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CONTINUUM LIMIT AND IMPROVED ACTION IN LATTICE THEORIES (I). Principles and ϕ^4 theory

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Corrections to continuum theory results stemming from finite lattice spacing can be diminished systematically by use of lattice actions that also include suitable irrelevant terms. We describe in detail the principles of such constructions for the example of the ϕ^4 theory.

1. Introduction

The inadequacy of perturbative calculations in field theories of the strong interactions has led to extensive computer simulation of euclidean lattice field theories. For definiteness, we think of a simple-hypercubic lattice, with periodic boundary conditions and equal lattice spacing a and box length $L = Na$ in all $D(=2$ or $=4)$ directions. Disregarding, in all of the following, statistical errors, mimicking the infinite continuum requires $a \ll \xi \ll L$, where ξ is the correlation length, i.e. the physical scale. Violation of these inequalities leads to systematic errors, of which we shall analyze only those due to the finiteness of the lattice spacing. We shall show how to decrease them in the regime where a/ξ can be treated as a small expansion parameter.

For simplicity, we consider a (in D dimensions) renormalizable massless theory with one coupling constant g_B. "Physical quantities" $P(g_B, L, a)$ computed on the lattice then obey [1]

$$\{-a[\partial/\partial a] + \bar{\beta}(g_B)[\partial/\partial g_B]\}P(g_B, L, a) = \mathrm{O}\big((a/\xi)^2 \ln(\xi/a)\big). \quad (1.1)$$

If the r.h.s. were zero, (1.1) would express that $P(\dots)$ depends only on the correlation length

$$\xi = a \exp\left[\int_{C_\xi}^{g_B} dg'/\bar{\beta}(g')\right], \quad (1.2)$$

rather than on a and g_B separately, and on L. Neglecting this latter dependence, i.e. choosing L sufficiently large, $P(g_B, \cdot, a)$ would thus "scale". Scaling is violated by the r.h.s. in (1.1), which leads to

$$P(g_B, \cdot, a) = \xi^{-\dim P}\mathrm{const}_P\left[1 + O\left((a/\xi)^2 \ln(\xi/a)\right)\right]. \tag{1.3}$$

Here const_P is universal but the correction is not and depends, not only on a/ξ and on P, but also on the precise lattice action used.

Choosing ξ large relative to a diminishes these corrections, but this is apt to lead to violation of $\xi \ll L$, i.e. to finite-size effects that have been found to be a major source of difficulties in practice [2]. They can be made less severe if a/ξ is allowed to be not very small, which is more acceptable if the correction in (1.3) is reduced to $O((a/\xi)^4 \ln(\xi/a))$. This can be achieved by employing an improved lattice action [3] that incorporates a finite number of judiciously chosen terms that compensate for the infinite number of $(a/\xi)^2 \ln(\xi/a)$ corrections.

That this reduction is possible for the lattice ϕ_4^4 theory we shall prove in this first paper of the series, since this is the simplest and most transparent renormalizable model. What this improvement, in the sense described, effects here is, e.g. the following. Define a normalization-invariant renormalized coupling constant g_{ren}, which is a "physical quantity" for this model. In the massless (and $L = \infty$) theory, g_{ren} is defined by a prescription that must involve an arbitrary mass scale μ, playing the rôle of L in (1.1). If we define (with an arbitrary lower limit of integration)

$$\bar{\rho}(g_B) = \int^{g_B} \mathrm{d}g' \bar{\beta}(g')^{-1}, \tag{1.4}$$

then

$$g_{\mathrm{ren}} = f_0\left[\bar{\rho}^{-1}(\bar{\rho}(g_B) + \ln(a\mu))\right] + \sum_{j=1}^{\infty} (a_\mu)^{2j} f_j(g_B, \ln(a\mu)), \tag{1.5}$$

where the function $f_0(z) = z + f_{02}z^2 + f_{03}z^3 + \cdots$ and the functions $f_j(g_B, \ln(a\mu))$ are computable as a power series. (They depend on the defining prescription for g_{ren} and on the precise lattice action.) Improvement of the lattice action results here in $f_1(g_B, \ln(a\mu)) \equiv 0$, besides uninteresting changes of the coefficients $f_{02}, f_{03} \cdots$ and b_k, $k \geqslant 2$, in the f_0 respectively $\bar{\beta}$ function. Using $\bar{\beta}(g_B) = b_0 g_B^2 + b_1 g_B^3 + \cdots$ with $b_0 = 3/(16\pi^2)$, as $a \searrow 0$ the argument of $f_0(..)$ in (1.5) is

$$z = \left(b_0 \ln\frac{1}{a\mu}\right)^{-1} - b_1 b_0^{-1}\left(b_0 \ln\frac{1}{a\mu}\right)^{-2} \ln\ln\frac{1}{a\mu} + O\left(\left(\ln\frac{1}{a\mu}\right)^{-2}\right) \tag{1.6}$$

This shows that (1.5) is compatible with the almost rigorously [4] proven vanishing of the renormalized coupling constant, as $a \searrow 0$, in the (for proof-technical reasons) massive ϕ_4^4 theory on the lattice with nearest-neighbor interactions.

Technically, the improvement procedure can be described as an extension of renormalization, by oversubtraction in the sense of Zimmermann [5] of, however, a peculiar type. As $a \searrow 0$, the diverging parts of (merely) superficially divergent one-particle-irreducible Feynman graphs are the lowest coefficients in the Taylor expansion in momenta and/or bare masses since the Taylor formula remainder is more convergent due to the differentiation w.r.t. these variables. In renormalization, these lowest Taylor expansion coefficients are replaced by numbers as prescribed by the renormalization convention In the improvement, additional Taylor expansion coefficients are replaced by the ones (obtained here with the help of analytical continuation from a higher space-dimension) of the corresponding continuum-theory functions. Hereby, the approach to the continuum theory as $a \searrow 0$ is speeded up. The terms inserted into the lattice action to perform that replacement are "irrelevant" ones of higher operator-dimension and involve also next-to-nearest neighbour couplings. For the usual purpose of computing e.g. mass ratios on the lattice, the "continuum theory" can be defined in such a way that the ordinary renormalization subtractions are unnecessary. In sect. 4 of this paper we show that the program sketched here can be carried out.

In sect. 2, we recall the important concept of the local effective lagrangian (LEL) [6] for lattice theories. The existence of a LEL furnishes the reason for the possibility of improvement, in principle to arbitrarily high order in a^2. An illustrative computation of small-a dependence is made in appendix A, and dimensional regularization of lattice theories, as a proof-technical tool, is commented upon in appendix B. Sect. 3 treats subtraction of the lattice action, first in a simple case related to renormalization (with some formulae relegated to appendix C). This prepares for the improvement process described in sect. 4. Sect. 5 contains concluding remarks.

2. Cutoff dependence and local effective lagrangian

2 1 LATTICE ACTION AND SMALL-a EXPANSION

The lattice action, for later purposes written in $4+\varepsilon$ rather than in 4 dimensions, of ϕ^4 theory (we choose, for simplicity, $L=\infty$ and return to $L<\infty$ only later) is

$$A_0 = a^{4+\varepsilon} \sum_{n \in \mathbb{Z}^{4+\varepsilon}} \left[-\tfrac{1}{2}\phi_n (K\phi)_n - \tfrac{1}{24} g_B \phi_n^4 - \tfrac{1}{2} m_{B0}^2 \phi_n^2 - \tfrac{1}{2}\Delta m_B^2 \phi_n^2 \right]. \tag{2 1}$$

Here

$$K = -\sum_{\mu=1}^{4+\varepsilon} \partial_\mu \partial_\mu^+ , \tag{2.2}$$

$$(\partial_\mu f)_n = (f_{n+\hat\mu} - f_n)/a , \qquad (\partial_\mu^+ f)_n = (f_n - f_{n-\hat\mu})/a ,$$

with $(n \pm \hat\mu)_\nu = n_\nu \pm \delta_{\mu\nu}$. $m_{B_0}^2 = a^{-2} f(g_B a^{-\varepsilon}, \varepsilon)$ is the bare mass squared which, if

$\Delta m_B^2 = 0$, leads to an infinite correlation length, i.e. zero physical mass. $f(g_B a^{-\varepsilon}, \varepsilon)$ can be calculated in perturbation theory for ε not negative-rational, and can be measured for integer ε. $\Delta m_B^2 \geqslant 0$ then characterizes the one-phase region, the only one we will consider. The more general case, however, can be treated similarly (cf. [7]).

Physical quantities can be extracted from Green functions. For our purposes, more convenient are the vertex functions (VFs), i.e. the one-particle-irreducible (full propagator) amputated parts of connected Green functions. Their Fourier transforms (with overall-momentum conserving delta function times $(2\pi)^{4+\varepsilon}$ taken out) possess (for $\varepsilon = 0$) to all orders in perturbation theory the asymptotic small-a expansion (momenta fixed in the first Brillouin zone)

$$\Gamma(p_1 \dots p_{2n}; g_B, \Delta m_B^2, a) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a^{2j} (\ln a)^k \bar{F}_{jk}(p_1 \dots p_{2n}; g_B, \Delta m_B^2), \quad (2.3)$$

whereby graphs with $\mathfrak{L}$ loops contribute only with $k \leqslant \mathfrak{L}$. The evidence for (2.3) will be discussed in sect. 5. In appendix A we verify the expansion (2.3) for one-loop graphs.

2.2 LOCAL EFFECTIVE LAGRANGIAN

The expansion (2.3) can be obtained directly from an LEL [8]. However, writing down a local lagrangian requires the adoption of a convention regarding its interpretation. An effective lagrangian of the type introduced by Zimmermann [5] could be employed. It is, however, defined only in perturbation theory, and in that frame we have available dimensional regularization [9], which is much more convenient for our purposes. It is then natural also to treat A_0 in $4 + \varepsilon$ dimensions (as it is written in (2.1)), and questions regarding this point are dealt with in appendix B Eq (2.3) is then replaced by

$$\Gamma(p_1 \dots p_{2n}; g_B, \Delta m_B^2, \varepsilon, a) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a^{2j - \varepsilon k} F_{jk}(p_1 \dots p_{2n}, g_B, \Delta m_B^2, \varepsilon), \quad (2.4)$$

whereby $\mathfrak{L}$-loop graphs contribute only with $k \leqslant \mathfrak{L}$. In $4 + \varepsilon$ dimensions we can write down the LEL naively [3]:

$$\begin{aligned} A_0 \approx\ & \tfrac{1}{2} Z_3 \phi \Box \phi - \tfrac{1}{24} Z_1 g_B \phi^4 - \tfrac{1}{2} Z_2 \Delta m_B^2 \phi^2 \\ & + a^2 \Big[\tfrac{1}{2} Z_4 \sum_\mu \phi \partial_\mu^4 \phi + \tfrac{1}{2} Z_5 \phi \Box^2 \phi + \tfrac{1}{6} Z_6 g_B \phi^3 \Box \phi \\ & + \tfrac{1}{720} Z_7 g_B^2 \phi^6 + \tfrac{1}{2} Z_8 \Delta m_B^2 \phi \Box \phi - \tfrac{1}{24} Z_9 \Delta m_B^2 g_B \phi^4 - \tfrac{1}{2} Z_{10} \Delta m_B^4 \phi^2 \Big] + \mathrm{O}(a^4) \\ \equiv\ & L_0 + a^2 L_1 + a^4 L_2 + \cdots , \end{aligned} \quad (2.5)$$

where $\Box = \sum_{\mu=1}^{4+\varepsilon} \partial_\mu^2$. L_j consists of all (engineering) dimension $4 + 2j + \varepsilon$ local monomials of ϕ and its derivatives having lattice symmetry and linear independence at zero momentum. The dimensionless coefficients have the form

$$Z_{\cdot}(g_{\mathrm{B}}a^{-\varepsilon}, \varepsilon) = \sum_{\mathfrak{L}=0}^{\infty} z_{\cdot\mathfrak{L}}(\varepsilon)(g_{\mathrm{B}}a^{-\varepsilon})^{\mathfrak{L}}, \tag{2.6}$$

where $\mathfrak{L}$ denotes the number of loops of the graphs entering the computation (see below). Hereby in (2.5) $z_{i0}(\varepsilon) = 1$ for $i = 1,2,3$, $= \frac{1}{12}$ for $i = 4$, $= 0$ for $i = 5$. 10 The $\approx$ sign in (2.5) signifies equality of the (Fourier transforms of the) VFs, in the sense of the expansion (2.4), computed with the lattice action on the left and with the LEL on the right-hand side, up to the error (neglected terms) quoted in each case.

Eq. (2.5) is the most compact description of the expansion (2.4) since all a-dependence stems only from the explicit one in the coefficients. An LEL similar to (2.5) also applies to lattices with unequal spacings in different directions (e.g., a continuum in one direction) but then with more terms due to correspondingly reduced symmetry. The evidence for (2.5) is discussed in sect. 5, and a sample computation is included in appendix B.

2.3 DETERMINATION AND PROPERTIES OF THE Z COEFFICIENTS

Granted the adequacy of the LEL (2.5), the coefficients Z_i can be obtained [5] by computing the VFs at zero momenta or, in the case of terms with derivatives, suitably differentiated at zero momenta and, moreover, in the case of terms with powers of Δm_{B}^2 having also the corresponding number of arguments $\frac{1}{2}\phi^2$, these can be computed at zero momenta, always in the massless ($\Delta m_{\mathrm{B}}^2 = 0$) lattice theory. Namely,

$$Z_1 = -g_{\mathrm{B}}^{-1}\Gamma(0000,, g_{\mathrm{B}}, 0, \varepsilon, a), \qquad \mathrm{Re}\,\varepsilon > 0, \tag{2.7a}$$

$$Z_2 = \Gamma(00,0; g_{\mathrm{B}}, 0, \varepsilon, a), \qquad \mathrm{Re}\,\varepsilon > 0, \tag{2.7b}$$

$$Z_3 = -\tfrac{1}{2}(\partial/\partial p^2)\Gamma(p(-p); g_{\mathrm{B}}, 0, \varepsilon, a)|_{p=0}, \qquad \mathrm{Re}\,\varepsilon > 0, \tag{2.7c}$$

and for instance

$$Z_7 = g_{\mathrm{B}}^{-2}\Gamma(000000,; g_{\mathrm{B}}, 0, \varepsilon, a), \qquad \mathrm{Re}\,\varepsilon > 2, \tag{2.7d}$$

where the number of zeros behind the first comma denotes the number of $\frac{1}{2}\phi^2$ arguments The expressions for the other Z_i in (2.5) are given in ref. [5]. In (2.7), $Z_{1,2,3}$ requires the evaluation of the r.h.s. for $\mathrm{Re}\,\varepsilon > 0$, and Z_7 requires $\mathrm{Re}\,\varepsilon > 2$. Generally, Z factors in L_j require for determination in the way indicated $\mathrm{Re}\,\varepsilon > 2j$, and are to be analytically continued from there (see appendix B). The reason for the validity of formulae (2.7) is that in the evaluation of the r.h.s., with the help of the continuum lagrangian (2.5), due to $\Delta m_{\mathrm{B}}^2 = 0$ and familiar rules of dimensional integration only the corresponding Born terms contribute, since all $\mathfrak{L} \geqslant 1$ graphs give vanishing contributions for dimensional reasons in the prescribed dimension range.

Since the LEL (2.5) yields by its definition finite VFs, its coefficients $Z_{1,2,3}$ have the 't Hooft form [10]: the corresponding $z_{\cdot\mathfrak{L}}(\varepsilon)$ in (2.6) are meromorphic, with poles $\varepsilon^{-1}\dots\varepsilon^{-\mathfrak{L}}$ but also with increasing powers of ε in their Laurent expansion at $\varepsilon = 0$. Thus, they are not minimal in the sense of ref. [10]. Likewise, the coefficients $Z_4\dots Z_7$ have the property that L_1, with $\Delta m_B^2 = 0$, is the sum of four linearly independent renormalized operators of dimension 6 in the sense of ref. [10] since they yield, upon insertion under dimensional integration rule, finite results (cf. (2.3)) as $\varepsilon \to 0$. ($Z_{8,9,10}$ do not lead to linearly independent additional operators and can be expressed in terms of the usual parametric functions (C.2a–c)). The finiteness properties of the terms in L_1 can be exploited, similarly as in the a^0 part in (1.5), (1.6), to "partially sum the logarithms" with $j = 1$ in (2.3), as was discussed for g_{ren} of the massive theory in ref. [6], and as can be done in all theories that are "asymptotically infrared free" as the ϕ_4^4 theory is. Similar considerations apply to L_2 etc. terms in (2.5).

An LEL of the type (2.5) also holds if "irrelevant" terms are added to the action (2.1), which are the lattice analogs of the terms in a^2L_1, a^4L_2 etc. Counting the number of available parameters then suggests that modifying A_0 to $A_0 + a^2A_1$ will allow us to achieve $L_1 \equiv 0$ in the corresponding LEL. Before we prove this in sect. 4, we treat a simpler case in sect 3.

3. Subtraction of the lattice action

3.1 DESCRIPTION AND EFFECTS OF SUBTRACTION

We compute VF graphs recursively order by order, starting with A_0 of (2.1), in $4+\varepsilon$ dimensions with $\mathrm{Re}\,\varepsilon > 0$, in the following way. From an $\mathfrak{L}$-loop graph, which for $2n$ external momenta is of order $g_B^{\mathfrak{L}+n-1}$, its Taylor expansion in the (independent) external momenta and Δm_B^2 (counted double) at zero momenta and $\Delta m_B^2 = 0$ of order $4-2n$ is subtracted. The monomials in ϕ and its derivatives and Δm_B^2 effecting this subtraction for the sum of all such graphs have a factor $a^{-\varepsilon\mathfrak{L}}$ and are added to the action, and used when going to $(\mathfrak{L}+1)$-loop graphs. If this procedure is followed from $\mathfrak{L} = 1$ on, no infrared divergences arise. Namely, what is generated in this way are the VFs of the theory with lagrangian (2.1) calculated by the rules of dimensional integration, in the $(4+\varepsilon)$-dimensional continuum rather than on the lattice, up to an error $O(a^{2-\varepsilon\mathfrak{L}})$. Indeed, the outlined procedure leads, in the $a \searrow 0$ limit, for $\mathrm{Re}\,\varepsilon < 2/\mathfrak{L}$ to UV convergent expressions, since power counting yields $\vartheta = 4 - 2n + \mathfrak{L}\,\mathrm{Re}\,\varepsilon$ as the degree of UV divergence for the $\mathfrak{L}$-loop graphs, such that the subtraction described is the minimal one giving UV convergence in the no-cutoff limit. Calling the action including all subtracting terms $A_0 + \Delta A_0$, the terms in ΔA_0 must have IR finite coefficients since otherwise they would not lead to finite VFs in the order in which these coefficients first contribute. That the VFs computed with action $A_0 + \Delta A_0$ have the property stated follows from the vanishing of the corresponding subtractions in the dimensionally integrated continuum theory, for the reason mentioned in sect. 2 in connection with (2.7). That the error term is $O(a^{2-\varepsilon\mathfrak{L}})$ follows from the insufficiency of the subtraction to yield finite VFs in the $a \searrow 0$ limit if $\mathrm{Re}\,\varepsilon > 2\mathfrak{L}^{-1}$.

Our result can be stated as

$$A_0 + \Delta A_0 \approx L_{00} + O(a^{2-\varepsilon\mathfrak{L}}), \tag{3.1}$$

where L_{00} has the terms of A_0 in (2.1) (disregarding the m_{B0}^2 one) but in the continuum. As to derivatives, we note: the terms of the same form to be added to $-\frac{1}{2}\phi K\phi$ on the lattice stem from differentiation of lattice-computed self-energy parts $\partial/\partial p^2 \Pi(p)|_{p=0}$. The negative lattice laplacian K of (2.2) simulates p^2 with $a^2\Sigma p_\mu^4$ error. This error means to bring in implicitly an unneeded dimension-6 term. It affects contributions outside the error limit stated in (3.1) only when it appears as the vertex in a graph with loops and thus in higher order. Then, however, the subtraction procedure described above absorbs all its effects, again up to the error stated in (3.1).

The subtraction procedure leading to (3.1) can also be described as follows: upon modifying $A_0 \to A_0 + \Delta A_0$ to $\mathfrak{L}$-loop order, one computes the LEL to this lattice action by the prescription outlined in connection with (2.7a)–(2.7c). This LEL deviates from L_{00} only in $(\mathfrak{L}+1)$-loop (and higher) order (cf. the beginning of this section). The lattice form of that continuum term, with change of sign, is the $(\mathfrak{L}+1)$-loop-order part of ΔA_0.

3.2 ANALYTIC CONTINUATION OF COEFFICIENTS

The $\vartheta > 0$ lattice VFs differentiated as described at zero momenta and $\Delta m_B^2 = 0$ yield, in each order, integral representations for the coefficients in ΔA_0. These coefficients are, therefore, meromorphic in ε and regular for $\mathrm{Re}\,\varepsilon > 0$ but have IR singularities at $\varepsilon = 0$ and $\varepsilon = -2/l$ with $1 \leqslant l \leqslant \mathfrak{L}$. The integral representations for the analytic continuation to $\mathrm{Re}\,\varepsilon < 0$ is obtained by subtracting from the lattice VF mentioned above, the same VF computed but, however, with L_{00} in (3.1) in the continuum and combining the momentum space integrands "under the integral sign". The simplest one-loop graph shows what is meant here (BZ = Brillouin zone):

$$\int_{\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K \Big(K^2 - \tfrac{1}{12}a^2 \sum_\mu K_\mu^4 + \cdots \Big)^{-2}$$

$$\to \int_{\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K \left[\Big(K^2 - \tfrac{1}{12}a^2 \sum_\mu K_\mu^4 + \cdots \Big)^{-2} - (K^2)^{-2} \right] - \int_{\mathrm{R}^{4+\varepsilon}-\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K (K^2)^{-2}$$

$$= \int_{\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K \left[\tfrac{1}{6}a^2 \sum_\mu K_\mu^4 (K^2)^{-3} + \cdots \right] - a^{-\varepsilon}\big[2\pi^2/(-\varepsilon) + \mathrm{const} + O(\varepsilon)\big], \tag{3.2}$$

whereby the BZ integral becomes IR divergent as $\mathrm{Re}\,\varepsilon \searrow -2$ and the $\mathrm{R}^{4+\varepsilon}-\mathrm{BZ}$ integral UV divergent as $\mathrm{Re}\,\varepsilon \nearrow 0$.

The integral representations for the coefficients at $\mathrm{Re}\,\varepsilon < 0$ mean that in every order, the Taylor expansion of the lattice graph is replaced by the Taylor expansion of the continuum graph by virtue of the subtraction (i.e. its analytical continuation). This explains the equivalence expressed in (3.1) also for $\mathrm{Re}\,\varepsilon < 0$.

3 3 REPARAMETRIZATION AND RENORMALIZATION

Since A_0 in (3.1) comprises terms of precisely the same form as in A_0, we can write

$$A_0 + \Delta A_0 = a^{4+\varepsilon}\sum_n\left[-\tfrac{1}{2}\mathcal{Z}_3\phi_n(K\phi)_n - \tfrac{1}{24}\mathcal{Z}_1 g_B\phi_n^4 - \tfrac{1}{2}\tilde{m}_{B0}^2\phi_n^2 - \tfrac{1}{2}\mathcal{Z}_2\Delta m_B^2\phi_n^2\right], \tag{3.3}$$

where the $\mathcal{Z}(g_B a^{-\varepsilon}, \varepsilon)$ have again the form (2.6) with $z\cdot \mathrm{O}(\varepsilon) = 1$, and $\tilde{m}_{B_0}^2 = a^{-2}\tilde{f}(g_B a^{-\varepsilon}, \varepsilon)$ is a new function defined as a power series. Due to the IR (or UV) singularities of the $\mathcal{Z}$ at $\varepsilon = 0$ the action (3.3) does not give finite results there as the r.h.s. of (3.1) shows. Now we define

$$\bar{g}_B = g_B\mathcal{Z}_1(g_B a^{-\varepsilon}, \varepsilon)\mathcal{Z}_3(g_B a^{-\varepsilon}, \varepsilon)^{-2}, \tag{3.4a}$$

$$\overline{\Delta m}_B^2 = \Delta m_B^2\mathcal{Z}_2(g_B a^{-\varepsilon}, \varepsilon)\mathcal{Z}_3(g_B a^{-\varepsilon}, \varepsilon)^{-1}, \tag{3.4b}$$

and, upon solving (3.4a), the three power series ($i = 1,2,3$)

$$Z_i(\bar{g}_B a^{-\varepsilon}, \varepsilon) = \mathcal{Z}_i(g_B a^{-\varepsilon}, \varepsilon)^{-1}. \tag{3.4c}$$

Then, setting

$$\bar{\phi} = \phi\mathcal{Z}_3(g_B a^{-\varepsilon}, \varepsilon)^{1/2}, \tag{3.4d}$$

(3.1) takes the form

$$A_0 \approx L_0 + \mathrm{O}(a^{2-\varepsilon E}), \tag{3.5}$$

where A_0 is as in (2.1) and L_0 as in (2.5), with ϕ, g_B, Δm_B^2 replaced by $\bar{\phi}$, $\bar{g}_B$, $\overline{\Delta m}_B^2$ in each case. This verifies the validity of (2.5) to order a^0 The use of (3.5) for renormalization is summarized in appendix C.

If the subtraction procedure that led to (3.1) is carried out in the $\Delta m_B^2 > 0$ theory using Taylor expansions in the momenta at zero momenta only, rather than in Δm_B^2 and momenta at $\varepsilon = 0$, no IR divergences arise. Setting then also $a = 0$ would mean computing with A_0 treated as an effective (continuum) action in the sense of Zimmermann [5]. Since no regularization is involved then, the result cannot be written in terms of a "true" lagrangian with Z factors. While the procedure we followed offers no particular advantage in ordinary renormalization, its extension to $O(a^2)$ terms leads directly to improvement.

4. Improvement of the lattice action

4.1 DESCRIPTION AND EFFECTS OF HIGHER SUBTRACTION

Again we compute VF graphs, starting with A_0 as described in sect. 3, but perform, with $\mathrm{Re}\,\varepsilon > 2\mathfrak{L}^{-1}$ rather than merely > 0, the Taylor expansion subtractions at zero momenta and $\Delta m_B^2 = 0$ to order $6-2n$ for a $2n$-point function. The coefficients of field monomials effecting the additional subtraction have factors $a^{2-\varepsilon\mathfrak{L}}$, and will turn out to be IR finite for ε as stated provided that already the $\mathfrak{L}=0$ two-point VF, the negative inverse bare propagator, is subjected to the last subtraction:

$$-4a^{-2}\sum_{\mu=1}^{4+\varepsilon} \sin^2\tfrac{1}{2}p_\mu a = -p^2 + \tfrac{1}{12}a^2\sum_\mu p_\mu^4 - \cdots$$

$$\to -4a^{-2}\sum_\mu\left[\sin^2\tfrac{1}{2}p_\mu a + \tfrac{1}{3}\sin^4\tfrac{1}{2}p_\mu a\right]$$

$$= -p^2 + \tfrac{1}{90}a^4\sum_\mu p_\mu^6 - \cdots . \qquad (4.1)$$

The lattice term effecting this is a next-to-nearest-neighbor coupling term, and the zeroth-order kinetic part becomes, in the notation (2.2),

$$-\tfrac{1}{2}\phi_n(K\phi)_n - \tfrac{1}{24}a^2\phi_n\left(\sum_\mu\left(\partial_\mu\partial_\mu^+\right)^2\phi\right)_n \equiv -\tfrac{1}{2}\phi_n(\tilde{K}\phi)_n . \qquad (4.2)$$

This improved negative lattice laplacian $\tilde{K}$ must also be used in effecting, as part of the lattice action, the higher-order subtractions of this $\partial/\partial p^2\Pi(p)|_{p=0}$ type for $2n=2$ graphs. (We can say that the p^2 in the propagator, and in effecting the self-energy subtraction, must be represented on the lattice "truncated SLAC improved".) No similar precaution is necessary for the other derivatives that appear.

The procedure described here leads to an action that can be written $A_0 + \Delta A_0' + a^2 A_1$. Here $\Delta A_0'$ is as in (3.1) or, more explicitly, (3.3) but with new coefficients $\mathfrak{Z}_l'(g_B a^{-\varepsilon}, \varepsilon) - 1$, $l = 1\ldots3$. A_1 consists of the lattice analogs of the terms in the square bracket representing L_1 in (2.5). The terms of the form $-\tfrac{1}{2}\phi(\tilde{K}-K)\phi$, yielding the improved bare propagator and the corrected self-energy subtractions mentioned after (4.2) are part of the first term in A_1 as just described.

The analog of (3.1) is

$$A_0 + \Delta A_0' + a^2 A_1 \approx L_{00} + \mathrm{O}(a^{4-\mathfrak{L}\varepsilon}), \tag{4.3}$$

because the continuum limit $a \searrow 0$ now leads to UV finite results for $\mathrm{Re}\,\varepsilon < 4\mathfrak{L}^{-2}$. For a reason analogous to that given in sect. 3, the coefficients in A_1 are meromorphic in ε and IR finite for $\mathrm{Re}\,\varepsilon > 2\mathfrak{L}^{-1}$. At $\varepsilon = 2\mathfrak{L}^{-1}$ they are IR singular. Their analytic continuation to $-2\mathfrak{L}^{-1} < \mathrm{Re}\,\varepsilon < 2\mathfrak{L}^{-1}$ is again obtained by subtracting the corresponding continuum graph "under the integral sign" (cf. (4.5) below). $\varepsilon = 0$ is a singular point, however, due to the $\varepsilon = 0$ singularities in $\Delta A_0'$ in (4.3) or also the form of the r.h.s.

The consistency of the subtraction prescription rests on the following. The final subtraction as described in an $\mathfrak{L}$-loop graph at $2\mathfrak{L}^{-1} < \mathrm{Re}\,\varepsilon < 4\mathfrak{L}^{-1}$ is, in the $a \searrow 0$ limit, a minimal one in the sense of Zimmermann [5]. The subtraction of an $\mathfrak{L}'$-loop subdiagram is then not minimal if $\mathfrak{L}' < 2(\mathrm{Re}\,\varepsilon)^{-1}$. However, it is not an oversubtraction [5] (such a one would here be IR divergent) but the replacement of the highest Taylor expansion coefficient on the lattice by the one in the continuum, as demanded for L_{00} in (4.3), by virtue of an analytic continuation as in (3 2) (see (4.5) below.) The difference is proportional to $a^{2-\varepsilon\mathfrak{L}'}$, i.e. has a convergence factor that makes minimal subtraction of the $\mathfrak{L}$-loop graph sufficient for the existence of the $a \searrow 0$ limit.

4 2 REPARAMETRIZATION OF THE RESULT

Since $A_0 + \Delta A_0'$ in (4.3) is of the form (3.3) with merely $\mathfrak{Z} \to \mathfrak{Z}'$, the transformation (3.4) can again be performed with new parameters $\bar{g}_\mathrm{B}'$ and $\overline{\Delta m_\mathrm{B}^{2'}}$. Then (4.3) becomes

$$A_0 + a^2 A_1' \approx L_0' + \mathrm{O}(a^{4-\varepsilon\mathfrak{L}}). \tag{4.4}$$

Here L_0' has (replacing $\bar{g}_\mathrm{B}'$ by g_B and $\overline{\Delta m_\mathrm{B}^{2'}}$ by Δm_B^2) singular coefficients $Z_1'(g_\mathrm{B} a^{-\varepsilon}, \varepsilon)$ of the form (2.6), with singularities again of the type to yield finite results at $\varepsilon = 0$, although different from the ones in (2.5) or (3.5) due to the additional term on the l.h.s. in (4.4). The coefficients in A_1' are meromorphic in ε and finite at $\varepsilon = 0$ The proof is recursive: the $\varepsilon = 0$ singular coefficient in $a^2 A_1'$ of lowest order would be incompatible with (4 4) in the order in which it first contributes, since on the r.h.s. of (4.4) terms with coefficients $a^{2-\varepsilon\mathfrak{L}}$ are never generated and a lattice action with finite coefficients (of lower order) cannot, in perturbation theory, yield an infinite result for an $a^{2-\varepsilon\mathfrak{L}}$ coefficient at $\varepsilon = 0$. The analytic continuation of the A_1' coefficients to $\mathrm{Re}\,\varepsilon < 2\mathfrak{L}^{-1}$ is again obtained by subtracting the corresponding continuum expression, which here have such extra factors as appear in L_0', "under the integral sign".

This explains the equivalence expressed by (4.4). The simplest $\mathfrak{L}=1$ example pertaining to (2.7d) illustrates this (cf. (3.2)):

$$\int_{\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K\left(K^2-\tfrac{1}{90}a^4\sum_\mu K_\mu^6+\cdots\right)^{-3}$$

$$\to \int_{\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K\left[\left(K^2-\tfrac{1}{90}a^4\sum_\mu K_\mu^6+\cdots\right)^{-3}-(K^2)^{-3}\right]$$

$$-\int_{\mathbb{R}^{4+\varepsilon}-\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K(K^2)^{-3}$$

$$=\int_{\mathrm{BZ}} \mathrm{d}^{4+\varepsilon}K\left[\tfrac{1}{30}a^4(K^2)^{-4}\sum_\mu K_\mu^6+\cdots\right]-a^{2-\varepsilon}[\mathrm{const}+\mathrm{O}(\varepsilon)] \tag{4.5}$$

Hereby the BZ integral becomes IR divergent at $\mathrm{Re}\,\varepsilon \searrow -2$ and the $\mathbb{R}^{4+\varepsilon}-\mathrm{BZ}$ integral UV divergent at $\mathrm{Re}\,\varepsilon \nearrow 2$.

The l.h.s. in (4.4) at $\varepsilon=0$ is the improved action. The values of its coefficients $C(g_B a^{-\varepsilon},\varepsilon)$ at $\varepsilon=0$, $C(g_B,0)$, can be obtained recursively by subtracting from the lattice VFs computed with the help of lower-order coefficients the part proportional to a^2. That part is polynomial in momenta and Δm_B^2 of order $6-2n$ for $2n$-point VFs and finite (i.e. there is no term e.g. with factor $a^2\ln a$) since otherwise (4.4) could not hold. The analytic continuation in this dimension we needed only for our proof

5. Concluding remarks

5.1 EXTENSION OF PROCEDURE

It is obvious that the process described in sect. 4 can be continued to arbitrarily high order in a^2, to yield instead of (4.4)

$$A_0+a^2A_1^{(k)}+a^4A_2^{(k)}+\cdots+a^{2k}A_k^{(k)}\approx L_0^{(k)}+\mathrm{O}(a^{2k+2-\varepsilon\mathfrak{L}})\,. \tag{5.1}$$

Hereby the improvement coefficients and the r.h.s. all depend, as indicated, on the order to which one carries out the improvement. From (5.1), (2.5) does not follow directly. An LEL can be deduced, however, if Pauli-Villars regularization rather than regularization by a lattice had been used, since the momentum integration range is then the same as in the continuum. This was used in the proof of the LEL existence for that case in ref. [8]. Since the calculations in appendix A and B do not show any deviation from the results of the corresponding ones with Pauli-Villars regularization in spite of the Brillouin zone limitation, we expect that the LEL (2.5) is correct to any order in g_B and a^2, with coefficients definable by formulae as explained in context with (2.7).

5.2 NON-PERTURBATIVE DETERMINATION OF IMPROVEMENT COEFFICIENTS

We have not found any useful PDEs obeyed by the improvement coefficients in (5.1), not even for $A_1^{(1)}$. The reason is that these coefficients are expressions of lattice effects and are themselves, upon inclusion of the factor a^2, of the order of the error in the usual PDEs, cf. (C.1). This contrasts with the properties of the coefficients in the LEL (2.5), as explained there, since in that case a^2 can be factored off.

Approximating the improvement coefficients by some orders of perturbation theory is possibly of value only if in the continuum limit, $g_B \searrow 0$, i.e. the theory is asymptotically free, as is the non-linear sigma model [11–13] and non-abelian gauge theory [14]. For improvement in the sense of sect. 1, however, the $C_1(g_B, 0)$ must be determined by Monte Carlo computations themselves. To this end, one would use the fact that these coefficients are also the same on a finite lattice since they compensate for the effect of space-time discretization and thus depend on local, rather than global, properties of the lattice. On the finite lattice, the simplest quantities that obey (1.1) and ought, upon correct improvement of the action, to obey it with a decreased r.h.s. are "generalized susceptibilities". These are Green functions with all momenta zero, or some momenta at the lowest discrete values available on the lattice with periodic boundary conditions. (For the ϕ_4^4 theory; in perturbation theory the zero mode would be annoying but it is harmless in non-perturbative treatments.)

5.3 ALTERNATIVE FORMS OF IMPROVED ACTION

The improved action (4.4) is by construction unique. It can be modified by admitting into the action terms that involve the source function J differently than in the term $a^{4+\epsilon}\sum_n J_n\phi_n \equiv \int J\phi$. The reason is that, by virtue of the Schwinger-Dyson (SD) equations derived from A_0 (which suffice for manipulations on the a^2 level) not all the terms in A_1 are linearly independent, provided one corrects this by admitting the contact terms that make the difference between classical field equations and SD equations. With $m_{B0}^2 + \Delta m_B^2 \equiv m_B^2$, the SD equations from A_0 in (2.1) are

$$K\phi + \tfrac{1}{6}g_B\phi^3 + m_B^2\phi - J = 0\,. \tag{5.2}$$

Here ϕ is to be interpreted as $\delta/\delta J$ (i.e. ϕ_n as $a^{-4-\epsilon}\partial/\partial J_n$) acting on the generating functional of Green functions that has in the action also the source term described before. Multiplying (5.2) from the left in turn by $K\phi$, ϕ^3, and J yields the three identities

$$(K\phi)^2 + \tfrac{1}{6}g_B\phi^3K\phi + m_B^2\phi K\phi - JK\phi + 4a^{-6-\varepsilon} = 0\,, \tag{5.3a}$$

$$\phi^3K\phi + \tfrac{1}{6}g_B\phi^6 + m_B^2\phi^4 - J\phi^3 - 3a^{-4-\varepsilon}\phi^2 = 0\,, \tag{5.3b}$$

$$JK\phi + \tfrac{1}{6}g_BJ\phi^3 + m_B^2J\phi - J^2 = 0\,. \tag{5.3c}$$

The vanishing is meant in the sense

$$\int \mathcal{D}\phi\,[(5.3\text{a}) \text{ or } (5.3\text{b}) \text{ or } (5.3\text{c})]\exp\left[A_0 + \int J\phi\right] = 0\,. \tag{5.4}$$

An easy recursive argument shows that (5.4) suffices to eliminate three terms, e.g. ϕ^6, $\phi^3K\phi$, and $J\phi^3$ from A_1 provided all the other terms of (engineering) dimension $6+\varepsilon$ that appear in (5.3a–c) are allowed to appear in A_1 with certain coefficients. These would have to be determined (e.g. in perturbation theory) by checking the improvement of a sufficient number of VFs or "physical quantities", since they are not obtained in an easy way from the coefficients computed when deriving (4.4). The advantage of eliminating terms not containing J in favour of those that do would be that contact term contributions need be evaluated only when making "measurements" while terms not containing J need be used in each Monte Carlo upgrading. In view of the limited interest of ϕ^4 theory we shall not pursue this matter here since it will reappear in the non-linear sigma model [13].

5.4 OTHER MODELS

The improvement technique expounded here for ϕ^4 theory is directly applicable to all lattice theories the perturbation expansions of which proceed in terms of propagators and vertices. For non-abelian gauge theory, such a form is proposed in ref. [15]. If fermions are also present, in (2.3), (2.4), (2.5) and correspondingly some other formulae also, odd powers of a and field monomials of odd dimension appear. The degeneracy problem [16] being one related to the corners of the Brillouin zone while improvement concerns the center, it could be dealt with by Wilson's method [16]. Complications may arise if the starting lattice action A_0 involves non-linear constraints. They are easily resolved in the case of the non-linear sigma model [11, 13].

Improvement is possible also in superrenormalizable lattice models. E.g. $\phi^4_{4+\varepsilon}$ theory becomes for $\varepsilon = -3$ the anharmonic oscillator on a "time" lattice. In this case of the improvement coefficients $C_1(g_Ba^{-\varepsilon}, \varepsilon)$ only a small number of expansion terms is needed.

5.5 SCOPE OF IMPROVEMENT

While the improved action does improve not only Green functions but also all other quantities that possess perturbative expansions, this is not assured for the other quantities, like spontaneous masses and the string tension in Yang-Mills theory. Since the determination of the improvement coefficients, subsect. 5.2, does not have to rely on perturbation theory, it appears a safe speculation that non-perturbative quantities like those mentioned are also improved in the sense of sect. 1 provided possible non-perturbative effects (cf. the discussion in ref. [15]) vanish faster than $O(a^2)$ in the continuum limit. If they do not, they could not be removed by using local terms only in an improved lattice action.

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Appendix A

SMALL-a EXPANSION OF ONE-LOOP GRAPHS

The two-propagators ("bubble") graph on the four-dimensional lattice at momentum $p = (p_1 p_2 p_3 p_4)$ requires to evaluate

$$F(p; \Delta m_B^2, a) = \tfrac{1}{4}a^4(2\pi)^{-4}\prod_{\mu=1}^{4}\left(\int_{-\pi/a}^{\pi/a} \mathrm{d}k_\mu\right)\left\{\sum_\mu\left[1-\cos(ak_\mu)\right]+\tfrac{1}{2}a^2\Delta m_B^2\right\}^{-1}$$

$$\times\left\{\sum_\mu\left[1-\cos\left(a(k_\mu+p_\mu)\right)\right]+\tfrac{1}{2}a^2\Delta m_B^2\right\}^{-1}. \tag{A.1}$$

Bringing the denominators into the exponent, carrying out the momentum integrations and using [17]

$$I_0\left((a^2-b^2)^{1/2}\right) = (2\pi i)^{-1}\int_{\mathcal{C}} \mathrm{d}t\,\Gamma(-t)\left(\tfrac{1}{2}a^{-1}b^2\right)^t I_t(a),$$

where the integration path is parallel to the imaginary axis with positive real part but encircles the origin at the left; and using as well the Mellin representation of the exponential function, yields

$$F(p; \Delta m_B^2, a) = \tfrac{1}{4}(2\pi i)^{-5}\int_{\mathcal{C}} \mathrm{d}w\,\Gamma(-w)\left(\tfrac{1}{2}a^2\Delta m_B^2\right)^w$$

$$\times\prod_{\mu=1}^{4}\left[\int_{\mathcal{C}} \mathrm{d}t_\mu\,\Gamma(-t_\mu)\left(2\sin^2\left(\tfrac{1}{2}p_\mu a\right)\right)^{t_\mu}\right]B\left(1+\sum t_\mu, 1+\sum t_\mu\right)$$

$$\times f(w, t_1 t_2 t_3 t_4), \tag{A.2}$$

where

$$f(w_1t_1t_2t_3t_4) = \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-4u} u^{1+w+\Sigma t_\mu} \prod_{\mu=1}^{4} I_{t_\mu}(u)\,. \tag{A.3}$$

In (A.2), the w and t integration paths are parallel to the imaginary axis, with $-1 < \mathrm{Re}\, w < 0$ and $-\frac{1}{4} < \mathrm{Re}\, t_\mu < 0$.

Splitting in (A.3) the integral at 1 and using in $0\ldots 1$ e.g. the power series, in $1\ldots\infty$ the Hankel asymptotic series for the modified Bessel functions yields $f(w, t_1 \ldots t_4)$ as a meromorphic function of its variables, the only singularities being for $\mathrm{Re}(w+\Sigma t_\mu) > -1$ simple poles at $0,1,2\ldots$ in the variable $w+\Sigma t_\mu$. Using, with $n = 0,1,2\ldots$

$$(2\pi i)^{-5}\int_{\subset} \mathrm{d}w\, \Gamma(-w) r^w \prod_{\mu=1}^{4}\left[\int \mathrm{d}t_\mu \Gamma(-t_\mu) s_\mu^{t_\mu}\right]$$

$$\times B\left(1+\sum t_\mu, 1+\sum t_\mu\right)\left(n - w - \sum t_\mu\right)^{-1}$$

$$= \int_0^1 \mathrm{d}x \left\{ \sum_{\substack{l=0\\ l\neq n}}^{\infty} (-1)^l (l!)^{-1}\left[r + (1-x)\sum s_\mu\right]^l (n-l)^{-1}\right.$$

$$+ (-1)^n (n!)^{-1}\left[r + x(1-x)\sum s_\mu\right]^n$$

$$\left.\times\left[\ln\left(r + x(1-x)\sum s_\mu\right) - \psi(1+n)\right]\right\}, \tag{A.4}$$

for $r = \frac{1}{2}a^2 \Delta m_\mathrm{B}^2$, $s_\mu = 2\sin^2(\frac{1}{2}p_\mu a)$; and the derivatives of (A.4) with respect to the s_μ, one finds in the small-a expansion of $F(p;\, \Delta m_\mathrm{B}^2, a)$ terms proportional to a^{2l} and $a^{2l}\ln a$, $l = 0,1,2\,\ldots$ with factors that are polynomial in Δm_B^2 and p_μ for both types of terms, with for the first ones in addition non-polynomial contributions from the logarithm in (A.4). This calculation can easily be extended to the general one-loop graph, with an analogous result.

If instead of lattice regularization a sharp-momentum-cutoff one (i.e. factors $\theta(\Lambda^2 - k^2)$ to the propagators) is used, in the large-Λ expansion also odd powers of Λ^{-1} appear [18], precluding the existence of an LEL to describe large-Λ behaviour. This effect is an "IR" one stemming from the sharpness of this cutoff [19].

Appendix B

REMARKS ON DIMENSIONAL INTEGRATION ON THE HYPERCUBIC LATTICE

Dimensional integration of Feynman graphs in the continuum can be treated rigorously by keeping the Lorentz-invariant scalar products of momenta fixed, since the Gram determinant condition plays no rôle. On the hypercubic lattice in $(4+\varepsilon)$-dimensional space, n momentum vectors $p_1 \dots p_n$ give rise to an infinite number of variables invariant under the lattice-symmetry operations, e.g. $\binom{n+1}{2}$ polynomials $\sum_\mu p_{i\mu} p_{j\mu}$, $\binom{n+3}{4}$ polynomials $\sum_\mu p_{i\mu} p_{j\mu} p_{k\mu} p_{l\mu}$ etc. A rigorous discussion would thus require the consideration of functions of an infinite number of variables, which we shall not attempt here. The reason is that what will be needed in sects. 3 and 4 are only VFs with all momenta zero, or differentiated at all momenta zero, and these expressions are unambiguous (in perturbation theory) once the manner of dimensional extrapolation from the integers is fixed by the usual convention. Namely, upon using for the propagators

$$\left[2a^{-2}\sum_{\mu=1}^{4+\varepsilon}\left(1-\cos(k_\mu a)\right)+\Delta m_{\mathrm{B}}^2\right]^{-1}=\tfrac{1}{2}a^2\int_0^\infty \mathrm{d}t \exp\left[-t\left(1+\tfrac{1}{2}a^2\Delta m_{\mathrm{B}}^2\right)\right] \times\prod_{\mu=1}^{4+\varepsilon}\exp\left[t\cos(k_\mu a)\right], \qquad \text{(B.1)}$$

the integrand of the proper-time integrations factorizes, e.g. it is a $(4+\varepsilon)$th power in the case of all external momenta zero. As an example, consider the "bubble" graph analyzed in appendix A, but in $4+\varepsilon$ dimensions at momentum zero. It yields

$$\begin{aligned} F(0;\Delta m_{\mathrm{B}}^2,\varepsilon,a) &= \tfrac{1}{4}a^{-\varepsilon}\int_0^\infty \mathrm{d}u\, u \exp\left[-\left(4+\varepsilon+\tfrac{1}{2}a^2\Delta m_{\mathrm{B}}^2\right)u\right] I_0(u)^{4+\varepsilon} \\ &= \Delta m_{\mathrm{B}}^{\varepsilon}\left(c_0(\varepsilon)+a^2\Delta m_{\mathrm{B}}^2 c_1(\varepsilon)+\cdots\right) \\ &\quad + a^{-\varepsilon}\left(d_0(\varepsilon)+a^2\Delta m_{\mathrm{B}}^2 d_1(\varepsilon)+\cdots\right), \end{aligned} \qquad \text{(B.2)}$$

again, e.g. by splitting the integration region, in conformity with (2.4).

Alternatively, one may take the ε extra dimensions in the continuum instead of on the lattice. This gives little conceptual relief at momenta non-zero, and at momenta zero (or differentiated there) gives slight numerical modification of the coefficients, e.g. (B.2) changes to

$$F(0;\Delta m_{\mathrm{B}}^2,\varepsilon,a)=\tfrac{1}{4}a^{-\varepsilon}\int_0^\infty \mathrm{d}u\, u\exp\left[-\left(4+\tfrac{1}{2}a^2\Delta m_{\mathrm{B}}^2\right)u\right] I_0(u)^4(2\pi u)^{-\varepsilon/2},$$

with an expansion as in (B.2) but with changed coefficients. Using this picture would require that we list more terms in e.g. (2.5) because of the different symmetry. For this reason we used the earlier symmetric picture. The final result, in the form of the l.h.s. in (4.4) or (5.1) at $\varepsilon = 0$, is the same, as is immediately verified e.g. in the example (4.5).

Appendix C

PDEs AND RENORMALIZATION

Changing in (2.5) the ϕ normalization, $g_B a^{-\varepsilon}$, a, and Δm_B^2 in such a way that L_0 remains unchanged leads to the Zinn-Justin PDE [1]

$$\left[-a\frac{\partial}{\partial a} + \bar\beta(g_B a^{-\varepsilon}, \varepsilon)\frac{\partial}{\partial(g_B a^{-\varepsilon})} + \bar\eta(g_B a^{-\varepsilon}, \varepsilon)\Delta m_B^2\frac{\partial}{\partial \Delta w_B^2} - 2n\gamma(g_B a^{-\varepsilon}, \varepsilon)\right]$$
$$\times \Gamma(p_1 \dots p_{2n}; g_B, \Delta m_B^2, \varepsilon, a) = O(a^{2-\varepsilon}), \tag{C.1}$$

with (abbreviating $g_B a^{-\varepsilon} \equiv \bar g$)

$$\bar\beta(\bar g, \varepsilon) = \varepsilon\left[\bar g^{-1} + \partial/\partial\bar g \ln\left(Z_1(\bar g, \varepsilon) Z_3(\bar g, \varepsilon)^{-2}\right)\right]^{-1}, \tag{C.2a}$$

$$\bar\gamma(\bar g, \varepsilon) = \tfrac{1}{2}\bar\beta(\bar g, \varepsilon)\,\partial/\partial\bar g \ln Z_3(\bar g, \varepsilon), \tag{C.2b}$$

$$\bar\eta(\bar g, \varepsilon) = -\bar\beta(\bar g, \varepsilon)\,\partial/\partial\bar g \ln\left(Z_2(\bar g, \varepsilon) Z_3(\bar g, \varepsilon)^{-1}\right) \tag{C.2c}$$

These parametric functions have $\varepsilon = 0$ limits since the VFs in (C.1) have. Integrating (C.1) at $\varepsilon = 0$ yields

$$Z_3(g, a\mu)^n \Gamma(p_1 \dots p_{2n}, g_B(g, a\mu), \Delta m_B^2(m^2, g, a\mu), 0, a)$$
$$= \Gamma_{\text{ren}}(p_1 \dots p_{2n}; g, m^2, \mu) + O(a^2 \ln a), \tag{C.3}$$

where, with $\bar\rho(g)$ from (1.4),

$$g_B(g, a\mu) = \bar\rho^{-1}(\bar\rho(g) - \ln a\mu) = \sum_{l=0}^{\infty} (l!)^{-1}\left[-\ln(a\mu)\bar\beta(g)\,\partial/\partial g\right]^l g. \tag{C 4a}$$

$$Z_3(g, a\mu) = \exp\left[-2\int_g^{g_B(\quad)} \mathrm{d}g'\bar{\beta}(g')^{-1}\bar{\gamma}(g')\right], \tag{C.4b}$$

$$\Delta m_B^a(m^2, g, a\mu) = m^2 \exp\left[\int_g^{g_B(\quad)} \mathrm{d}g'\bar{\beta}(g')^{-1}\bar{\eta}(g')\right]. \tag{C.4c}$$

Eq. (C.3) with (C.4a–c) describes "mass-independent" renormalization in the sense of 't Hooft [10] and Weinberg [20] though in Zinn-Justin convention [1], which yields, in terms of the functions in (2.3),

$$\Gamma_{\mathrm{ren}}(p_1 \dots p_{2n}; g, m^2, \mu) = \sum_{k=0}^{\infty} (-\ln\mu)^k \bar{F}_{0k}(p_1 \dots p_{2n}; g, m^2). \tag{C.5}$$

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