ON BARE AND INDUCED MASSES OF SUSSKIND FERMIONS

P. MITRA[1] and P. WEISZ[2]
II. Institut für Theoretische Physik der Universität Hamburg, Fed. Rep. Germany

Received 15 March 1983

It is shown that the mass matrix for Susskind fermions on the lattice cannot have more than two distinct eigenvalues if cubic symmetry is enforced. If the standard interaction is replaced by one proposed by Becher and Joos, degeneracy-lifting mass counterterms are induced. The Λ-parameter is calculated.

Among the different ways of putting fermions on a lattice, Susskind's method [1,2] is probably the most attractive one, at least from a theoretical point of view. Unlike Wilson's method [3], it retains a "chiral" symmetry. (see the discussion below). Unfortunately it also retains a part of the species "doubling" seen with naive fermions, and makes practical calculations somewhat involved. However, the recent discovery [4] that Susskind fermions have a deep geometrical foundation – they satisfy the natural lattice version of the equation derived by Kähler in his programme of rewriting the Dirac equation in terms of differential forms – has given a new boost to the method.

The original Susskind description used one-component fermionic variables attached to the sites of a lattice. It turns out that the free theory admits a more convenient formulation on the block lattice with spacing twice as large as the original one. One can use either cochains [4] or matrices [5] associated with the new sites. the two pictures being equivalent. A 4×4 matrix is a transparent way of representing the four four-component fermions arising in this (euclidean) description and will be used by us.

Gauge interactions can be introduced in two inequivalent ways, both of which lead to the desired classical continuum limit. One can stick to the original lattice and have gauge matrices sitting on its links. One can also work on the block lattice and have gauge matrices only on the bigger links. The former choice makes for a bigger invariance group: one can have independent gauge transformations not only at the sites of the block lattice, but also at the sites of the original lattice excluded from the block lattice. This seems to us to be more in line with the principle of extending a symmetry from a global one to a maximally local one, and was in fact the interaction originally considered [1,2]. The second way of bringing in interactions [4] has the advantage of being technically easier to handle. It has been argued that this is the more natural interaction because here one does not have different gauge transformations for different spinor and flavour components. However, when for nonzero lattice spacing all these components are spread out on different sites, different transformations do not look all that unnatural. In any case, these are just questions of aesthetics. The proof of the pudding is in the eating.

In this letter we are interested in masses – those put in by hand, as well as those generated by counterterms. Bare masses were considered in ref. [6]. We discuss this question here from the point of view of a rotational symmetry of the theory. If we insist on this symmetry on the lattice, then the degeneracy of the fermions can be only partially lifted.

As regards masses induced by quantum corrections, the one-loop investigations of ref. [2] indicated that the full interaction introduced on the original lattice does not generate masses. We find that the interaction introduced on the block lattice does not share this nice feature: a mass counterterm is indeed produced. We discuss how this is compatible with the "chiral" symmetry.

[1] Alexander von Humbolt Fellow.
[2] Heisenberg Foundation Fellow.

A by-product of our perturbative studies is the determination of the Λ-parameter for the block-lattice interaction. The value differs significantly from that associated with the original interaction and is in fact close to the value for the pure gauge theory.

We start with Gliozzi's form [5] of the massless Susskind action on a four-dimensional, hypercubic, euclidean lattice of unit spacing:

$$S = \sum_x \mathrm{tr}\, \bar\psi'(x)\{\gamma_\mu \nabla_\mu \psi'(x) + \gamma_5 \Delta_\mu \psi'(x) \gamma_5 \gamma_\mu\}. \quad (1)$$

Here ψ', $\bar\psi'$ are 4×4 matrices defined on the sites x, the γ-matrices are hermitian, and the derivatives are defined by

$$\nabla_\mu \psi'(x) = \tfrac{1}{2}\{\psi'(x+e_\mu) - \psi'(x-e_\mu)\},$$

$$\Delta_\mu \psi'(x) = \tfrac{1}{2}\{\psi'(x+e_\mu) + \psi'(x-e_\mu) - 2\psi'(x)\}. \quad (2)$$

The second term on the right-hand side of (1) distinguishes it from the naive action. It vanishes in the continuum limit, as can be checked by putting in the lattice spacing. On the lattice its presence is essential if the 16-fold degeneracy of naive fermions is to be avoided. Besides mixing the different columns of our matrices, this term also seems to break parity. That this is an illusion is best seen by going over to the new matrix variables

$$\psi = \psi\, Q_+ - \gamma_5 \psi' Q_-, \qquad \bar\psi = Q_+ \bar\psi' + Q_- \bar\psi' \gamma_5, \quad (3)$$

where

$Q_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$.

One can rewrite (1) as

$$S = \sum_x \mathrm{tr}\, \bar\psi(x)\{\gamma_\mu \nabla_\mu \psi(x) + \Delta_\mu \psi(x) \gamma_\mu\}, \quad (4)$$

where the γ_5 has disappeared. This action is invariant under "μ-parity" (any μ)

$$\psi(x) \to \gamma_\mu \psi(\mathcal{P}_\mu x), \qquad \bar\psi(x) \to \bar\psi(\mathcal{P}_\mu x)\gamma_\mu, \quad (5)$$

where

$$(\mathcal{P}_\mu x)_\nu = (-)^{\delta_{\mu\nu}-1} x_\nu. \quad (6)$$

A mass term can be introduced in (4). It will have the form $\mathrm{tr}\, \bar\psi \psi M$ with M an arbitrary hermitian 4×4 matrix. Such mass terms have been considered in ref. [6]. The term $\mathrm{tr}\, \bar\psi' \psi'$ which corresponds to $\mathrm{tr}\, \bar\psi \psi \gamma_5$ in the redefined version arises naturally from the original Susskind formulation, but other mass terms can be introduced and correspond to particular non-nearest-neighbour couplings. M can be diagonalized and in the naive continuum limit one obtains four independent fermions. However, on the lattice the Δ_μ piece in the action couples these. It also reduces the rotational symmetry of the action, with the result, as we shall presently see, that a general mass matrix breaks this symmetry. For the rotation described by

$$x \to A^{-1}x, \quad (A^{-1}x)_\mu = x_\nu, \quad (A^{-1}x)_\nu = -x_\mu,$$

$$\text{rest unchanged}, \quad (7)$$

the straightforward lattice analogue of the transformation of continuum spinors is

$$\psi(x) \to S(A)\, \psi(A^{-1}x), \qquad \bar\psi(x) \to \bar\psi(A^{-1}x)\, S^{-1}(A), \quad (8)$$

with

$$S(A) = (1/\sqrt{2})(1 - \gamma_\mu \gamma_\nu). \quad (9)$$

This leaves the first piece of the action, as also all mass terms, invariant, but the so called irrelevant Δ_μ piece changes. The above transformation can be modified to

$$\psi(x) \to S(A)\, \psi(A^{-1}x)\, R(A),$$

$$\bar\psi(x) \to R^{-1}(A)\, \bar\psi(A^{-1}x)\, S^{-1}(A), \quad (10)$$

with

$$R(A) = \mathrm{i}\gamma_5(\gamma_\mu - \gamma_\nu)/\sqrt{2} \quad (11)$$

so that the full action (4) stays invariant, but then some mass matrices will break this invariance. In fact the most general mass matrix that is strictly invariant is

$$M_{(1)} = m\mathbf{1} + \tfrac{1}{2} m' \sum_\varphi \gamma_\varphi. \quad (12)$$

Note that this does not include the original Susskind mass term $\mathrm{tr}\, \bar\psi \psi \gamma_5$. This mass term anticommutes rather than commutes with $R(A)$, which means that it changes sign under the rotation (7). This change of sign can however be compensated by a "chiral" transformation

$$\psi(x) \to \gamma_5 \psi(x) \gamma_5, \qquad \bar\psi(x) \to -\gamma_5 \bar\psi(x) \gamma_5, \quad (13)$$

so that the action remains invariant under the joint action of (13) and (10). A mass matrix of the form

$\gamma_5 \Sigma_\varphi \gamma_\varphi$ also anticommutes with $R(A)$, but the resulting change of sign cannot be compensated by the "chiral" transformation. The general structure of the mass matrix that keeps the action invariant under the above joint transformation is

$$M_{(2)} = m'' \gamma_5 + \tfrac{1}{2} m' \sum_\varphi \gamma_\varphi . \tag{14}$$

The second piece, which we have met before, stays invariant separately under the two transformations.

The fact that one can have a mass term which is invariant under (13) means incidentally that this is not a normal chiral transformation. In fact if we go to the naive continuum limit and redefine the fields in such a way that the m' term becomes the usual mass term m' tr $\bar{\psi}'' \psi''$, one finds that in that basis the transformation (13) is a pure rotation in the species ("flavour") space and has nothing to do with chirality. Consequently it is not quite accurate to say that Susskind fermions have chiral invariance.

It may be of interest to look at the eigenvalues of the mass matrices. $M_{(1)}$ has two double eigenvalues $m \pm m'$, and describes in the naive continuum limit two fermions of mass $m + m'$, and two of mass $|m - m'|$. If $m \neq m'$, the degeneracy is partly lifted. On the other hand $M_{(2)}$ has the double eigenvalues $\pm (m''^2 + m'^2)^{1/2}$ and describes four degenerate fermions of mass $(m''^2 + m'^2)^{1/2}$ in the naive continuum limit.

Of course, the insistence on these rotational symmetries may be unreasonably restrictive. While it is believed that a cubic symmetry on the lattice is sufficient to guarantee continuous rotational invariance in the continuum, there is no argument saying that it is necessary. If we give up these restrictions, an arbitrary mass matrix M can be introduced and one can have arbitrarily different masses for the fermions in the continuum limit. We examined the effect of mass terms on various diagrams to see if the rotational symmetry is broken in the continuum limit but failed to find anything positive.

So much for bare masses. Can any mass term be produced by quantum corrections? If either m'' or m is zero, then it stays zero because of invariance under (10) or under (13) together with (10). On the other hand the m' term is invariant under both of these symmetries and can in principle be generated even if all mass terms are zero to begin with. The question is, whether it is in fact generated.

The answer depends in the way in which the interaction is introduced. One-loop calculations with the original Susskind interaction [2] do not provide any evidence for the generation of such counterterms. We however worked with a slightly different interaction [4] defined directly on the block lattice. The action is still given by (4), with (2) replaced by

$$\nabla_\mu \psi(x) = \tfrac{1}{2} \{u(x,\mu)\, \psi(x+e_\mu) - u^{-1}(x-e_{\mu},_\mu)\, \psi(x-e_\mu)\},$$

$$\Delta_\mu \psi(x) = \tfrac{1}{2} \{u(x,\mu)\, \psi(x+e_\mu) + u^{-1}(x-e_{\mu},_\mu)\, \psi(x-e_\mu) - 2\psi(x)\}. \tag{15}$$

To do weak coupling perturbations one sets $u(x,\mu) = \exp[ig\frac{1}{2}\lambda^i A^i_\mu(x)]$ and expands in g. Here $\frac{1}{2}\lambda^i$ denotes the ith generator of the gauge group SU(N).

We examined the one-loop contributions to the fermion self-energy and the vacuum polarization tensor. We worked in a covariant ξ-gauge and obtained for the fermion self-energy $\Sigma(0)$ at zero mass ($M = 0$) and zero momentum the ξ-independent result

$$\Sigma(0) = \frac{N^2-1}{2N} \frac{3g^2}{16}\; \mathbf{1} \otimes \sum_\varphi \gamma_\varphi^{\mathrm{T}} \times \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \frac{1}{\Sigma_\mu \sin^2 \frac{1}{2} l_\mu} . \tag{16}$$

The first matrix, viz., $\mathbf{1}$ acts in spinor space while the other is in the species ("flavour") space. In our matrix notation, then, the counterterm has the expected structure tr $\bar{\psi}\psi\Sigma_\varphi\gamma_\varphi$. This result means that the action (4) with the interaction given by (15) is not self-consistent and a mass term of the above form has to be added. If the physical mass is to be zero, a fine tuning of the coefficient of the term is necessary. This is precisely what happens with Wilson fermions [7], so that the real practical superiority of Susskind fermions to Wilson fermions collapses when the interaction is introduced as in (15).

We have also calculated the vacuum polarization tensor to one-loop level. To extract the Λ-parameter it is sufficient [8] to extract the terms quadratic in the momenta, keeping a mass parameter, e.g. m'', nonzero at first. We obtain

$$\pi^{ij}_{\mu\nu}(p) = \delta^{ij}\, 4g^2(p_\mu p_\nu - p^2\delta_{\mu\nu}) \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \times \frac{1-\frac{1}{3}\cos l_1 \cos l_2}{(m''^2 + 4\Sigma_\varphi \sin^2 \frac{1}{2} l_\varphi)^2} + \text{higher degree terms.} \tag{17}$$

Comparing with previous work [8], we then obtain the ratio

$$\frac{\Lambda_{\min}}{\Lambda_{(\text{Becher-Joos})}} = 12.51, \quad \text{for } N = 3, \qquad = 7.77, \quad \text{for } N = 2. \tag{18}$$

It is curious that these values are very close to those for the pure gauge theory,

$$\frac{\Lambda_{\min}}{\Lambda_{(\text{pure gauge})}} = 10.85, \quad \text{for } N = 3, \qquad = 7.46, \quad \text{for } N = 2, \tag{19}$$

in contrast to the case of the original Susskind interaction:

$$\frac{\Lambda_{\min}}{\Lambda_{(\text{original Susskind})}} = 28.78, \quad \text{for } N = 3, \qquad = 34.44, \quad \text{for } N = 2. \tag{20}$$

Of course, these numbers have no intrinsic meaning and must be coupled with nonperturbative calculations, e.g. of the Monte Carlo type, including fermion loop effects.

References

[1] L. Susskind, Phys. Rev. D16 (1977) 3031.
[2] H.S. Sharatchandra, H.J. Thun and P. Weisz, Nucl. Phys. B192 (1981) 205.
[3] K. Wilson, Phys. Rev. D10 (1974) 2445.
[4] P. Becher and H. Joos, Z. Phys. C15 (1982) 343.
[5] F. Gliozzi, Nucl. Phys. B204 (1982) 419;
H. Kluberg-Stern, A. Morel, O. Napoly and B. Petersson, Saclay preprint D. Ph.–T 82/56 (July 1982);
cf. also P. Becher, Phys. Lett. 104B (1981) 221.
[6] P. Mitra, Phys. Lett. 123B (1983) 77.
[7] L. Karsten and J. Smit, Nucl. Phys. B183 (1981) 103.
[8] P. Weisz, Phys. Lett. 100B (1981) 331.