ON THE CONTINUUM LIMIT OF A Z_4 LATTICE GAUGE THEORY

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The continuum limit of a Z_4 gauge plus matter lattice theory is identified with massless scalar and vector fields with quartic self-interactions ϕ^4 and $(A_\mu A_\mu)^2$, $\Sigma_\mu A_\mu^4$, respectively. The analysis is based on the mean field approximation after gauge fixing.

1. Introduction. In a recent paper [1] a method based on the mean field approximation combined with a loop expansion around it, has been suggested by Brézin and Drouffe as a general approach to investigate the field theoretic content of lattice theories in the neighbourhood of a critical point [2]. They applied the method to a Z_2 gauge model coupled to matter. They showed that the scaling limit of this model is a continuum one component massless scalar field with quartic self-interaction.

In this note we apply the same approach to a Z_4 lattice gauge theory coupled to bosonic matter. We are motivated by the fact that a richer structure of the gauge group will be reflected in a more complex structure of the continuum limit. After gauge fixing, the mean field approximation leads us to the detection of a second order transition point (critical point), end of a first order line. The choice of a particular gauge is necessary in order to avoid the restriction imposed by Elitzur's theorem [3], namely the vanishing of the expectation value of any non-gauge invariant quantity. We then study the correlation functions which are relevant to the long distance limit of the model at this critical point, and obtain the vertices and propagators of the associated field theory. These correspond to massless scalar and vector fields, with quartic self-interactions of the type ϕ^4 and $(A_\mu A_\mu)^2$, $\Sigma_\mu A_\mu^4$, respectively, and uncoupled among themselves. The interaction $\Sigma_\mu A_\mu^4$ has cubic symmetry, but not euclidean symmetry. Thus, euclidean invariance is not fully obtained at the critical point. This should be related to the lack of a formal continuum limit for the lattice action.

We do not include the loop expansion around the mean field approximation [4]. This would lead to a quantum field theory with vertices and propagators as given by the tree approximation considered here.

2. Critical point. Let us begin by recalling the definition of the model. With each site i of a d-dimensional hypercubic lattice and each oriented link i, μ (μ is a unit vector along the positive axis μ) we associate, respectively, the variables $h_i = \exp(\mathrm{i}\phi_i)$ and $U_{i,\mu} = \exp(\mathrm{i}\phi_{i,\mu})$, with $\phi_i = \pi n_i/2$, $\phi_{i,\mu} = \pi n_{i,\mu}/2$, $n_i, n_{i,\mu} = 0, 1, 2, 3$. The Z_4 locally invariant action is defined by

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$$S = S_G + S_{GM}, \tag{1}$$

where

$$S_G = \beta_p \sum_{i,\mu\nu} \mathrm{Re}(U_{i,\mu}\, U_{i+\mu,\nu}\, U^*_{i+\nu,\mu}\, U^*_{i,\nu}) \tag{2}$$

and

$$S_{GM} = \beta_\ell \sum_{i,\mu} \mathrm{Re}(h_i^* U_{i,\mu}\, h_{i+\mu}). \tag{3}$$

β_p and β_ℓ are non-negative real parameters (plaquette and link coupling constants, respectively). For later convenience, we shall express the pure gauge part of the action, S_G, in terms of two-dimensional real unit vectors [5]. Defining

$$\hat{k}_{i,\mu} \equiv (\cos\phi_{i,\mu}, \sin\phi_{i,\mu}) \equiv k^\alpha_{i,\mu}, \qquad \alpha = 1,2$$

it is easy to obtain

$$S_G = \beta_p \sum_{i,\mu\nu} [(\hat{k}_{i,\mu}\cdot\hat{k}_{i+\nu,\mu})(\hat{k}_{i+\mu,\nu}\cdot\hat{k}_{i,\nu}) - (\hat{k}_{i,\mu}\cdot\hat{k}_{i+\mu,\nu})(\hat{k}_{i+\nu,\mu}\cdot\hat{k}_{i,\nu}) + (\hat{k}_{i,\mu}\cdot\hat{k}_{i,\nu})(\hat{k}_{i+\nu,\mu}\cdot\hat{k}_{i+\mu,\nu})]. \tag{2'}$$

As we mentioned in the introduction, we are interested in the detection of a second order transition point, end of a first order line. In a straightforward application of the mean field approximation this point disappears, therefore making it necessary to first fix the gauge. We shall work in the unitary gauge, defined by the condition that the action depends only upon the gauge variables. After gauge fixing, the gauge–matter piece of the action becomes

$$S_{GM} = \beta_\ell\, \hat{\mathbf{1}}\cdot \sum_{i,\mu} \hat{k}_{i,\mu}, \tag{3'}$$

where $\hat{\mathbf{1}}$ is a unit vector along the positive direction 1 in internal space. In the mean field approximation, the interaction among the dynamical variables is replaced by an external field $K_{i,\mu} = |K_{i,\mu}|(\cos \pi n_{i,\mu}/2, \sin \pi n_{i,\mu}/2)$, $n_{i,\mu} = 0, 1, 2, 3$, coupled to the gauge variables through the independent link action

$$S_0 = \sum_{i,\mu} K_{i,\mu}\cdot\hat{k}_{i,\mu}. \tag{4}$$

By using standard techniques, one obtains for the generating functional of vertex functions the expression

$$\Gamma(\{x_{i,\mu}\}) = -\beta_p \sum_{i,\mu\nu} [(x_{i,\mu}\cdot x_{i+\nu,\mu})(x_{i+\mu,\nu}\cdot x_{i,\nu}) - (x_{i,\mu}\cdot x_{i+\mu,\nu})(x_{i+\nu,\mu}\cdot x_{i,\nu}) + (x_{i,\mu}\cdot x_{i,\nu})(x_{i+\nu,\mu}\cdot x_{i+\mu,\nu})]$$
$$- \beta_\ell \sum_{i,\mu} x_{i,\mu} \cos \pi n_{i,\mu}/2 + \sum_{i,\mu}[(1+x_{i,\mu})\ln(1+x_{i,\mu}) + (1-x_{i,\mu})\ln(1-x_{i,\mu})], \tag{5}$$

where $x_{i,\mu} = \langle \hat{k}_{i,\mu}\rangle_0$ and $x_{i,\mu} = |x_{i,\mu}|$. $\langle\ \rangle_0$ denotes the expectation value with $\exp(iS_0)$ as weight factor.

The vertex functions are obtained by taking derivatives of Γ with respect to the averaged link variables $x_{i,\mu}$, and evaluating them at the configuration $\{x^*_{i,\mu}\}$ which minimizes Γ. This configuration is easily obtained if we consider the translationally invariant situation $x_{i,\mu} = x = x\,(\cos \pi n/2, \sin \pi n/2)$. In this case, (5) reduces to

$$\Gamma(x)/Nd = -\tfrac{1}{4}\beta x^4 - \beta_\ell x \cos \pi n/2 + (1+x)\ln(1+x) + (1-x)\ln(1-x), \tag{6}$$

where N is the number of lattice sites and $\beta = 2\beta_p\,(d-1)$. The minima of $\Gamma(x)$ are obtained by setting $n = 0$ and by solving the equation

$$\beta_\ell = -\beta x^{*3} + \ln[(1+x^*)/(1-x^*)]. \tag{7}$$

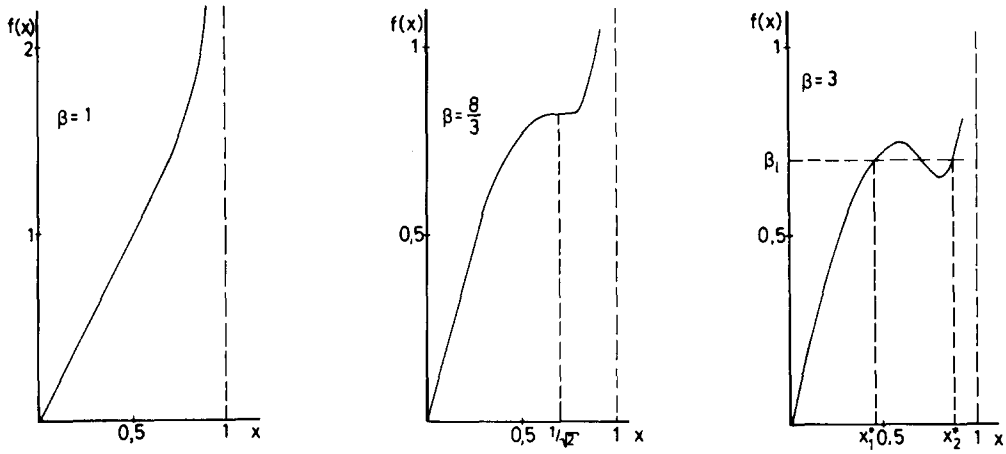

Fig. 1. $f(x) = -\beta x^3 + \ln[(1+x)/(1-x)]$ for different values of β.

In fig. 1 we plot the rhs of (7) for different values of β. For $\beta > 8/3$ there exists a first order transition line in the (β, β_ℓ)-plane defined by the condition $\Gamma(x_1^*) = \Gamma(x_2^*)$. This line ends at the critical point

$$C \equiv (\beta_c, \beta_\ell^c) = (8/3,\ 2[-\sqrt{2}/3 + \ln(1+\sqrt{2})]\ , \qquad \boldsymbol{x}_c^* = (1/\sqrt{2})\ \hat{1}, \tag{8}$$

where the transition is of second order. In fact, $\Gamma''(1/\sqrt{2}) = 0$ for $\beta = \beta_c$. In the neighbourhood of the critical point the equation for the first order line is approximately given by

$$(\beta - \beta_c)/2\sqrt{2} + (\beta_\ell - \beta_\ell^c) = 0. \tag{9}$$

(see fig. 2).

3. Continuum limit. In this section we shall consider the vertex functions which are relevant to the long distance behaviour of the model, namely the two-, three- and four-point functions. Their study will lead us to the identification of the continuum field theories associated with the discrete theory at its critical point.

Let $\boldsymbol{x}^* = x^* \hat{\mathbf{1}}$ be a minimum of $\Gamma(\{\boldsymbol{x}_{i,\mu}\})$. For the two-link inverse two-point function a straightforward calculation leads to the result

$$\Gamma^{(2)\,\alpha\beta}_{i_1,\mu_1;i_2,\mu_2} \equiv \left.\frac{\partial^2\Gamma(\{\boldsymbol{x}_{i,\mu}\})}{\partial x^\alpha_{i_1,\mu_1}\,\partial x^\beta_{i_2,\mu_2}}\right|_{\boldsymbol{x}_{i,\mu}=\boldsymbol{x}^*} = \begin{bmatrix} -\beta x^{*2}P_{i_1,\mu_1;i_2,\mu_2} + \dfrac{2}{1-x^{*2}}\,\delta_{i_1,\mu_1;i_2,\mu_2} & 0 \\ 0 & \beta_p x^{*2}(\mp P_{i_1,\mu_1;i_2\,\mu_2} + 2(d-1)\,\delta_{i_1,\mu_1;i_2\,\mu_2}) \end{bmatrix}. \tag{10}$$

The – and + signs refer to the choice of links indicated in figs. 3a and 3b, respectively, and

$$P_{i_1,\mu_1;i_2,\mu_2} = 1, \quad \text{if the links } i_1,\mu_1 \text{ and } i_2,\mu_2 \text{ belong to the same plaquette}$$
$$= 0, \quad \text{otherwise.} \tag{11}$$

Notice that $\Gamma^{(2)\,11}_{i_1,\mu_1;i_2,\mu_2}$ is the same as the corresponding quantity in ref. [1]. The Fourier transform of (10) is given by

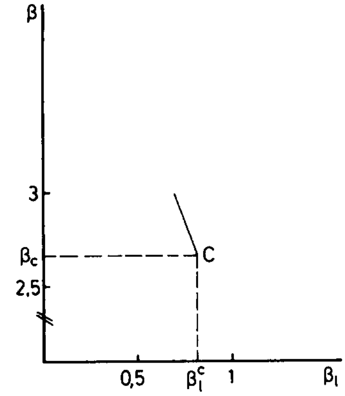

Fig. 2. First order transition line in the neighbourhood of the critical point.

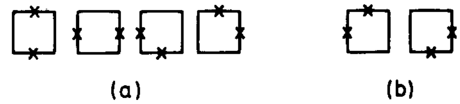

Fig. 3. Choice of links in the calculation of the two-point function.

$$\Gamma^{(2)\alpha\beta}_{\mu_1\mu_2}(q)=\sum_{r_2}\exp[iq\cdot(r_1-r_2)]\;\Gamma^{(2)\alpha\beta}_{i_1,\mu_1;i_2\mu_2}$$

$$=\begin{bmatrix} -2\beta_p x^{*2}\left(\delta_{\mu_1\mu_2}\sum_{\mu\neq\mu_1}\cos q_\mu+2(1-\delta_{\mu_1\mu_2})\cos\tfrac12 q_{\mu_1}\cos\tfrac12 q_{\mu_2}\right)+\dfrac{2\delta_{\mu_1\mu_2}}{1-x^{*2}} & 0 \\ 0 & -2\beta_p x^{*2}\left(\delta_{\mu_1\mu_2}\sum_{\mu\neq\mu_1}\cos q_\mu+2(1-\delta_{\mu_1\mu_2})\sin\tfrac12 q_{\mu_1}\sin\tfrac12 q_{\mu_2}-\delta_{\mu_1\mu_2}(d-1)\right)\end{bmatrix}, \tag{12}$$

where r_j is the position vector of the center of the link i_j, μ_j. At the critical point and for small q (long distances) this expression reduces to

$$\Gamma^{\alpha\beta}_{\mu_1\mu_2;c}(q)=\frac{2}{3(d-1)}\begin{bmatrix} 4d(\delta_{\mu_1\mu_2}-1/d)+\delta_{\mu_1\mu_2}(q^2-2q^2_{\mu_1})+\frac12(q^2_{\mu_1}+q^2_{\mu_2}) & 0 \\ 0 & q^2(\delta_{\mu_1\mu_2}-q_{\mu_1}q_{\mu_2}/q^2\end{bmatrix}+\mathrm{O}(|q|^4). \tag{13}$$

A simple analysis of this matrix reveals the continuum limit that is associated with each of the two directions in internal space:

(i) $\Gamma^{(2)\,11}_{\mu_1,\mu_2;c}(q)$ is not an euclidean tensor. However, the diagonalization of the matrix $\Gamma^{(2)\,11}_{\mu_1\mu_2;c}(0)$ leads to one zero eigenvalue in the direction $(\mu_1,\mu_2,\ldots,\mu_d)=(1,1,\ldots,1)$ and to $d-1$ non-zero eigenvalues in the transverse directions. Therefore, there is only one mode relevant to the long distance limit of the discrete model, namely a massless real scalar field ϕ with an inverse propagator given by

$$\Gamma^{(2)\,11}_{c}(q)=\frac{1}{d^2}\sum_{\mu_1,\mu_2}\Gamma^{(2)\,11}_{\mu_1,\mu_2;c}(q)=\frac{4}{3d^2}q^2+\mathrm{O}(|q|^4). \tag{14}$$

(ii) $\Gamma^{(2)\,22}_{\mu_1,\mu_2;c}(q)$ is a tensor, which can be identified with the inverse propagator of a massless vector (V) field A_μ in the unitary gauge.

The nature of the interactions among these fields is obtained from the three-and four-point functions evaluated at zero momenta and projected along the direction $(1,1,\ldots,1)$ when they involve the massless scalar field.

The three-point function vanishes identically when evaluated at zero momenta and at the critical point (8). This is a trivial consequence of the simultaneous vanishing of the two-point function at this point when evaluated at zero momentum [6]. An explicit calculation confirms this fact.

At $d = 4$ the relevant interactions also allow for a term proportional to the momentum i.e. a $\phi\phi$V interaction with derivative coupling. However, a straightforward calculation leads to the vanishing of this three-point vertex when evaluated at non-zero momenta, namely

$$\Gamma^{(3)\alpha\beta\gamma}_{\mu_1\mu_2\mu_3}(q_1, q_2, q_3)(2\pi)^d \delta^d(q_1+q_2+q_3) = 0, \tag{15}$$

$\alpha\beta\gamma = 112, 121, 211$. This fact is related to the real, and therefore neutral, character of the ϕ field.

For the four-point function, at zero momenta and at the critical point, we obtain

$$\Gamma^{(4)\alpha\beta\gamma\delta}_{\mu_1\mu_2\mu_3\mu_4;c}(q_1, q_2, q_3, q_4)|_{q_i=0} = N\{-[4/3(d-1)]\,[\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}(1-\delta_{\mu_2\mu_3})A^{\alpha\beta\gamma\delta} + (1-\delta_{\mu_1\mu_2})(\delta_{\mu_3\mu_1}\delta_{\mu_4\mu_2}B^{\alpha\beta\gamma\delta}$$
$$+ \delta_{\mu_3\mu_2}\delta_{\mu_4\mu_1}C^{\alpha\beta\gamma\delta})] + \delta_{\mu_1\mu_2}\delta_{\mu_2\mu_3}\delta_{\mu_3\mu_4}[64x_c^{*\alpha}x_c^{*\beta}x_c^{*\gamma}x_c^{*\delta}/x_c^{*4} + \tfrac{16}{3}(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma})]\}, \tag{16}$$

where

$$A^{1111} = A^{2222} = A^{2211} = A^{1122} = B^{1111} = B^{2222} = B^{1212} = B^{2121} = C^{1111} = C^{2222} = C^{1221} = C^{2112} = 4, \tag{17}$$

and the rest of the components being zero. From the last two equations,

$$\sum_{\mu_1\mu_2\mu_3\mu_4} \Gamma^{(4)1111}_{\mu_1\mu_2\mu_3\mu_4;c} = 64Nd, \qquad \Gamma^{(4)1112}_{\mu_1\mu_2\mu_3\mu_4;c} = \Gamma^{(4)2221}_{\mu_1\mu_2\mu_3\mu_4;c} = \sum_{\mu_3\mu_4} \Gamma^{(4)2211}_{\mu_1\mu_2\mu_3\mu_4;c} = 0, \tag{18a, b}$$

and

$$\Gamma^{(4)2222}_{\mu_1\mu_2\mu_3\mu_4;c} = \frac{16}{3(d-1)}(\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} + \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4} + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}) + 16\,\frac{d}{d-1}\sum_{\mu} \delta_{\mu_1\mu}\delta_{\mu_2\mu}\delta_{\mu_3\mu}\delta_{\mu_4\mu}. \tag{18c}$$

Eq. (18b) shows the vanishing of the $\phi\phi\phi$V, VVVϕ and $\phi\phi$VV couplings, while (18a) shows that the scalar field has a quartic self-interaction. The reason for the vanishing of the would-be seagull term $\phi\phi$VV is again in the neutral character of the scalar field. Thus, to the order considered, the correlation functions for this field are the same as those corresponding to a single component ϕ^4 theory. Finally, let us consider (18c). The first term corresponds to a quartic self-interaction of the form $(A_\mu A_\mu)^2$ for the massless vector field. This interaction is clearly non-gauge invariant what is consistent with the fact that we are working in a fixed gauge. As to the second term, we notice that it corresponds to an interaction of the form

$$\sum_{\mu\nu\rho\sigma} \delta_{\mu\nu}\delta_{\mu\rho}\delta_{\mu\sigma} A_\mu A_\nu A_\rho A_\sigma = \sum_{\mu=1}^{d} A_\mu^4.$$

Since its associated coupling constant is dimensionless for $d = 4$ and has positive dimensions for $d < 4$, it can not be ignored in the above mentioned loop expansion. However, notice that this interaction is not euclidean invariant, but only cubic. This means that a full euclidean invariance is not obtained at the critical point. Presumably, this is related to the fact that the original action has no classical or formal continuum limit. However, this point deserves further research.

In summary, by applying to a Z_4 lattice gauge model with bosonic matter the mean field approximation in the unitary gauge, we have detected a critical point, end of a first order line, where a second order phase transition occurs. At this point, the long distance limit of the relevant correlation functions of the model has the same structure as that of the correlation functions associated with continuum massless one-component scalar and vector fields with quartic self-interactions of the type ϕ^4 and $(A_\mu A_\mu)^2$, $\Sigma_\mu A_\mu^4$, respectively, and uncoupled among themselves.

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