi\mu}\Big) \tag{4.3}$$

The k-integration converges, since there are enough powers of k^2 in (4.3) to compensate the zeroes of the denominator. Due to the definition of M (2.23), the numerator (4.3) decomposes into a term proportional to a^2D^2 and the leading piece of a^2M^2 which is proportional to r^2. The first one belongs to class 1), but it is cancelled by the tadpole graphs of Fig. 7c. Their result is:

$$-\frac{8g^2}{m}\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}\frac{M(ak)}{(D(ak))^2} \cdot \Big(-\Sigma^R_{AA}(ak) + F^2(ak)\Sigma^R_{FF}(ak) + 2\sum_\mu (ak)\Sigma^R_{\psi\mu}(ak)\Big) \tag{4.4}$$

After performing the same rescaling procedure as above, and by singling out the r-independent term in the numerator of (4.4), we arrive at the same result as from the first part of (4.3). The remainders of (4.3) and (4.4) are proportional to r^2/a^2 and r/a^3, respectively, and belong to class 2). This takes care of the leading part of (4.2) as well as the full contribution of the tadpole graphs.

Next-to-leading terms of (4.2) are obtained by expanding the integrand in powers of a (after having done the rescaling $k_\mu \to k'_\mu = ak_\mu$). Interesting contributions only come from the expansion of the numerator. First we have terms proportional to amr from the combinations aM or a^2M^2: they lead to converging and p^2-independent integrals of the type 2) (together with the factor a^{-2} in front of the integral they go as $r/a\cdot m$). Next we note that there are no contributions proportional to ap_μ: such terms always come together with an odd functions of k'_μ and thus make the integral vanish. Finally we come to terms of the form a^2p^2 and a^2m^2. The former ones arise from expanding $F^2(k'-ap)$. Near $k'_\mu=0$ they can be written as

$$a^2p^2(-a^2\Sigma^R_{AA}(k') - a^2m^2\Sigma^R_{FF}(k')) \tag{4.5}$$

Hence the k'-integration becomes divergent near $k'=0$, and we have to treat the integral in the same way as we did in the previous section (combining denominators, shifting from k'_μ to l_μ and dividing the region of l-integration into a small-l and a large-l part). As before, the large-l region leads to a p^2-independent integral, whereas the small-l region gives a function of p^2 plus powers of $\ln a^2$. Thus we have found the term of type 3), and we have to convince ourselves that—after subtraction at $p^2=0$—it agrees with the continuum limit. This is easily done by noticing that the continuum version of (4.2) can be obtained by replacing under the integral all lattice elements by their continuum expressions and by further using the Ward identities between Σ^R_{AA}, Σ^R_{FF}, Σ^R_ψ. Comparing this with (4.5) and making use of the fact that the limit $a\to 0$ (at small k'_μ) of Σ^R_{AA} and Σ^R_{FF} in (4.5) agrees with the continuum theory, we arrive at the desired result.

We are still left with a few constant terms of class 2). They arise from further terms in the expansion of the numerator (proportional to a^2m^2) or from expanding the denominators. In all cases, the resulting integrals are convergent near $k'_\mu=0$, and the p^2-dependence drops out. This then completes the discussion of (4.2).

Before we go on it is useful to compare the pattern of cancellation of divergencies with the continuum theory. There quadratic divergencies cancel within the diagrams of Fig. 7a, and tadpole graphs add up to zero. On the lattice the quadratic divergencies, which correspond to terms of class 1), remain after summing up all diagrams of Fig. 7a, but they cancel against the nonzero tadpole contributions. What remains are divergent constants proportional to r or r^2 that will lead to a mass counterterm A^2, and logarithmically divergent terms of the type 3).

Next we have to study the graphs of Fig. 7b. In the continuum theory these diagrams are finite. With our previous experience, we therefore expect no terms of class 1), and no powers of $\ln a^2$ in terms of type 3). This will, in fact, be true. The sum of all diagrams of Fig. 7b is:

$$-2^6 g^4 \int_{-\pi/a}^{\pi/a} \frac{d^4k\,d^4k'}{(2\pi)^8} \cdot \frac{1}{(D(ak)D(ak-ap)D(ak')D(ak-ak')D(ap-ak+ak')}$$

$$\cdot [M(ak-ak')M(ak'-ap)(F(ak)-F(ak-ap+ak') - F(ak'))^2 + M(ak-ap-ak')M(ak-ap)(F(ak) - F(ak-ak')-F(ak'))^2] \quad (4.6)$$

In deriving (4.6), we have made use of our results for the three-point vertices of the previous section (left hand subgraph of Fig. 7b). In particular, we have not yet included any subtractions for triangle subgraphs: for the leading terms in the limit $a \to 0$ we had found that they agree with the continuum limit, which implies that no subtractions were necessary.

The analysis of (4.6) proceeds in very much the same way as before: rescaling of the integration momenta k_μ and k'_μ leads to an overall factor a^{-2} in front of the integral, and the integrand consists of pieces which are finite near $a=0$ ($a^2D(k)$, $aM(k)$, a^2F^2 etc.). The leading term is obtained by keeping only the r-piece of the M's in the numerator and by putting $a=0$ everywhere in the integrand. The resulting integral converges near k, $k-k' \sim 0$, and there is no p^2-dependence. Next-to-leading terms of the form a.r.m come from expanding the aM-terms and are constant integrals, independent of p^2. Picking the am-parts of aM, we still are left with convergent integrals without any p^2-dependence. Next we expand the bracket-terms, which contain F-factors, in powers of a. From the second line of (4.6) we have (for $k_\mu \sim k'_\mu \sim 0$):

$$a^2 p^2 \left[a^2m^2 + amr\tfrac{1}{2}((k-k')^2 + k'^2) + \frac{r^2}{4}(k-k')k'^2 \right] \quad (4.7)$$

The first piece leads to an integral of the form: a^2p^2 times an integral which diverges $\sim a^{-2}$. After a careful treatment of the small $(k;k')$-region one finds that this equals the continuum result. The second piece of (4.7) is of the form: $a \cdot p^2$ times logarithmic divergent integral, and therefore can be neglected, whereas the last part of (4.7) leads to p^2·constant. Alltogether, (4.7) provides contributions of the type 3) but no terms proportional to $\ln a^2$.

This then completes our analysis of diagrams of the vertex Γ_{AA}. We summarize our results by defining the necessary renormalization conditions. Equation (2.6) requires that Γ^R_{AA} vanishes at $p^2=0$. Because of the nonvanishing terms of type 2), which are coming from all three classes of diagrams of Fig. 7, it is necessary to introduce a nonzero mass for the A-field:

$$-\tfrac{1}{2}\delta m^2_{AA} A^2_A \quad (4.8)$$

In each order of perturbation theory, starting with the two-loop approximation, this mass parameter has to be adjusted such that Γ^R_{AA} vanishes at $p^2=0$. In our approximation, it equals the sum of all constant pieces of type 2), including terms of the order a^{-2} or a^{-1}. As we have seen, all these pieces are proportional to the Wilson parameter r; the change in the Lagrangian therefore reads:

$$\Delta' \mathscr{L}_r = -\tfrac{1}{2}\delta m^2_{AA} A^2_R \quad (4.9)$$

The two-loop contribution to the wave function renormalization constants follows from (2.6).

After having discussed in somewhat more detail how the limit $a \to 0$ of Γ_{AA} is obtained, we will be rather brief about the remaining two-point functions and describe the results only qualitatively. The tadpole graphs of Fig. 7c contribute, apart from Γ_{AA}, to the two-point functions $\Gamma^{(2)}_{\psi\psi}$ and Γ_{AF}. In the limit $a \to 0$ they provide terms of both type 1) and type 2), and in contrast to Γ_{AA} where part of the tadpole contributions were cancelled by similar terms in graphs of Fig. 7a, they all survive. This then leads to a redefinition of the bare mass parameter m, but since the contribution of the tadpole graphs to $\Gamma^{(2)}_{\psi\psi}$ and Γ_{AF} is the same, this shift does not break supersymmetry.

Diagrams of Fig. 7a contribute to $\Gamma^{(1)}_{\psi\psi}$ and Γ_{FF}, but not to Γ_{AF}. In the limit $a \to 0$, there are no terms that go with inverse powers of a. In case of $\Gamma^{(1)}_{\psi\psi}$, we have terms of the form $\gamma \cdot p \cdot f(p^2)$ where $f(p^2)$ also contains powers of $\ln a^2$ multiplied by constant factors and p^2-independent finite constants. This is quite analogous to the one-loop result (3.12): after renormalization all constant terms independent of p^2 drop out, and the result agrees with the continuum theory. In the case of Γ_{FF} we find a function of p^2 which has the properties which we have described under 2) at the beginning of this section. After multiplication with Z_F, powers of $\ln a^2$ as well as p^2-independent constants drop out, and the result agrees with the continuum theory. It is important to note that again the wave function renormalization constants Z_A, Z_F, and Z_ψ are different from each other: this confirms what we had found already in the one-loop approximation.

Finally, we come to the diagrams of Fig. 7b and their contributions to the vertexfunctions Γ_{AF}, $\Gamma^{(1)}_{\psi\psi}$, $\Gamma^{(2)}_{\psi\psi}$ and Γ_{FF}. In the continuum theory, these diagrams are finite, once the one-loop results for the 3-point subgraphs have been inserted. Correspondingly, the lattice graphs in the limit $a \to 0$ do not contain terms with inverse powers of a or $\ln a^2$ (also no terms of type 2)). After the necessary subtractions coming from the wave function renormalization constants are made, the result agrees with the continuum theory. Again, the contributions to Z_A, Z_F, and Z_ψ come out different from each other.

B) Higher Vertices

Next we make a few remarks about 3-point and 4-point vertices. There are again three groups of diagrams which we have illustrated in Figs. 8a–c. In the continuum theory, these functions are UV-finite. For the vertices A^3, AB^2, A^4, A^2B^2, and B^4 individual diagrams have logarithmic divergencies, but after summation they drop out. Our previous experience with the limit $a \to 0$ of lattice diagrams indicates that we should expect the following terms:

1) terms which do not have a power r in front. They are finite functions of p^2 and agree with the continuum theory. In particular, powers of $\ln a^2$ should not be present which would require a subtraction and, hence, new counterterms in the Lagrangian.

2) constant (as a function of p^2) terms proportional to r (or r^2). They require new terms in the Lagrangian (e.g. A^3 and A^4) and may come with inverse powers of a.

We have not performed an explicit calculation of all these vertexfunctions, but the correctness of our description follows from the "rule" which we have mentioned after (3.33) and which states that we are allowed to take the limit $a \to 0$ *inside* the integral, if only the continuum integrals are convergent. But rather than simply applying this rule to the diagrams of Fig. 8, we now wish to give an argument why we believe that the rule holds in general. We take, as an example, the diagrams of Fig. 8a and see how they contribute to the vertex Γ_{AAA}. For the propagator insertions of internal lines we use the expressions Σ^R_{AA} etc. (explained after (4.2)). In particular we make use of the fact, that in the limit of small a and small momentum they agree with the continum theory. The sum of all diagrams of Fig. 8a will be of the form:

$$\text{sum of diagrams of Fig. 8a} = \int_{-\pi/a}^{\pi/a} d^4k \frac{\text{numerator}}{D^4} \tag{4.10}$$

Here the numerator is some combination of M, F^2, $\Sigma^R_{AA}\ldots$, and it has the dimension 5, whereas the denominator has dimension 8. For dimensional reasons, each piece of the numerator must be proportional to a factor M. For this argument, we first shall ignore terms proportional to r, i.e. for each M we simply substitute m. The integral (4.10) must then be proportional to m, which we put in front. So the remainder has dimension 4. Now we perform the usual rescaling procedure, as a result of which the numerator has only the combinations a^2m^2, a^2F^2, etc., and there is no factor a in front of the integral. In the limit $a \to 0$, only the region near $k_\mu = 0$ is of interest. From our knowledge of the continuum theory we can conclude that at $k_\mu \to 0$ the numerator either approaches a constant (which must

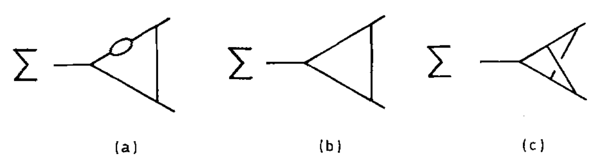

Fig. 8a–c. Two-loop diagrams which contribute to three-point functions. The $\sum$ indicates that one has to sum over all possible internal lines

be bilinear in a^2p^2 and a^2m^2) or vanishes as const.$\cdot k^2$, but not faster than that. This is because for small k_μ and small a all elements in the numerator of (4.11) can be replaced by their continuum analogues. Since in the continuum theory the integral is UV finite, there can, at most, be one power of k^2 in the numerator. This then implies that our integral (4.10), in the limit $a \to 0$, necessarily diverges. If we apply our usual procedure of dividing the integral into small and large values of the integration variable we find, for the small momentum part, two types of divergent integrals. If the numerator approaches a constant near $k=0$ (which then is bilinear in a^2p^2, a^2m^2) our integral is of the form:

$$a^4 m \int_{|k|<\delta} d^4k \frac{f(p^2,m^2)}{(a^2D)^4} \tag{4.11}$$

where the integral diverges as a^{-4}, and the numerator agrees with the continuum theory. A calculation simular to (3.35) then shows that (4.11) leads to the correct behavior of the continuum theory. Similarly, if the numerator goes as constant$\cdot k^2$ (where the constant is a^2p^2 or a^2m^2), we find

$$a^2 m \int_{|k|<\delta} d^4k \frac{\hat{f}(p^2,m^2)k^2}{(a^2D)^4} \tag{4.12}$$

The integral now diverges as a^{-2}, and the calculation of (3.35) leads to agreement with the continuum theory. The large-k region always leads to finite integrals, but because of the factor a^2 in front, they do not contribute. In summary, this justifies the description of 1).

As to the terms proportional to r (point 2) above) they are either constant (as a function of p^2) or nonleading. This can be seen by repeating the argument above and replacing $am \to r\sum(1-\cos k_\mu)$. One then always looses a power of a but gains a factor which near $k_\mu = 0$ is of order k^2 and thus improves the convergence near $k_\mu = 0$.

These arguments not only apply to the contribution of Fig. 8a to Γ_{AAA}, but, with appropriate changes, also to the diagrams of Figs. 8b and c and to all other vertexfunctions. We therefore conclude that the only new counterterms that we need are proportional to the Wilson parameter r, for example:

$$\frac{r}{a}(Z_{AAA}-1)A_R^3,\ r(Z_{AAAA}-1)A_R^4 \tag{4.13}$$

C) *Summary of Results*

Let us finally summarize the results of this section. The analysis of the two-point functions has confirmed the main result of the previous section: the renormalization constants Z_A, Z_F, and Z_ψ which in the continuum theory coincide are no longer equal. This forces us, in order to keep renormalized masses equal, to adjust bare masses in a way which destroys supersymmetry. In addition to this, there is a (common) shift of all bare masses, resulting from the tadpole graphs.

Finally, the operator A^2 has to be added to the Lagrangian, which was not present in the continuum theory. Its coefficient is proportional to the Wilson parameter r, and its perturbation expansion starts at the two-loop level.

Higher order (three-point, four-point) vertexfunctions also confirm our previous result: the bare coupling constants have to be adjusted in a nonsupersymmetric way, in order that the renormalized coupling constants are supersymmetric. Furthermore, new operators such as $(1/a)A^3$, A^4, etc., are needed, and their coefficients are proportional to r.

One should expect that the picture which has emerged from our one and two-loop calculations remains true also in higher order perturbation theory:

a) the wave function renormalization constants Z_A, Z_F and Z_ψ are different from each other. This can be traced back to the fact that lattice diagrams, which in the continuum theory would be logarithmically divergent, are sensitive also to the large momentum behavior and thus are sensitive to the different nature of bosons and fermions. One feels that this is rather independent of the way in which form fermions are put into the lattice.

b) There are new operators of dimension $\leqq 4$ which are not present in the continuum theory. Their coefficients are proportional to the Wilson parameter r and, hence, seem to be more strongly dependent upon the Wilson method of dealing with the fermion problem.

We conclude that a lattice version of the Wess–Zumino model which in the limit $a \to 0$ has the correct continuum limit is given by the following renormalized Lagrangian:

$$\mathscr{L}_r = \mathscr{L}_{0R} + \mathscr{L}_{mR} + \mathscr{L}_{gR} + \Delta \mathscr{L}_{0R} + \Delta \mathscr{L}_r \tag{4.14}$$

Here the first three terms represent the "naive" Lagrangian of (2.15)–(2.18) with the bare quantitites being replaced by the renormalized ones. $\Delta \mathscr{L}_{0R}$ stands for the counterterms of (3.27). The final piece $\Delta \mathscr{L}_r$ which comes in only at the two-loop level consists of the new operators which had not been there in the continuum theory:

$$\Delta \mathscr{L}_r = -\tfrac{1}{2}\delta m_{AA}^2 A_R^2 + \frac{r}{a}(Z_{AAA} - 1)A_R^3 + r(Z_{AAAA} - 1)A_R^4 + \cdots \tag{4.15}$$

where the dots stand for similar operators of dimension $\leqq 4$ containing the pseudoscalar field B. If we rewrite (4.14) in terms of bare quantities (as we did in (3.35)–(3.37)), we easily see the breakdown of supersymmetry.

V Conclusions

In this paper we have constructed, on the basis of renormalized perturbation theory, a lattice version of the Wess–Zumino model. In the two-loop approximation we have shown that the renormalized vertexfunctions have the correct continuum limit. Our result has been presented in terms of the renormalized Lagrangian (4.14), where the Z's are functions of renormalized mass, coupling constant g, lattice spacing a and the Wilson parameter r. Outside of perturbation theory our calculations suggest the following lattice version of the Wess–Zumino model:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_m + \mathscr{L}_g + \mathscr{L}_r' \tag{5.1}$$

where the first term agrees with the "naive" lattice Lagrangian (2.16). The remaining three terms deviate from what one would have expected in the tree approximation:

$$\mathscr{L}_m = m_{AF}(AF + GB) - \tfrac{1}{2} m_\psi \bar{\psi}\psi \tag{5.2}$$

$$\mathscr{L}_g = g_{AAF} F(A^2 - B^2) + g_{AGB^2} GAB - g_{A\bar{\psi}\psi} \bar{\psi}(A + i\gamma_5 B)\psi \tag{5.3}$$

Here the masses are no longer the same for all fields, and also the coupling constants cannot be equal. Finally, the term $\mathscr{L}_r'$ in (5.1) is a consequence of Wilson's method of eliminating the additional fermionic degrees of freedom on the lattice:

$$\mathscr{L}_r' = -\tfrac{1}{2} m_{AA}^2 (A^2 + B^2) + \lambda_{AAA} A^3 + \lambda_{AB^2} AB^2 + g_1 A^4 + g_2 A^2 B^2 + g_3 B^4 \tag{5.4}$$

All parameters m_{AA}, λ's, and g's (the λ's have dimension of mass) strongly depend upon r (in perturbation theory, they are proportional to r or r^2). None of the operators in (5.4) was present in the continuum theory.

The Lagrangian contained in (5.1)–(5.4) has the most general form of a field theory which describes the interaction of a (Majorana) spinor with a scalar, a pseudoscalar, and two auxiliary fields. All operators of dimension $\leqq 4$ are present. One has, however, to keep in mind that the masses and coupling constants are not independent from each other. In perturbation theory, they can be computed as functions of one common mass m_R, a coupling g_R, the lattice spacing a and the parameter r. Beyond perturbation theory they still have to be thought of as being functions of these parameters, but this functional dependence has to be found by methods other than perturbation theory.

It is also important to mention that our lattice Lagrangian is not unique. One expects that it is possible to add "irrelevant" operators (i.e. terms with positive powers of a) which would change the parameters in (5.1)–(5.4) but not the general structure. So it is conceivable that one might find a lattice version which looks more "elegant" than ours. An example in this direction is our form of the r-term in (2.19): we have introduced such a term not only for the fermion fields but also for the other fields (at the same time we introduced the next-to-nearest neighbour derivative for the scalar fields). As we remarked at the end of Sect. II, this leads to the same denominator for all lattice propagators and thus preserves the cancellation of quadratic divergencies on the lattice. It is, however, likely that we also could have proceeded in a less symmetric fashion, ending up with the same form for $\mathscr{L}$.

We finally list a few problems which in our opinion deserve further investigation. There is first the question whether and how our Lagrangian could be used for a study of the strong coupling regime of the Wess–Zumino model. As we have said before, the parameters in (5.1)–(5.4) are not all free but have to be considered as functions of fewer parameters. How this can be done in practice is not quite clear yet. Secondly, it is necessary to extend our analysis to other supersymmetric models which are more likely to be realistic. In this context it also would be interesting to find whether the lattice version of the $N=2$ model [11] escapes the effect that we have found: finite quantum corrections could spoil the continuum limit. Finally, the fact that all the terms in (5.4) so strongly depend upon the Wilson parameter r suggests to repeat our analysis with another method of handling the fermion problem on the lattice, namely the geometric fermions in the Dirac–Kahler formalism [19].

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