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INTERACTION OF A FERMION WITH A MONOPOLE I

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It is proposed that some low-energy properties of the interaction of one or more fermions with a monopole are sensitive to, and hence can be used as probes for, extremely small distances. One possible example of such properties is studied in detail.

1. Introduction

It has been realized recently [1] that there is no zero-energy bound state for fermions in the field of a fixed SU(5) grand unified monopole [2–4]. Therefore, in this case, the Rubakov effect [5] (monopole catalysis of proton decay at strong-interaction cross sections) is expected to be absent. This leads to the following two possibilities:

(a) This absence of the Rubakov effect holds for all grand unified theories; or

(b) the Rubakov effect is present for some grand unified theories, and absent for others.

In scenario (b), the Rubakov effect provides a deep probe of the structures of the underlying group and Higgs system [6]. Furthermore, since monopoles can form bound states with various atoms and molecules [7], the presence of the Rubakov effect implies that the passage of a monopole through a proton-decay detector is likely to lead to spectacular events. Such events will give a wealth of information about interactions at extremely short distances.

What can we expect for scenario (a)? Although such spectacular events do not appear, we can nevertheless look for deep probes into such very short distances of the order of 10^{-28} cm. This is the motivation for the present considerations.

As a step in this direction, we study in this series of papers several interrelated but distinct problems of the interaction between spin-$\frac{1}{2}$ fermions and a monopole. All these problems are to be treated on the level of the c-number field equations.

In this first paper, we study the scattering of a spin-$\frac{1}{2}$ fermion by a fixed Dirac dyon. Our interest is not restricted to the dyon, but is also due to the following reasoning. Six years ago, Kazama, Yang, and Goldhaber [8] investigated the scattering by a fixed Dirac monopole [9], and found it necessary to introduce the

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artifice of an infinitesimal extra magnetic moment for the fermion. When this same procedure is applied to the present case of a dyon, the result turns out to be completely different: the limit does not exist when the extra magnetic moment approaches zero with the dyon charge kept fixed. If the extra magnetic moment and the dyon charge approach zero simultaneously, then the result depends on the way the limit is taken. The possible limiting theories are parametrized by one real constant. This main result of the present paper is obtained in sect. 4.

The Dirac monopole has a singularity which is absent for grand unified monopoles. The indeterminacy of this real parameter thus means that the parameter is sensitive to the phenomena at short distances. It is accordingly a promising candidate for the deep probe of extremely short distances.

2. Monopole harmonics and radial equations

Let Ze be the electric charge of the spin-$\frac{1}{2}$ fermion, while $Z_\mathrm{d}e$ and g are the electric and magnetic charges of the dyon respectively. Thus the wave function for the fermion in the field of the fixed dyon depends on the quantities

$$q = Zeg\,, \tag{1}$$

$$\zeta = ZZ_\mathrm{d}e^2\,. \tag{2}$$

We shall always exclude the case $q = 0$. The Dirac quantization condition [9] is that $2q$ is an integer, which may be positive or negative, but imposes no restriction on the value of ζ. While Z is an integer, Z_d need not be. In particular, the dyon reduces to a monopole in the limit $\zeta \to 0$.

The hamiltonian is given by

$$H_0 = \boldsymbol{\alpha} \cdot (-i\boldsymbol{\nabla} - Ze\boldsymbol{A}) + \beta M - \zeta/r\,, \tag{3}$$

where M is the mass of the fermion, and the vector potential $\boldsymbol{A}$ is defined in terms of two or more functions in a corresponding number of overlapping regions [10]. A convenient choice of the Dirac matrices is

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

The basic idea of Kazama et al. [8] is to replace this H_0 by

$$H = H_0 - \frac{\kappa q \beta\, \boldsymbol{\sigma} \cdot \boldsymbol{r}}{2Mr^3}\,, \tag{5}$$

and eventually take the limit $\kappa \to 0+$ or $\kappa \to 0-$.

Since H is rotationally invariant, it is possible to carry out partial wave decomposition in terms of monopole harmonics [8, 10]. For the present purpose, only the simplest partial waves are of interest. They are those where the upper two and the lower two components have identical angular dependences:

$$\psi = \begin{bmatrix} f(r)\eta_m \\ g(r)\eta_m \end{bmatrix}, \tag{6}$$

where $f(r)$ and $g(r)$ are scalar functions of the radial variable r, while η_m has two components

$$\eta_m = \begin{bmatrix} -\left(\dfrac{|q|-m+\frac{1}{2}}{2|q|+1}\right)^{1/2} Y_{q,|q|,m-\frac{1}{2}} \\ \left(\dfrac{|q|+m+\frac{1}{2}}{2|q|+1}\right)^{1/2} Y_{q,|q|,m+\frac{1}{2}} \end{bmatrix}. \tag{7}$$

In (7), Y are the monopole harmonics and m is in the range $-(|q|-\frac{1}{2})$ to $|q|-\frac{1}{2}$. In the simplest case $|q|=\frac{1}{2}$, there is only one such η. To get a complete orthonormal set of such two-component sections, these η must be supplemented by an infinite number of other eigensections of the total angular-momentum operators J^2 and J_z.

The substitution of (6) and (7) into the Dirac equation

$$H\psi = E\psi, \tag{8}$$

gives the radial equations

$$\begin{bmatrix} M-E-\dfrac{\zeta}{r}-\dfrac{\kappa|q|}{2Mr^2} & -i\dfrac{q}{|q|}\left(\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{1}{r}\right) \\ -i\dfrac{q}{|q|}\left(\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{1}{r}\right) & -M-E-\dfrac{\zeta}{r}+\dfrac{\kappa|q|}{2Mr^2} \end{bmatrix} \begin{bmatrix} f(r) \\ g(r) \end{bmatrix} = 0. \tag{9}$$

3. Behavior for small r

In order to understand the limiting behavior of small κ, it is essential to analyze the behavior of $f(r)$ and $g(r)$ for small r. Let [12]

$$r = |\kappa q|\rho/(2M), \tag{10}$$

$$f = (\kappa q/|\kappa q|)F/r, \qquad g = -iG/r, \tag{11}$$

then (9) becomes

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}\rho} &= \left[\frac{\kappa|q|(M-E)}{2M} - \frac{\kappa}{|\kappa|}\frac{\zeta}{\rho} - \frac{1}{\rho^2}\right]F, \\ \frac{\mathrm{d}F}{\mathrm{d}\rho} &= \left[\frac{\kappa|q|(M+E)}{2M} + \frac{\kappa}{|\kappa|}\frac{\zeta}{\rho} - \frac{1}{\rho^2}\right]G. \end{aligned} \tag{12}$$

For $\kappa \to 0$ with fixed ρ, the first terms on the right-hand sides of (12) are negligible.

Thus the approximation for small r is

$$\frac{\mathrm{d}G}{\mathrm{d}\rho}=\left(-\frac{\kappa}{|\kappa|}\frac{\zeta}{\rho}-\frac{1}{\rho^2}\right)F, \qquad \frac{\mathrm{d}F}{\mathrm{d}\rho}=\left(\frac{\kappa}{|\kappa|}\frac{\zeta}{\rho}-\frac{1}{\rho^2}\right)G. \tag{13}$$

The first step is to solve eq. (13) exactly with the boundary conditions

$$F(0)=G(0)=0. \tag{14}$$

Let

$$\begin{aligned} F&=\mathrm{e}^{-x/2}(\Phi_1-\Phi_2),\\ G&=-\mathrm{e}^{-x/2}(\Phi_1+\Phi_2), \end{aligned} \tag{15}$$

where

$$x=2/\rho, \tag{16}$$

then

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}x}=\frac{\kappa}{|\kappa|}\frac{\zeta}{x}\Phi_2, \qquad \frac{\mathrm{d}\Phi_2}{\mathrm{d}x}=-\frac{\kappa}{|\kappa|}\frac{\zeta}{x}\Phi_1+\Phi_2. \tag{17}$$

The elimination of Φ_2 gives

$$\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}\Phi_1-x\frac{\mathrm{d}}{\mathrm{d}x}\Phi_1+\frac{\zeta^2}{x}\Phi_1=0, \tag{18}$$

and hence, because of (14),

$$\begin{aligned} \Phi_1&=x^{i\zeta}\Psi(i\zeta,1+2i\zeta;x)\\ &=\frac{\Gamma(-2i\zeta)}{\Gamma(-i\zeta)}x^{i\zeta}\Phi(i\zeta,1+2i\zeta;x)+\frac{\Gamma(2i\zeta)}{\Gamma(i\zeta)}x^{-i\zeta}\Phi(-i\zeta,1-2i\zeta;x), \end{aligned} \tag{19}$$

where Φ and Ψ are confluent hypergeometric functions [13]. The substitution into (17) then yields

$$\begin{aligned} \Phi_2&=\frac{\kappa}{|\kappa|}\zeta x^{i\zeta}\Psi(1+i\zeta,1+2i\zeta;x)\\ &=\frac{\kappa}{|\kappa|}i\left[\frac{\Gamma(-2i\zeta)}{\Gamma(-i\zeta)}x^{i\zeta}\Phi(1+i\zeta,1+2i\zeta;x)-\frac{\Gamma(2i\zeta)}{\Gamma(i\zeta)}x^{-i\zeta}\Phi(1-i\zeta,1-2i\zeta;x)\right]. \end{aligned} \tag{20}$$

With (11), (15), (19) and (20), the behavior of $f(r)$ and $g(r)$ in the range

$$|\kappa q|/M \ll r \ll 1/M, \tag{21}$$

is found to be

$$\begin{aligned} f(r)&\sim\frac{\kappa q}{|\kappa q|}r^{-1}(\tfrac{1}{2}\cosh(\pi\zeta))^{-1/2}\cos\left[\zeta\ln\frac{4Mr}{|\kappa q|}+\frac{\kappa}{|\kappa|}\frac{\pi}{4}+\phi_0(\zeta)\right],\\ g(r)&\sim ir^{-1}(\tfrac{1}{2}\cosh(\pi\zeta))^{-1/2}\cos\left[\zeta\ln\frac{4Mr}{|\kappa q|}-\frac{\kappa}{|\kappa|}\frac{\pi}{4}+\phi_0(\zeta)\right], \end{aligned} \tag{22}$$

where

$$\phi_0(\zeta) = \arg[\Gamma(\tfrac{1}{2} + i\zeta)]. \tag{23}$$

In writing down (22), the Legendre duplication formula for the Γ function has been used. This behavior is consistent with the Coulomb wave function for a Dirac particle [14]. The right-hand sides of (22), and hence $f(r)$ and $g(r)$ for all r, do not approach any limiting values when $\kappa \to 0+$ or $\kappa \to 0-$ with fixed $\zeta \neq 0$.

The important point here is that the magnitude of κ appears on the right-hand sides of (22) in an essential way when $\zeta \neq 0$, although not when $\zeta = 0$. This is the underlying reason why the present result for the dyon is qualitatively different from that of ref. [8] for the monopole.

4. The monopole

Consider next the limiting process $\kappa \to 0$, $\zeta \to 0$ simultaneously. This is a natural way to study the monopole. Let us study the various terms in the arguments of the cosines in (22). For a fixed r, clearly

$$\zeta \ln(4Mr/|q|) \to 0,$$

in this limit. Since $\phi_0(\zeta) \to 0$ as $\zeta \to 0$, the interesting term is

$$-\zeta \ln|\kappa|. \tag{24}$$

As $\zeta \to 0$ and $\kappa \to 0$ simultaneously, this quantity may or may not have a limiting value. Furthermore, it can be made to approach any real value by a suitable choice of the limiting process. This is the parameter discussed in the introduction.

This limiting process need to be discussed in some detail. Since the sign of κ appears in the arguments of the cosines in (22), it is essential to restrict the limiting processes to $\kappa \to 0+$ or $\kappa \to 0-$ in order to get a finite theory for the monopole. In other words, κ should approach zero through either positive values or negative values, but not in a way that oscillates between the positive and negative values. In both of these two cases $\kappa \to 0+$ and $\kappa \to 0-$, one of the many possible choices of the continuous parameter is, up to modulo 2π because of the zeros of f,

$$\omega = \lim_{r \to 0} \lim_{\substack{\zeta \to 0 \\ \kappa \to 0}} 2\frac{\kappa}{|\kappa|} \arg\left(\frac{\kappa q}{|\kappa q|} + \frac{g}{f}\right). \tag{25}$$

Explicitly, this angle is

$$\omega = -2 \lim_{\substack{\zeta \to 0 \\ \kappa \to 0}} (\zeta \ln|\kappa|) + \tfrac{1}{2}\pi\kappa/|\kappa|, \tag{26}$$

modulo 2π.

Once this angle ω is specified, the hamiltonian (3) with $\zeta = 0$ can be used to describe the interaction between a spin-$\frac{1}{2}$ fermion and a monopole, leading to, in particular,

$$\begin{bmatrix} M-E & -i\frac{q}{|q|}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}\right) \\ -i\frac{q}{|q|}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}\right) & -M-E \end{bmatrix} \begin{bmatrix} f(r) \\ g(r) \end{bmatrix} = 0\,, \tag{27}$$

together with the boundary condition

$$2 \lim_{r\to 0} \arg\left[\frac{\kappa q}{|\kappa q|} + \frac{g(r)}{f(r)}\right] = \frac{\kappa}{|\kappa|}\omega\,, \tag{28}$$

modulo 2π. The key point here is that (27) implies that f and g are 90° out of phase and hence this boundary condition (28) is actually *independent of the sign of* κ. Accordingly, without loss of generality, κ can be taken to be positive and the boundary condition is

$$2 \lim_{r\to 0} \arg\left[\frac{q}{|q|} + \frac{g(r)}{f(r)}\right] = \omega\,, \tag{29}$$

or equivalently

$$\lim_{r\to 0} \frac{g(r)}{f(r)} = i\frac{q}{|q|}\tan\tfrac{1}{2}\omega\,. \tag{30}$$

In this way, the possible theories for the interaction of a spin-$\frac{1}{2}$ fermion and a monopole are classified in terms of this angular parameter ω. (For some purposes such as charge conjugation, $\omega' = \omega \pm \frac{1}{2}\pi$ may be more convenient.)

5. Bound-state energy

As an especially simple application of this angle ω, consider a fermion-monopole bound state. This involves solving the radial equations (27) for $|E| < M$. the solution is

$$\begin{aligned} f(r) &= N\frac{q}{|q|} r^{-1} \exp\left[-(M^2-E^2)^{1/2} r\right]\,, \\ g(r) &= iN\left(\frac{M-E}{M+E}\right)^{1/2} r^{-1} \exp\left[-(M^2-E^2)^{1/2} r\right]\,, \end{aligned} \tag{31}$$

where N is the normalization constant. The boundary condition (30) then gives the bound-state energy

$$E = M\cos\omega\,, \tag{32}$$

if $0 < \omega < \pi$; there is no bound state for $-\pi \leqslant \omega \leqslant 0$.

It is seen from (26) that the limiting processes used by Kazama et al. [8] correspond to the following special values of ω:

$$\kappa > 0 \leftrightarrow \omega = \tfrac{1}{2}\pi\,, \qquad \kappa < 0 \leftrightarrow \omega = -\tfrac{1}{2}\pi\,, \tag{33}$$

with the former giving a zero-energy bound state [12].

6. Discussions

Although κ modifies the properties of the fermion while ζ modifies those of the monopole, they should both be considered to be short-distance cut-offs. Removal of these cut-offs does not lead to a unique theory but instead to a class of theories parametrized by the angle ω. All properties related to these partial waves under consideration depend significantly on this angle, and include not only the bound-state energy as given by (32), but also phase shifts, effective potentials [7] etc. Accordingly, when monopoles becomes available in the laboratory, there are in principle many ways to determine this angle ω.

It is less clear precisely what short-distance property this angle measures. In a later paper of this series, scattering from an SU(5) monopole [4] will be studied. Within the context of the c-number field theory, this angle probes the structure within the monopole radius. While this is believed to be also the case for a more complete theory, other longer distances can be expected to give significant modifications.

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References

[1] T.F. Walsh, P. Weisz and T.T. Wu, DESY 83-022, to be published
[2] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438;
H. Georgi, H.R. Quinn and S. Weinberg, Phys. Rev. Lett. 33 (1974) 451;
H. Georgi, Particles and fields: 1974 (APS/DPF Williamsburg), ed. C.E. Carlson (AIP, New York, 1975) p. 575;
M.S. Chanowitz, J. Ellis and M.K. Gaillard, Nucl. Phys. B128 (1977) 506
[3] Z. Horvath and L. Palla, Phys. Lett. B69 (1977) 197;
M. Daniel, G. Lazarides and Q. Shafi, Nucl. Phys. B170 [FS1] (1980) 156;
D.M. Scott, Nucl. Phys. B171 (1980) 95, 109
[4] C.P. Dokos and T.N. Tomaras, Phys. Rev. D21 (1980) 2940
[5] V.A. Rubakov, ZhETF Pisma 33 (1981) 658; JETP Lett. 33 (1981) 644; Nucl. Phys. B203 (1982) 311;
C.G. Callan, Phys. Rev. D25 (1982) 2141; D26 (1982) 2058; Nucl. Phys. B212 (1983) 391
[6] P.W. Higgs, Phys. Lett. 12 (1964) 132; Phys. Rev. Lett. 13 (1964) 508;
F. Englert and R. Brout, Phys. Rev. Lett. 13 (1964) 321;
G.S. Guralnik, C.R. Hagen and T.W.B. Kibble, Phys. Rev. Lett. 13 (1964) 585

[7] Th.W. Ruijgrok, J.A. Tjon and T.T. Wu, DESY 83-036, to be published
[8] Yoichi Kazama, Chen Ning Yang and Alfred S. Goldhaber, Phys. Rev. D15 (1977) 2287
[9] P.A.M. Dirac, Proc. Roy. Soc. A133 (1931) 60
[10] Tai Tsun Wu and Chen Ning Yang, Nucl. Phys. B107 (1976) 365
[11] H.J. Lipkin, W.I. Weisberger and M. Peshkin, Ann. of Phys. 53 (1969) 203
[12] Yoichi Kazama and Chen Ning Yang, Phys. Rev. D15 (1977) 2300
[13] Bateman Manuscript Project, Higher transcendental functions, ed. A. Erdélyi (McGraw-Hill, New York, 1953), vol. I, chapter 6
[14] M.E. Rose, Relativistic electron theory (John Wiley, New York, 1961)