IMPROVED CONTINUUM LIMIT IN THE LATTICE O(3) NON-LINEAR SIGMA MODEL

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Received 18 March 1983

First Monte Carlo results are reported for the two-dimensional O(3) non-linear sigma model with an action perturbatively improved up to one-loop order. We find a markedly improved scaling behaviour of the correlation length and of the magnetic susceptibility. The universal parts of the β- and γ-functions are approached fast by the numerically calculated functions.

In massless renormalizable lattice theories (asymptotically free, to be reasonably assured of a non-trivial continuum limit) all physical quantities are, in the continuum limit, proportional to the appropriate power of the correlation length (inverse mass gap) ξ with universal coefficients. For non-zero lattice spacing $a \neq 0$ this "scaling" is violated by *non-universal* terms of order $O((a/\xi)^2 \ln(\xi/a)^{\cdot})$ [1]. By including in the lattice action judiciously chosen "irrelevant" terms (next-to-nearest-neighbour couplings and other higher dimensional terms having, for dimensional reasons, an extra factor a^2). these violations can be reduced to $O((a/\xi)^4 \ln(\xi/a)^{\cdot})$ [2–5]. More explicitly, n-point Green-functions calculated on the lattice with "standard" action obey [6]

$$[-a\, \partial/\partial a + \bar{\beta}(g)\, \partial/\partial g + n\bar{\gamma}(g)]\, G_n(p_1, ..., p_n; g, a)$$
$$= O(a^2 \ln a^{\cdot})\,. \tag{1}$$

Improvement, as described, leads to $O(a^4 \ln a^{\cdot})$ on the right-hand side. The $\bar{\beta}$- and $\bar{\gamma}$-functions have in the O(N) non-linear sigma model the universal parts [7]:

$$\beta_{\text{univ}}(g) = -[(N-2)/2\pi]\, g^2 - [(N-2)/4\pi^2]\, g^3\,, \tag{2a}$$

$$\gamma_{\text{univ}}(g) = [(N-1)/2\pi]\, g\,. \tag{2b}$$

In this model, the improved action has the following form [4]:

$$S_{\text{imp}} = g^{-1} a^2 \sum_j \Big[(-\tfrac{1}{2}\phi K\phi + J\phi)$$
$$+ a^2 J[c_1 \phi(\phi K\phi) + c_2 K\phi] + a^2 c_3 (J\phi)^2 + a^2 c_4 J^2$$
$$+ a^2 c_5 (K\phi)^2 + a^2(-\tfrac{1}{24} + c_6) \sum_\mu (\partial_\mu \partial_\mu^+ \phi)^2$$
$$+ a^2 c_7 (\phi K\phi)^2 + a^2 c_8 \sum_\mu (\phi \partial_\mu \partial_\mu^+ \phi)^2$$
$$+ a^2 c_9 \sum_{\mu\nu} \left(\frac{\partial_\mu + \partial_\mu^+}{2} \phi \cdot \frac{\partial_\nu + \partial_\nu^+}{2} \phi \right)^2 \Big]\,. \tag{3}$$

Here the O(N)-vectors $\phi \equiv \phi_j$ on lattice sites j are normalized to unity: $\phi^2 = 1$, and the following notations are used:

$$\partial_\mu \phi_j = a^{-1}(\phi_{j+\hat{\mu}} - \phi_j)\,, \quad \partial_\mu^+ \phi = a^{-1}(\phi_j - \phi_{j-\hat{\mu}})\,,$$

with

$$(j \pm \hat{\mu})_\nu = j_\nu \pm \delta_{\mu\nu}$$

$$K = -\sum_{\mu=1}^{2} \partial_\mu \partial_\mu^+\,. \tag{4}$$

[1] Supported by Deutsche Forschungsgemeinschaft.
[2] Supported by Bundesministerium für Forschung und Technology.

The Green-functions are obtained by differentiating $Z_{\mathrm{imp}} = \int [\mathrm{d}\phi] \exp(S_{\mathrm{imp}})$ with respect to the source vector J. The irrelevant terms in (3) involving the source J are characteristic for models defined in terms of constraints [8] [‡1]. The improvement coefficients $c_i(g, N)$ can be determined in perturbation theory, in $1/N$ expansion or, in principle, also by Monte Carlo checks of scaling ("trial and error"). In lowest (one loop) order perturbation theory, defining $c_i = c_{i1}g + c_{i2}g^2 + \dots$, one finds (for details see ref. [8]):

$$c_{11} = \delta - c_{31} - c_{71} ,$$

$$c_{21} = 2\gamma(N-1) - \beta - (1 + 1/N)\delta + (1 + 1/N)(c_{31} + c_{71}) ,$$

$$c_{41} = -(N-1)\gamma - (1 + 1/N)c_{31} ,$$

$$c_{51} = \delta + \beta - (N-1)\gamma - (1 + 1/N)c_{71} ,$$

$$c_{61} = \alpha + (3/N)(\epsilon - \delta) ,$$

$$c_{81} = 3(\delta - \epsilon) , \quad c_{91} = -4\delta \tag{5}$$

with

$$\alpha = \int_K \frac{(\partial^4/\partial K_1^4) N(K_1)}{48 \hat{N}(K)} = -0.0061880 \dots$$

$$\beta = \int_K \left(\frac{(\partial^2/\partial K_1^2) N(K_1)}{4 [\hat{N}(K)]^2} - \frac{1}{2K^4} \right) - \frac{2+\pi}{16\pi^4} = -0.017308 \dots$$

$$\gamma = \int_K \left(\frac{1}{4[\hat{N}(K)]^2} - \frac{1}{4K^4} \right) - \frac{2+\pi}{32\pi^4} = 0.0011994 \dots$$

$$\delta = \int_K \left(\frac{(\partial^2/\partial K_1^2) N(K_1) \cdot (\partial^2/\partial K_2^2) N(K_2)}{48 [\hat{N}(K)]^2} - \frac{1}{12K^4} \right) - \frac{2+\pi}{96\pi^4} = -0.0048591 \cdots \tag{6}$$

$$\epsilon = \int_K \left(\frac{[(\partial^2/\partial K_1^2) N(K_1)]^2}{48 [\hat{N}(K)]^2} - \frac{1}{12K^4} \right) - \frac{2+\pi}{96\pi^4} = -0.0019429 \dots \tag{6 con'd}$$

whereby

$$N(K_\mu) = 2(1 - \cos K_\mu) + \tfrac{1}{3}(1 - \cos K_\mu)^2 ,$$

$$\hat{N}(K) = \sum_{\mu=1}^{2} N(K_\mu) , \quad \int_K \equiv \prod_{\mu=1}^{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}K_\mu}{2\pi} ,$$

$$K^4 \equiv (K_1^2 + K_2^2)^2 .$$

The denominators in the one-loop integrals in (6) are "truncated SLAC-improved". For the nine improvement coefficients in (5) only seven relations are obtained due to two linear relations [4] between the nine irrelevant terms in (3). This follows from the Schwinger–Dyson equations derived from the unimproved action. In the numerical computations we found the choice $c_3 = c_7 = 0$ most convenient.

We have carried out a Monte Carlo investigation of the improved O(3) model. In this letter we concentrate on the mass gap, the magnetic susceptibility and the numerical determinations of the β-function. Our results are obtained on a 50^2 lattice with periodic boundary conditions and we always compare with corresponding results for the standard action on the same size lattice. (For previous Monte Carlo calculations in the O(3) model see refs. [9–14]. A more detailed account of our Monte Carlo results will be published elsewhere [15].) For the standard action we used the heat bath method, whereas for the improved action we rely on the Metropolis method. With four Metropolis trials for the updating of a single spin our computer program with the improved action is about 5 times slower in the updating than the program with standard action. Measurements need in the present form about the same time for both actions, therefore the final factor between the two actions was roughly 2.5.

The numerical results for the mass gap m (inverse correlations length ξ) are summarized in fig. 1. For the improved action there is a "scaling window" for $\xi \geqslant (1.5\text{–}2)a$ $(0.7 \leqslant \beta \equiv g^{-1} \leqslant 1.1)$. In contrast, no real scaling is seen on our 50^2 lattice for the standard

[‡1] The action proposed in ref. [3] does not improve the $\geqslant 4$-point functions to one-loop order.

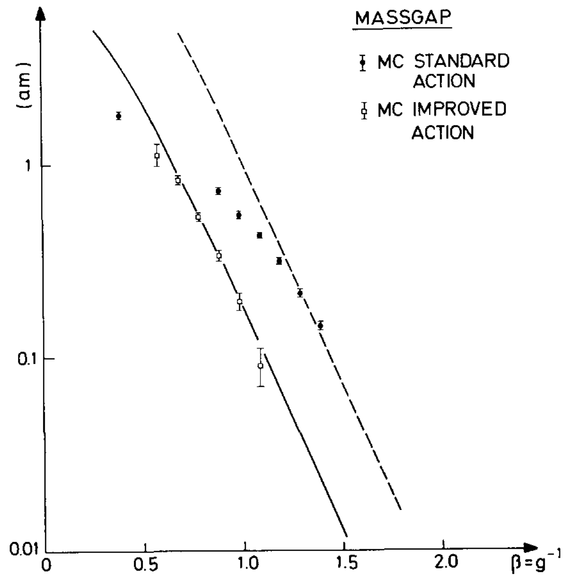

Fig. 1. The mass gap in lattice units am for the standard action (full circles) and improved action (open squares). The full line is the scaling curve $am = 100\,\beta\exp(-2\pi\beta)$ for the improved action. The dashed line is the scaling curve for the standard action $am = 600\,\beta\exp(-2\pi\beta)$ obtained in ref. [15] on a 200^2 lattice.

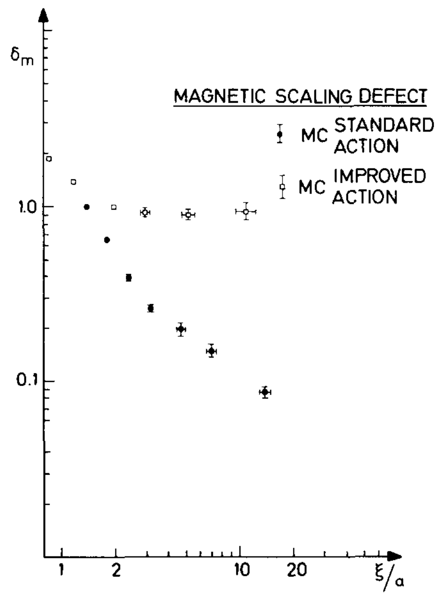

Fig. 2. The scaling defect δ_m of magnetic susceptibility, as defined in the text, for the standard action (full circles) and improved action (open squares) as a function of the correlation length ξ/a.

action. From refs. [12,14] we know, however, that on a large (200^2) lattice there is a small scaling window for $\xi \geqslant 6a$ ($1.4 \leqslant \beta \leqslant 1.6$). Our points at $\beta = 1.3$ and 1.4 for the standard action are consistent with refs. [9,12,14].

The "scaling defect" $\delta_m = \mathrm{const.}\,\beta^4\exp(-4\pi\beta)\chi_m$ of the magnetic susceptibility χ_m is shown in fig. 2 as a function of the correlation length. Here the difference between the two actions is more striking: the large scale breaking observed previously [10,11] for the standard action disappears almost completely due to the improvement. The standard action may do better for $\beta \geqslant 1.4$ [14], but then a much larger lattice is needed.

Neglecting finite size effects, the lattice β-function in (1) can be calculated from the mass gap m as

$$\bar{\beta}(g) = -[\partial\ln(am)/\partial g]^{-1}\,. \qquad (7)$$

Alternatively, we may use, for instance, the two-point function

$$S_2 = \delta^2 Z[J]/\delta J_1\delta J_2|_{J=0}\,. \qquad (8)$$

In order to eliminate $\bar{\gamma}(g)$, we measure S_2 for different lattice momenta $ap = (2\pi/L)k$ (L = periodicity length in lattice units, $k = 0, 1, 2, ...$) and take the ratios $R_2 = S_2(ap)/S_2(ap = 0)$. ($S_2(ap = 0)$ is the magnetic susceptibility χ_m.) Eq. (1) gives

$$\bar{\beta}(g) = \frac{\partial R_2}{\partial\ln k}\bigg/\frac{\partial R_2}{\partial g}\,. \qquad (9)$$

Our numerical results for the expressions (7) and (9) are summarized in table 1. It is seen that the numerically obtained lattice β-function for the improved action (3) is, within the statistical error of about 5–10%, equal to the universal part β_{univ} of the β-function. For the standard action $\bar{\beta}(g)$ is quite different, although the mass and the two-point function give similar deviations from β_{univ}, especially for the larger values of $\beta = g^{-1}$. For smaller β values $\bar{\beta}(g)$ obtained from the two-point function and from the mass begin to deviate also from each other. Table 1 and fig. 2 together imply that for the improved action also the wave function renormalization $\bar{\gamma}$-function has, in the considered range, the universal value.

In conclusion, in the O(3) model the one-loop per-

Table 1
The values of the lattice β-function $\bar{\beta}_{7,9}$ as obtained from eqs. (7), (9) compared to the universal part β_{univ} of the β-function (2a) for the O(3) model. $k = apL/2\pi$ gives the values of lattice momenta in (9).

	$\beta = g^{-1}$	β_{univ}	$\bar{\beta}_7$	$\bar{\beta}_9$	
standard action	1.0	−0.184	−0.34 ± 0.01	(k = 10) − 0.38 ± 0.01	(k = 16) − 0.37 ± 0.01
	1.1	−0.151	−0.30 ± 0.01	(k = 8) − 0.31 ± 0.01	(k = 12) − 0.30 ± 0.01
	1.2	−0.125	−0.20 ± 0.01	(k = 6) − 0.18 ± 0.01	(k = 12) − 0.19 ± 0.01
	1.3	−0.106	−0.15 ± 0.02	(k = 6) − 0.14 ± 0.01	(k = 12) − 0.14 ± 0.01
improved action	0.9	−0.231	−0.24 ± 0.01	(k = 6) − 0.25 ± 0.01	(k = 8) − 0.24 ± 0.01
	1.0	−0.184	−0.15 ± 0.03	(k = 6) − 0.17 ± 0.02	(k = 10) − 0.19 ± 0.02

turbative improvement has yielded also improved Monte Carlo results. Moreover, from the practical point of view it seems better to use the improved action than to go to larger lattices with the standard action. With more computer time available one should try to optimize the constants in the action (3). In four dimensional lattice gauge theory a similar program of improving the action can be attempted [5].

We would like to thank all members of the Institut für Theoretische Physik, Universität Heidelberg for the possibility to do some calculations on the IBM 370/4341 and the staff of the RHKL at the University of Kaiserslautern for assistance with the Siemens 7561 computer.

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