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CONTINUUM LIMIT AND IMPROVED ACTION IN LATTICE THEORIES
(II). O(N) non-linear sigma model in perturbation theory

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The method of paper (I) of this series is applied to the O(N) non-linear sigma model. Due to the use of non-manifestly invariant perturbation theory the improvement part of the action, computed explicitly to one-loop order, is not manifestly O(N) invariant. It can be brought into manifestly O(N) invariant form by use of linear identities among dimension-four operators, which follow from the field equations of the unimproved action. The adequacy of the resulting two-parameter family of manifestly O(N) invariant improved actions is verified to one-loop order.

1. Introduction

The non-linear sigma model is defined by a linear continuous symmetry, here O(N), and a non-linear constraint. To apply perturbation theory this constraint must be resolved: to write the action in terms of canonically independent fields. The linear symmetry hereby turns into a non-linear one which is not manifest in the perturbation expansion. In two dimensions where the O(N) symmetry cannot be broken [1], perturbation theory leads to IR divergences that cancel only in invariant linear combinations [2] after, to define perturbation theory for the model, e.g. an external symmetry-breaking field H is introduced and then allowed to approach zero at the end.

In $2+\varepsilon$ dimensions ($\mathrm{Re}\,\varepsilon > 0$), convenient for the improvement technique of paper (I) of this series [3], perturbation theory does not require an IR regulator but leads into the phase of broken symmetry due to spontaneous magnetization at low temperature. The subtractions described in sect. 4 of (I) lead to the action $A_0 + \Delta A_0 + a^2 A_1$ where ΔA_0 and $a^2 A_1$ contain not-manifestly-O(N) invariant terms. In ΔA_0, the non-symmetry is related to renormalization of the coupling constant [4] which occurs in perturbation theory in a non-symmetric way, and the analog of the transformation described in subsect. 3.3 of (I) removes ΔA_0. The remaining improvement part a^2A_1' (cf. subsect. 4.2 of (I)) is still not manifestly symmetric though of the form found by Brézin, Zinn-Justin, and Le Guillou [4] to be compatible with the O(N) Ward identities. The non-symmetric form here reflects merely the freedom of choice concerning that part, due to several identities among dimension-4 operators derivable from the field equations of A_0, analogous to the identities in ϕ^4 theory described in subsect. 5.2 of (I). By use of these identities, a^2A_1' can be brought into manifestly symmetric form, with still two parameters free due to two identities among manifestly invariant dimension-4 operators.

In sect. 2 we construct the subtracted action perturbatively to one-loop order. It is to this order related to the local effective action (LEL) in $2+\varepsilon$ dimensions merely by change of sign. The dim $2+\varepsilon$ and dim $4+\varepsilon$ terms we find are indeed the ones allowed by the Ward identities [4]. Their coefficients to this order are computed in appendix A and B. In sect. 3, identities among dim-4 operators are derived. We also show that the self-contractions that make the difference between normal and ordinary operator products affect only the next loop-order, with some algebra relegated to appendix C. The two-parameter family of improved and manifestly O(N) invariant actions, with coefficients as follow from sects. 2 and 3, is in sect. 4 directly derived by requiring the improvement of Green functions to one-loop order and extended to all orders. Sect. 5 contains concluding remarks.

2. Perturbatively subtracted lattice action

2.1 STRATEGY

We apply the prescriptions of sect. 4 of (I). Since the field has dimension zero, or $\frac{1}{2}\varepsilon$ in $2+\varepsilon$ dimensions, we need to subtract all VFs four times. This is best done as follows. For the action subtracted up to $\mathfrak{L}-1$ loop order, one constructs the effective action in the sense of Coleman and Weinberg [5] up to 4th order in momenta, with $\mathrm{Re}\,\varepsilon > 2\mathfrak{L}^{-1}$ for the 4th-order term, to $\mathfrak{L}$-loop order (the $(\mathfrak{L}-1)$-loop orders vanish by construction). This effective action, rewritten for the lattice and with the "classical field" replaced by the lattice field operator, is with change of sign the $\mathfrak{L}$-loop-order subtraction. In the present model, however, it is advisable to perform the reparametrization described in subsect. 4.2 of (I) and a further transformation described here in subsect. 4.3 in each loop order separately before proceeding to the next loop order.

2.2 GENERAL FORM OF THE MOMENTUM-EXPANDED EFFECTIVE ACTION

We use the functional method [6]. The generating functional $\Gamma(\pi)$ of the VFs (in our case, π has $N-1$ components) can, upon Taylor-expanding all VFs to 4th order in momenta at all momenta zero, be written in the continuum

$$\Gamma(\pi) = -\int \mathrm{d}x\Big[V(\pi) + \tfrac{1}{2}\partial_\mu\pi\, Z_{\mu\ \nu}(\pi)\,\partial_\nu\pi + \tfrac{1}{2}\partial_\mu\partial_\nu\pi\ \partial_\kappa\pi\ S_{\mu\nu\ \kappa}(\pi) + \tfrac{1}{8}\partial_\mu\partial_\nu\pi\ \partial_\kappa\partial_\lambda\pi\ T_{\mu\nu\ \kappa\lambda}(\pi) + \tfrac{1}{4}\partial_\mu\partial_\nu\pi\ \partial_\kappa\pi\ \partial_\lambda\pi\ U_{\mu\nu\ \kappa\ \lambda}(\pi) + \tfrac{1}{24}\partial_\mu\pi\ \partial_\nu\pi\ \partial_\kappa\pi\ \partial_\lambda\pi\ W_{\mu\ \nu\ \kappa\ \lambda}(\pi) + \mathrm{O}(\mathrm{mom}^5)\Big]. \qquad (2.1)$$

In the presence of hypercubic lattice symmetry (called military symmetry by R. Jost), Z is proportional to $\delta_{\mu\nu}$, $S \equiv 0$, and T, U, W consist of terms proportional to the tensors $\delta_{\mu\nu}\delta_{\kappa\lambda}$, $\delta_{\mu\kappa}\delta_{\nu\lambda}$, $\delta_{\mu\lambda}\delta_{\nu\kappa}$, and $\delta_{\mu\nu}\delta_{\mu\kappa}\delta_{\mu\lambda}$.

For the action

$$A_0 = \sum\left[-\tfrac{1}{2}\pi K\pi + S(\pi)\right]a^{2+\varepsilon} - \text{measure term}(\pi), \qquad (2.2)$$

the functions in (2.1) are obtained [6] by momentum-expanding one-particle-irreducible lattice graphs made of propagators $[K - S''(\pi)]^{-1}$ and vertices that are third and higher derivatives of $S(\pi)$, with π space-"time" independent throughout the graph. Uncontracted π-indices at vertices are external arguments, at which infinitesimal momenta, to first or higher powers, enter these graphs. Thus, in (2.1) the coefficient functions are obtained by momentum-expanding graphs as just described with the indicated number (up to four) of external arguments.

2.3 SECOND ORDER IN MOMENTA

For our model, (2.2) is

$$A_0 = a^{2+\varepsilon}\sum_{n\in\mathbb{Z}^{2+\varepsilon}}\left[-\tfrac{1}{2}\boldsymbol{\pi}_n\cdot(\tilde{K}\boldsymbol{\pi})_n - \tfrac{1}{2}g^{-1}\sigma_n(\tilde{K}\sigma)_n - a^{-2-\varepsilon}\ln\sigma_n\right], \qquad (2.3)$$

where $\boldsymbol{\pi} = (\pi_i,\ i = 1\dots N-1)$, $\sigma \equiv (1 - g\boldsymbol{\pi}^2)^{1/2}$ and (see subsect. 4.1 of (I))

$$\tilde{K} = \sum_{\mu=1}^{2+\varepsilon}\left[-\partial_\mu\partial_\mu^+ + \tfrac{1}{12}a^2(\partial_\mu\partial_\mu^+)^2\right]. \qquad (2.4)$$

A constant magnetic field H would play the rôle of Δm_B^2 in the ϕ^4 model of (I), and a source term $h\sigma$ would also bring in VFs with composite-operator arguments to which the subtraction technique extends in an obvious way. (In contrast to Δm_B^2 in (2.5) of (I), a non-constant source term h appears also at non-zero momentum i.e. differentiated.) For simplicity, we omit such terms in this section. (See, however, subsect. 3.1 below).

Comparing (2.3) with (2.2) we have

$$S''_{ij}(\pi) = -g\sigma^{-1}\pi_i\tilde{K}\left(\sigma^{-1}\pi_j\right) + \sigma^{-1}\left[\delta_{ij} + g\sigma^{-2}\pi_i\pi_j\right]\tilde{K}\sigma\,, \tag{2.5}$$

whereof the last term vanishes for π space-"time" independent. The propagator is, therefore,

$$[K - S'']_{ij}^{-1} = \tilde{K}^{-1}\left[\delta_{ij} - g\pi_i\pi_j\right]. \tag{2.6}$$

In

$$\Gamma^{1\,\mathrm{loop}}(\pi) = -\tfrac{1}{2}\mathrm{Tr}\ln\left[K - S''(\pi)\right] - \text{measure term}\ (\pi) \tag{2.7}$$

the last term already effects the subtraction at zero momenta [7].

Differentiating (2.7) twice leads to the one-loop contributions to Z and T in (2.1). The vertices follow from (2.5), and we obtain the graphs shown in fig. 1. Hereby, a broken line denotes $\tilde{K}$ of (2.4) and a dashed solid line the propagator (2.6). The number of derivatives of σ is indicated by strokes, and the two lettered arguments are the external ones carrying infinitesimal external momentum, to second order yielding Z, to 4th order T in (2.1). Dots indicate dummy arguments contracted by the propagator (2.6).

Diagrams A and B of fig. 1 contribute to both Z and T in (2.1) and can, using $\sigma'_i = -g\sigma^{-1}\pi_i$, be combined:

$$\tilde{K} - g\tilde{K}\sigma^{-1}\pi_r\tilde{K}^{-1}(\delta_{rs} - g\pi_r\pi_s)\sigma^{-1}\pi_s\tilde{K} = \sigma^2\tilde{K}. \tag{2.8}$$

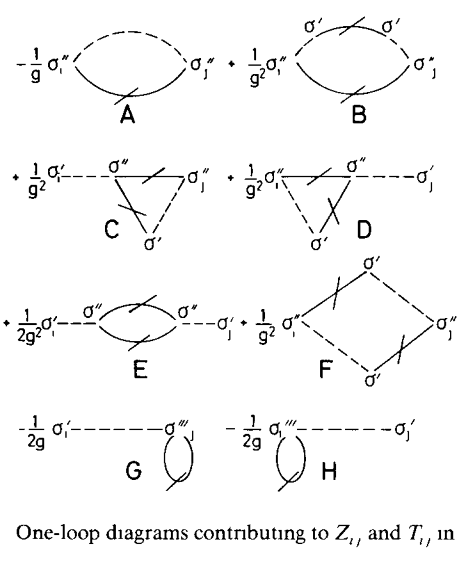

Fig 1 One-loop diagrams contributing to Z_{ij} and T_{ij} in (2.1)

Furthermore, noting that (2.6) contains the inverse of the matrix structure of

$$\sigma''_{ij} = -g\sigma^{-1}\left(\delta_{ij} + g\sigma^{-2}\pi_i\pi_j\right), \tag{2.9}$$

we have

$$\sigma^2\sigma''_{ir}(\delta_{rs} - g\pi_r\pi_s)\sigma''_{sj} = g^2\left(\delta_{ij} + g\sigma^{-2}\pi_i\pi_j\right), \tag{2.10}$$

such that the (A + B) contribution to (2.1) to 2nd order in momenta

$$= -\tfrac{1}{2}g\sum_{\mu}\left(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + g^{-1}\partial_\mu\sigma\partial_\mu\sigma\right)\times\left(p^2 \text{ coefficient of fig. 2a}\right). \tag{2.11}$$

(A solid line denotes the bare improved propagator $\tilde{K}^{-1}$). Due to $\tilde{K}^{-1}\tilde{K} = 1$, diagrams C and D contribute from $\tilde{K}$ to second order only in momenta since the triangle reduces to a tadpole, and gives

$$(\text{C} + \text{D}) \text{ contribution to } (2.1) = \sigma^{-1}\Delta\sigma\cdot(\text{diagram fig. 2b}), \tag{2.12}$$

where Δ is the laplacian Diagram E contributes only to T in (2.1). Diagram F, again due to $\tilde{K}^{-1}\tilde{K} = 1$, is momenta independent and gives zero. Finally, upon simple algebra,

$$(\text{G} + \text{H}) \text{ contribution to } (2.1) = -\tfrac{1}{2}(N+1)\sigma^{-1}\Delta\sigma\cdot(\text{diagram fig. 2b}). \tag{2.13}$$

Collecting, we have

$$\Gamma^{0\text{ loop}}(\pi) + \Gamma^{1\text{ loop}}(\pi) \text{ to second order in momenta}$$

$$= -\tfrac{1}{2}\int \mathrm{d}x\left(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + g^{-1}\partial_\mu\sigma\partial_\mu\sigma\right)\left[1 + g\cdot\left(p^2 \text{ term of fig. 2a}\right)\right]$$

$$-\tfrac{1}{2}(N-1)\int \mathrm{d}x\, \sigma^{-1}\Delta\sigma\cdot \text{fig. 2b}. \tag{2.14}$$

The two coefficients are evaluated in appendix A, (A.2) and (A.4). They have first-order poles at $\varepsilon = 0$

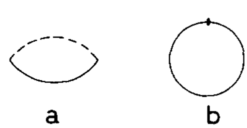

Fig 2 Graphs needed for one-loop renormalization coefficients

It is now seen that the terms obtained are, as they must be, the two terms that were found by Brézin, Zinn-Justin, and Le Guillou [4] to be compatible with the O(N) Ward identities. Noting that

$$\partial/\partial g\left[-\tfrac{1}{2}g^{-1}\int\sigma\Delta\sigma\right]=\tfrac{1}{2}g^{-2}\int\sigma^{-1}\Delta\sigma, \tag{2.15}$$

we can rewrite RHS of (2.14)

$$= -\tfrac{1}{2}\int\left(\partial_\mu\pi\cdot\partial_\mu\pi+\hat{g}^{-1}\partial_\mu\hat{\sigma}\partial_\mu\hat{\sigma}\right)Z_3, \tag{2.16a}$$

$$\hat{g}=g+(N-1)g^2a^{-\varepsilon}\cdot(\mathrm{A}.4)+\cdots\equiv gZ_1Z_3^{-1},$$

$$\hat{\sigma}\equiv\left(1-\hat{g}\pi^2\right)^{1/2}, \tag{2.16b}$$

$$Z_3=1+ga^{-\varepsilon}\cdot(\mathrm{A}\ 2)+\cdots. \tag{2.16c}$$

Our subtraction prescription requires us to subtract the lattice forms (where $-\Delta$ becomes $\tilde{K}$) of the 1-loop terms in (2.14) from A_0; however, the analog of the transformation in subsect. 4.2 of (I) brings them with opposite sign to the LEL side again, such that (2.16) is, to one-loop order and, by the arguments of ref. [4], to all orders the form of L_0' of eq. (4.4) of (I).

The finite parts of the one-loop terms in the coefficients in (2.16) differ from the ones obtained with the unimproved propagator. This difference determines the "Λ-ratio" $\Lambda_{\mathrm{imp}}/\Lambda_{\mathrm{stan}}$ that shows up when comparing computer results for the improved action with the corresponding ones for the unimproved (standard) action, at the same value of g. This ratio is given in (A.7).

2.4 FOURTH ORDER IN MOMENTA

Diagrams A and B (which combine as before) contribute to T in (2.1) by expanding fig. 2a to fourth order in momentum. Diagram E contributes to T with fig. 3a taken at zero momentum. We shall not write out in full all the diagrams contributing to U and W in (2.1). It may immediately be seen, however, that as far as momentum structure is concerned, in addition to figs. 2a and 3a, only the expansion of the diagram fig. 3b (cf. figs. 1C, D) to second order in momenta, and of the diagram fig. 3c (cf. figs. 1B, F) to fourth order in momenta are needed; whereby only the terms linear in p and q, respectively linear in p, q, r, and s are relevant since each of these momenta should appear to first order at least. The calculation is given in appendix B.

Performing the trivial but lengthy O(N) algebra (which can be streamlined using the method of Brown and Duff [6]) and collecting all results, we find

$$\begin{aligned}
&\Gamma^{1\,\text{loop}}(\pi) \text{ to 4th order in momenta}\\
&= a^{2-\varepsilon}\int \mathrm{d}x \Bigg[-\tfrac{1}{48}c_1(\varepsilon) g \sum_{\mu} \left(\partial_\mu^2 \pi \cdot \partial_\mu^2 \pi + g^{-1}\partial_\mu^2\sigma\partial_\mu^2\sigma \right) \\
&\quad + \tfrac{1}{12}c_2(\varepsilon) g^2 \sum_{\mu\nu} \left(\partial_\mu\pi\cdot\partial_\nu\pi + g^{-1}\partial_\mu\sigma\partial_\nu\sigma \right)^2 \\
&\quad - \tfrac{1}{48}c_2(\varepsilon) g^2 \left[\sum_{\mu} \left(\partial_\mu\pi\cdot\partial_\mu\pi + g^{-1}\partial_\mu\sigma\partial_\mu\sigma \right) \right]^2 \\
&\quad + \tfrac{1}{16}c_3(\varepsilon) g^2 \sum_{\mu} \left(\partial_\mu\pi\cdot\partial_\mu\pi + g^{-1}\partial_\mu\sigma\partial_\mu\sigma \right)^2 \\
&\quad + \tfrac{1}{4}c_4(\varepsilon) g \sum_{\mu} \left(\partial_\mu\pi\cdot\partial_\mu\pi + g^{-1}\partial_\mu\sigma\partial_\mu\sigma \right)\sigma^{-1}\Delta\sigma \\
&\quad + \tfrac{1}{4}(N-1)c_5(\varepsilon)(\sigma^{-1}\Delta\sigma)^2 \Bigg] . \qquad (2.17)
\end{aligned}$$

The coefficients $c(\varepsilon)$ are defined in (B.1)–(B.5). Again one notes that precisely the same dim-4 terms compatible with the Ward identities found by Brézin, Zinn-Justin, and Le Guillou [4] appear, considering that under lattice symmetry also the structures of the first and fourth term in (2.17) are allowed. In one-loop order, the term $\Delta\pi\cdot\Delta\pi + g^{-1}\Delta\sigma\Delta\sigma$ of ref. [4] has zero coefficient.

The terms in (2.17), rewritten as lattice terms and with opposite sign, yield the improvement part $a^2 A_1'$ in (4.4) of (I) to one-loop order. When transferring (2.17) onto the lattice, its terms should, by virtue of the derivation of (2.17), be interpreted as normal products in the sense of Zimmermann [8]. While massless self-contractions

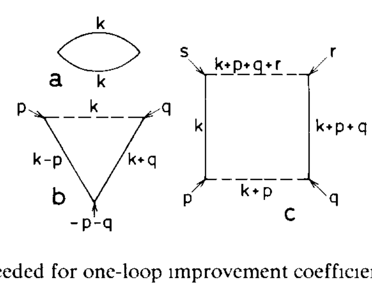

Fig 3 Graphs needed for one-loop improvement coefficients (besides fig 2a)

are zero in dimensional integration, they are not zero on the lattice and ought to be put in. We shall show in subsect. 3.3, however, that these extra terms are two-loop ones and thus outside the one-loop approximation. Therefore we can ignore them.

Choosing the lattice terms invariant under the lattice symmetry operations ($\partial_\mu \to -\partial_\mu^+$, and $\frac{1}{2}\pi$ rotations) the simplest transcription yields at this stage the one-loop improved lattice action

$$A_{\mathrm{imp}}^{1\ \mathrm{loop}} = a^{2+\varepsilon}\sum\Bigg\{-\tfrac{1}{2}\boldsymbol{\phi}\cdot\tilde{K}\boldsymbol{\phi} - a^{-2-\varepsilon}\ln\sigma$$
$$+a^{2-\varepsilon}\Bigg[\tfrac{1}{48}c_1(\varepsilon)g\sum_\mu \partial_\mu\partial_\mu^+\boldsymbol{\phi}\cdot\partial_\mu\partial_\mu^+\boldsymbol{\phi}$$
$$-\tfrac{1}{12}c_2(\varepsilon)g^2\sum_{\mu\nu}\left(\tfrac{1}{4}(\partial_\mu+\partial_\mu^+)\boldsymbol{\phi}\cdot(\partial_\nu+\partial_\nu^+)\boldsymbol{\phi}\right)^2$$
$$+\tfrac{1}{48}c_2(\varepsilon)g^2(\boldsymbol{\phi}\cdot K\boldsymbol{\phi})^2 - \tfrac{1}{16}c_3(\varepsilon)g^2\sum_\mu\left(\boldsymbol{\phi}\cdot\partial_\mu\partial_\mu^+\boldsymbol{\phi}\right)^2$$
$$+\tfrac{1}{4}c_4(\varepsilon)g\boldsymbol{\phi}\cdot(K\boldsymbol{\phi})\sigma^{-1}K\sigma - \tfrac{1}{4}(N-1)c_5(\varepsilon)(\sigma^{-1}K\sigma)^2\Bigg]\Bigg\}. \tag{2.18}$$

Hereby, we wrote $\boldsymbol{\phi} \equiv (\boldsymbol{\pi}, g^{-1/2}\sigma)$ and used the fact that in the improvement parts in the square bracket, the unimproved lattice laplacian in (2.2) of (I) could be used to sufficient accuracy such that in (2.18), at most, the next-nearest-neighbor couplings appear. Lattice symmetry invariance could also have been achieved in ways other than in (2.18), however, the difference to (2.18) would have consisted of $a^{4-\varepsilon}$ terms that would have an effect on improvement coefficients only in two-loop order when they themselves appear in loops. The $c(\varepsilon)$ are meromorphic, and regular at $\varepsilon = 0$. The $c(0)$ are listed in (B.6).

Rather than calculating two-loop graphs with the one-loop-improved action (2.18), as would be the analog of the procedure in sect. 4 of (I), we shall first transform (2.18) into manifestly O(N) invariant form by the tools of the following section, and take up the question of higher-loop orders only in subsect. 4.3.

3. Operator identities and normal products

3.1 NOT-MANIFESTLY-INVARIANT OPERATOR IDENTITIES

The action (2.3) amended by source terms $a^{2+\varepsilon}\Sigma_n(\boldsymbol{J}_n\cdot\boldsymbol{\pi}_n + g^{-1/2}h_n\sigma_n)$ leads to the field equation

$$-\tilde{K}\boldsymbol{\pi} + \sigma^{-1}\boldsymbol{\pi}\tilde{K}\sigma + \boldsymbol{J} - g^{1/2}h\sigma^{-1}\boldsymbol{\pi} + a^{-2-\varepsilon}g\sigma^{-2}\boldsymbol{\pi} = 0. \tag{3.1}$$

It is convenient to interpret $\boldsymbol{\pi}$ as $a^{-2-\epsilon}\partial/\partial \boldsymbol{J}$ acting on the functional integral with sources, whereupon (3.1) becomes the Schwinger-Dyson equation. Multiplying (3.1) from the left by $g\boldsymbol{\pi}$ and rearranging yields

$$\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) - g(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma) + g(\boldsymbol{J}\cdot\boldsymbol{\pi} + g^{-1/2}h\sigma) + a^{-2-\epsilon}g\sigma^{-2} + (N-2)a^{-2-\epsilon}g = 0\,, \qquad (3.2)$$

which is the "counting identity" [9] of the model.

From (2.3) with source terms included follows

$$\partial/(\partial g)(A_0 + \text{source terms}) = \tfrac{1}{2}g^{-2}a^{2+\epsilon}\sum\left[\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) + a^{-2-\epsilon}\sigma^{-2}g\boldsymbol{\pi}^2\right] \qquad (3.3)$$

If we had included the h source term in the subtraction steps in subsects. 2.3 and 2.4, $\sigma^{-1}\Delta\sigma$ in (2.12) would have been replaced, as expected from ref. [4], by the negative of the first term in (3.2) remembering that, on the a^0 level, $\tilde{K}$ is the required transcription of $-\Delta$ onto the lattice. This confirms that $\boldsymbol{\pi}$ and $g^{-1/2}\sigma$ would be renormalized to one-loop order, and by the argument of ref. [4] to all orders, in the same way. (In this respect, $g^{-1/2}h\sigma$ is not analogous to $-\frac{1}{2}\Delta m_{\mathrm{B}}^2\phi^2$ of the ϕ^4 theory, since ϕ^2 there requires an independent renormalization factor as does $\boldsymbol{\pi}^2 - \langle\boldsymbol{\pi}^2\rangle$ in the present model [4, 10].) The two last terms in (3.2) and the last one in (3.3) are two-loop ones: multiplying (3.2) by $a^{-\epsilon}$, which is dimensionally its factor as integrand in (2.14), allows us to write e.g. the last term as $a^{-2-\epsilon}(ga^{-\epsilon}) + a^{-2-\epsilon}(ga^{-\epsilon})(\sigma^{-2}-1)$, showing it to be a two-loop term in comparison with $-a^{2-\epsilon}\ln\sigma$ in (2.3) which is a one-loop one.

Multiplying (3.2) from the left by $g(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma)$ yields

$$g(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma)\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) - g^2(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma)^2 + g^2(\boldsymbol{J}\cdot\boldsymbol{\pi} + g^{-1/2}h\sigma) \times(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma) + \text{2-loop terms} = 0\,. \qquad (3.4)$$

Among the two-loop terms herein is also the commutator

$$\left[g(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma),\, g\boldsymbol{J}\cdot\boldsymbol{\pi}\right] = a^{-2-\epsilon}g^2(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma) - a^{-2-\epsilon}g\sigma^{-1}\tilde{K}\sigma\,. \qquad (3.5)$$

Namely, multiplying the RHS by $a^{2-\epsilon}$, dimensionally in (2.17) the coefficient of the first term in (3.4), yields

$$(ga^{-\varepsilon})^2[\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi}+g^{-1}\sigma\tilde{K}\sigma]-(ga^{-\varepsilon})[a^{-\varepsilon}\sigma^{-1}\tilde{K}\sigma],$$

identifying these as a^0-level two-loop terms since the square brackets herein are zero and one-loop terms, respectively.

Multiplying (3.2) from the left by the same expressions with all signs except that of the first term reversed yields

$$\begin{aligned}&[\sigma^{-1}(\tilde{K}\sigma-g^{1/2}h)]^2-g^2(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi}+g^{-1}\sigma\tilde{K}\sigma)^2-g^2;(\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma)^2;\\&\quad+2g^2(\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma)(\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi}+g^{-1}\sigma\tilde{K}\sigma)+\text{2-loop terms}=0.\end{aligned}\tag{3.6}$$

Here the semicolon sign means ordering J to the left of π and σ. The difference relative to the ordinary square, with the factor $a^{2-\varepsilon}$ included, is

$$\begin{aligned}&-a^{2-\varepsilon}g^2[(\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma),\boldsymbol{J}\cdot]\boldsymbol{\pi}\\&\quad=-(ga^{-\varepsilon})^2[\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma]+(ga^{-\varepsilon})^2[g^{-1/2}h\sigma^{-1}],\end{aligned}\tag{3.7}$$

where the two last square brackets are a^0-level zero-loop terms. Also all other commutators arising in the step from (3.2) to (3.5) are two-loop terms as seen already from the "loop counting" factors $ga^{-\varepsilon}$. Eqs. (3.4) and (3.6) will be used in subsect. 4.1.

3.2 MANIFESTLY INVARIANT OPERATOR IDENTITIES

Multiplying (3.1) from the left by J and rearranging yields

$$\begin{aligned}&-(\boldsymbol{J}\cdot\tilde{K}\boldsymbol{\pi}+g^{-1/2}h\tilde{K}\sigma)+(\boldsymbol{J}^2+h^2)+(\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma)\sigma^{-1}(\tilde{K}\sigma-g^{1/2}h)\\&\quad+a^{-2-\varepsilon}g(\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma)\sigma^{-2}-a^{-2-\varepsilon}g^{1/2}h\sigma^{-1}=0.\end{aligned}\tag{3.8}$$

Multiplying (3.2) from the left by $\boldsymbol{J}\cdot\boldsymbol{\pi}+g^{-1/2}h\sigma$ and subtracting the result from (3.8) gives, with the use of (3.7), in the notation introduced in (2.18)

$$-\hat{\boldsymbol{J}}\cdot\tilde{K}\boldsymbol{\phi}+\hat{\boldsymbol{J}}^2+g\hat{\boldsymbol{J}}\cdot\boldsymbol{\phi}\boldsymbol{\phi}\cdot\tilde{K}\boldsymbol{\phi}-g;(\hat{\boldsymbol{J}}\cdot\boldsymbol{\phi})^2;-a^{-2-\varepsilon}(N-1)g\hat{\boldsymbol{J}}\cdot\boldsymbol{\phi}=0,\tag{3.9}$$

with $\hat{\boldsymbol{J}}\equiv(\boldsymbol{J},h)$ and the semicolon sign as in (3.6). It will in subsect. 4.3 turn out to be significant that in this invariant identity also the "higher-loop" term is so.

Multiplying (3.1) from the left by $\tilde{K}\boldsymbol{\pi}$ gives

$$-(\tilde{K}\boldsymbol{\pi})^2 + \boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi}\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h + a^{-2-\varepsilon}g\sigma^{-1})$$
$$+\boldsymbol{J}\cdot\tilde{K}\boldsymbol{\pi} + \tfrac{5}{2}(2+\varepsilon)(N-1)a^{-4-\varepsilon}, \tag{3.10}$$

where (2.4) has been used. Multiplying (3.2) from the left by $\boldsymbol{\pi}\cdot\tilde{K}\boldsymbol{\pi} + g^{-1}\sigma\tilde{K}\sigma$ and subtracting the result from (3.10) yields, using (3.5)

$$-(\tilde{K}\boldsymbol{\phi})^2 + g(\boldsymbol{\phi}\cdot\tilde{K}\boldsymbol{\phi})^2 - g(\hat{\boldsymbol{J}}\cdot\boldsymbol{\phi})(\boldsymbol{\phi}\cdot\tilde{K}\boldsymbol{\phi})$$
$$+\hat{\boldsymbol{J}}\cdot\tilde{K}\boldsymbol{\phi} - a^{-2-\varepsilon}(N-1)g\boldsymbol{\phi}\cdot\tilde{K}\boldsymbol{\phi}$$
$$+\tfrac{5}{2}(2+\varepsilon)(N-1)a^{-4-\varepsilon} = 0, \tag{3.11}$$

again an entirely invariant identity. If in (3.10) and (3.11) the unimproved laplacian K instead of $\tilde{K}$ had been used, in the last terms the factor $\frac{5}{2}$ would have been replaced by 2.

3.3 NORMAL PRODUCTS

The terms in the one-loop improvement part (2.18) are designed to implement certain subtractions for $\mathrm{Re}\,\varepsilon > 2$, and corresponding substitutions for $-2 < \mathrm{Re}\,\varepsilon < 2$ by analytic continuation, as described in subsect. 2.1. The way the coefficients were determined in subsect. 2.4 implies that those terms in (2.18) should be interpreted as normal products in the sense of Zimmermann [8]. However, the difference between ordinary and normal products in (2.18) contributes only in the two-loop order and can thus be disregarded in (2.18). Namely, in contractions two π operators are replaced by const $a^{-\varepsilon}$, which combines to a "loop-counting" factor $ga^{-\varepsilon}$ with the factor g that had to be present for dimensional balance relative to an operator product with two π operators less. Depending on the number of derivatives on the π operators, the contraction will be a two-loop order a^0 or a^2 part. The contractions of interest in (2.18) are worked out in appendix C, and the formulae (C.6) and (C.8)–(C.11) show the described effect

For example, the difference on the LHS of (C.6) would be relevant in the normal ordering of (2.18) (at $\varepsilon = 0$) multiplied by g^2a^2, whereupon the first term on the RHS of (C.6) becomes (comparing with (2.1)) a two-loop a^0 term and the second and third terms become (comparing with (2.18)) two-loop a^2 terms. One notes that on the lattice, contractions of "rotationally invariant" terms contain also "rotationally non-invariant" ones.

The only term which requires attention is the last one in (2.18) since, to lowest order,

$$\sigma^{-1}K\sigma = -g\boldsymbol{\pi}\cdot K\boldsymbol{\pi} + \tfrac{1}{2}g(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + \partial_\mu^+\boldsymbol{\pi}\cdot\partial_\mu^+\boldsymbol{\pi}).$$

The contraction of the square hereof contains $\langle \pi_i \pi_j \rangle$ of (C.4) with the factor $g^2(K\pi_i)(K\pi_j)$. Such a term cannot be disregarded since it has no $\varepsilon \to 0$ limit. However, in subsect. 4.1 we shall replace the normal-ordered first term in (3.6) by the negative of the normal-ordered other terms, and the third term in (3.6) then contains a contraction $g^2 J_i J_j \langle \pi_i \pi_j \rangle$ that, to lowest order, balances the contraction of the first term of (3.6) to lowest order due to the lowest-order form $K\pi_i = J_i$ of (3.1). Thus, the IR problem is deferred and will be taken up in subsect. 4.3.

4. Manifestly O(N) invariant improved action

4.1 TRANSFORMATION OF (2.18) INTO INVARIANT FORM

Since the improvement terms in (2.18) are one-loop ones, the generating functional of one-loop improved GFs is

$$G_{\text{imp}}^{1\text{ loop}}\{\hat{J}\} = \text{const} \int \mathcal{D}\pi \left(1 + A_{\text{imp}}^{1\text{ loop}} - A_0\right) \exp\left[A_0 + a^{2+\varepsilon} \sum \hat{\boldsymbol{J}} \cdot \boldsymbol{\phi}\right]. \tag{4.1}$$

Here the identities (3.4) and (3.6) can be used to replace the two last terms of (2.18) by invariant ones, with the result to one-loop accuracy

$$G_{\text{imp}}^{1\text{ loop}}\{\hat{J}\} = \text{const} \int \mathcal{D}\boldsymbol{\phi} \prod \delta(\boldsymbol{\phi}^2 - g^{-1})$$
$$\times \exp\left\{a^{2+\varepsilon} \sum \left(-\tfrac{1}{2} \boldsymbol{\phi} \cdot \tilde{K} \boldsymbol{\phi} + \hat{\boldsymbol{J}} \cdot \boldsymbol{\phi}\right) + a^2 A_{1\text{ inv}}^{1\text{ loop}}\right\}, \tag{4.2a}$$

with

$$A_{1\text{ inv}}^{1\text{ loop}} = a^{2+\varepsilon} \sum a^{-\varepsilon} g \Bigg\{ \tfrac{1}{48} c_1(\varepsilon) \sum_{\mu} \partial_\mu \partial_\mu^+ \boldsymbol{\phi} \cdot \partial_\mu \partial_\mu^+ \boldsymbol{\phi}$$
$$- \tfrac{1}{12} c_2(\varepsilon) g \sum_{\mu\nu} \left[\tfrac{1}{4}(\partial_\mu + \partial_\mu^+) \boldsymbol{\phi} \cdot (\partial_\nu + \partial_\nu^+) \boldsymbol{\phi}\right]^2$$
$$+ \left[\tfrac{1}{48} c_2(\varepsilon) + \tfrac{1}{4} c_4(\varepsilon) - \tfrac{1}{4}(N-1) c_5(\varepsilon)\right] g (\boldsymbol{\phi} \cdot K \boldsymbol{\phi})^2$$
$$- \tfrac{1}{16} c_3(\varepsilon) g \sum_{\mu} (\boldsymbol{\phi} \cdot \partial_\mu \partial_\mu^+ \boldsymbol{\phi})^2$$
$$+ \left[-\tfrac{1}{4} c_4(\varepsilon) + \tfrac{1}{2}(N-1) c_5(\varepsilon)\right] g \hat{\boldsymbol{J}} \cdot \boldsymbol{\phi} \boldsymbol{\phi} \cdot K \boldsymbol{\phi}$$
$$- \tfrac{1}{4}(N-1) c_5(\varepsilon) g {:}(\hat{\boldsymbol{J}} \cdot \boldsymbol{\phi})^2{:} \Bigg\}. \tag{4.2b}$$

Hereby, according to subsect. 3.3 and noting (C.8), all terms may be interpreted as either normal or ordinary products, with the exception of the last term which should be read as a normal product, i.e. with its contraction subtracted to avoid perturbative IR difficulties. In (4.2b), K could be used instead of $\tilde{K}$ since the difference amounts to $O(a^4)$ terms that have $O(a^0)$ and $O(a^2)$ effects only in two-loop order.

The invariant identities (3.9) and (3.11) show that the one-loop improved action is not unique, but that we can obtain a two-parameter family of such actions This leads us to write down the ansatz

$$A_{1\,\mathrm{inv}} = a^{2+\varepsilon}\sum\Big\{\hat{J}\cdot[\bar{c}_1 g\phi\phi\cdot K\phi + \bar{c}_2 K\phi] + \bar{c}_3 g{:}(\hat{J}\cdot\phi)^2{:} + \bar{c}_4\hat{J}^2 + \bar{c}_5(K\phi)^2$$

$$+\bar{c}_6\sum_\mu(\partial_\mu\partial_\mu^+\phi)^2 + \bar{c}_7 g(\phi\cdot K\phi)^2 + \bar{c}_8 g\sum_\mu(\phi\cdot\partial_\mu\partial_\mu^+\phi)^2$$

$$+\bar{c}_9 g\sum_{\mu\nu}\left[\tfrac{1}{4}(\partial_\mu+\partial_\mu^+)\phi\cdot(\partial_\nu+\partial_\nu^+)\phi\right]^2\Big\} \tag{4.3a}$$

with

$$\bar{c}_i \equiv \bar{c}_i(ga^{-\varepsilon}, N, \varepsilon) = \sum_{\ell=1}^{\infty}\bar{c}_{i\ell}(\varepsilon, N)(ga^{-\varepsilon})^{\ell}. \tag{4.3b}$$

Our result so far (4.2b) yields a two-parameter set of meromorphic coefficients $\bar{c}_{i1}(\varepsilon, N)$ regular at $\varepsilon = 0$. We shall discuss (4.3) after verifying the correctness of these coefficients directly.

4.2 CHECK OF THE INVARIANT ANSATZ TO ONE-LOOP ORDER

We compute from (4.3a) upon functional differentiation w.r.t. J and h GFs and then VFs to one-loop order, write them in momentum space and expand them at fixed momenta w.r.t. small-a behaviour. There are a^0 and $a^{-\varepsilon}$ parts, which are "continuum" ones and ignored. (Compare the formulae (C.5) and (2.3) of (I).) Due to the use of the improved laplacian $\tilde{K}$ of (2.4) in A_0, there is no a^2 part, but an $a^{2-\varepsilon}$, a^4, $a^{4-\varepsilon}$ etc. part. Setting the $a^{2-\varepsilon}$ part zero gives constraints on the $\bar{c}_{i1}(\varepsilon, N)$, solvable by functions that are finite at $\varepsilon = 0$ (cf. end of subsect. 4.2 of (I)).

The calculation leads to the same graphs figs. 1–3 that were discussed in subsect. 2.4 and an additional one, and we list only the constraints obtained from the GFs (i.e. corresponding VFs) given by

$\langle\pi_i(x)\pi_j(x)\rangle$:

$$\bar{c}_{61} - \tfrac{1}{48}c_1(\varepsilon) = 0,$$

$$\bar{c}_{21} + \bar{c}_{41} + \bar{c}_{51} = 0, \tag{4.4a}$$

$$g^{-1}\langle\sigma(x)\sigma(y)\rangle: \qquad \bar{c}_{31}+\bar{c}_{41}+\tfrac{1}{4}(N-1)c_5(\varepsilon)=0, \tag{4.4b}$$

$$g^{-1}\langle\pi_i(x)\pi_j(y)\sigma(z)\sigma(u)\rangle: \quad \bar{c}_{11}+\bar{c}_{21}-\tfrac{1}{2}(N-1)c_5(\varepsilon)+\tfrac{1}{4}c_4(\varepsilon)=0, \tag{4.4c}$$

$$\langle\pi_i(x)\pi_j(y)\pi_k(z)\pi_l(u)\rangle: \qquad \bar{c}_{81}+\tfrac{1}{16}c_3(\varepsilon)=0,$$

$$\bar{c}_{91}+\tfrac{1}{12}c_2(\varepsilon)=0,$$

$$\bar{c}_{51}+\bar{c}_{71}-\tfrac{1}{48}c_2(\varepsilon)-\tfrac{1}{4}c_4(\varepsilon)+\tfrac{1}{4}(N-1)c_5(\varepsilon)=0. \tag{4.4d}$$

Hereby, for (4.4c) the graph fig. 4 had to be evaluated at zero momentum, where it reduces to fig. 3a. In (4c, d) are listed only those constraints that are not already so. The $c_i(\varepsilon)$ are the ones in (B.1)–(B.5). Further functions, e.g. $g^{-1/2}\langle\pi_i(x)\pi_j(y)\sigma(z)\rangle$, $\langle\pi_i\pi_j\pi_k\pi_l\sigma\rangle$, $\langle\pi\pi\pi\pi\pi\pi\rangle$ give no additional constraints, and functions $\langle\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\rangle$ etc. need no improvement. In fact, no further function gives additional constraints, for the following reason. Improvement of the (momentum)^4-term is needed for graphs of type fig. 3c, of the (momentum)^2-term for graphs of type fig. 3b, and of the (momentum)^0-term for graphs of type fig. 3a, whereby the general type is obtained from the prototype figs. 3a–c by insertion of elements shown in figs. 5a, b, etc. (E.g., inserting fig. 5a into fig. 3a yields fig. 4.) Since infinitesimal momenta can enter into the figs. 3c, b, a type graphs at most at four, respectively two, respectively zero corners, all other corners carrying zero momenta, due to $\tilde{K}\tilde{K}^{-1}=1$ no coefficient functions other than those calculated for figs 3c, b, a themselves occur. A functional algorithm similar to the one used in subsect. 2.3 then shows that no other constraints on the $\bar{c}_{i1}$ than those in (4.4) arise.

The identities (3.9) and (3.11) give the two invariances

$$\Delta\bar{c}_{11}=-\Delta\bar{c}_{21}=-\Delta\bar{c}_{31}=\Delta\bar{c}_{41}=x, \tag{4.5a}$$

$$\Delta\bar{c}_{11}=-\Delta\bar{c}_{21}=\Delta\bar{c}_{51}=-\Delta\bar{c}_{71}=y, \tag{4.5b}$$

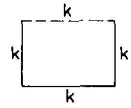

Fig 4 Graph needed in eq (4 4c)

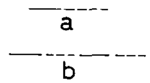

Fig 5 Insertions into graphs of fig 3

where x and y are arbitrary. These are indeed invariances of the eqs. (4.4). Upon comparing (4.2b) with (4.3a) it is found that the coefficients in (4.2b) obey the constraints (4.4). (The improved action of ref. [11] has at $\varepsilon = 0$ in (4.3a) $\bar{c}_6 = \frac{1}{48} g c_1(0)$, $\bar{c}_5 = -\frac{1}{4} g c_5(0)$, the other coefficients zero. Hereby the invariant one-loop two-point function is improved, not, however, e.g. the invariant one-loop four-point function.)

4.3 GENERALITY OF THE ANSATZ (4.3)

Eq (4.3a) comprises in the curly bracket all (lattice) local terms of (engineering) dimension $4+\varepsilon$ that can be formed polynomially from ϕ and $\hat{J}$, with factors g, in an O(N) invariant way. That (4.3a) with suitable coefficients of the form (4.3b) will improve all VFs to arbitrarily high-loop order is shown by the following recursive argument. We have already shown improvement for $\mathfrak{L} = 1$. VFs computed with the $\mathfrak{L} = 1$ improved action of form (4.3a) to two-loop order have a^0, $a^{-\varepsilon}$, and $a^{-2\varepsilon}$ terms, the last ones in non-O(N)-invariant form as in subsect. 2.3. As there, we can transform them by ϕ and g redefinition into O(N) invariant form. There remain, according to sect. 4 of (I), in the small-a expansion local polynomial terms with factors $a^{2-2\varepsilon}$, the polynomials being the ones appearing in (2.17), also including the term $(\Delta\pi)^2 + g^{-1}(\Delta\sigma)^2$ accidentally missing there, since these are all local solutions of the relevant Ward identities [4]. The corresponding lattice terms with opposite sign are then, with the help of the identities (3.4) and (3.6), brought into manifestly O(N) invariant form to yield the $\mathfrak{L} = 2$ terms in (4.3b). Hereby the terms that would appear in addition in (3.4) and (3.6), due to use of the action $A_0 + a^2 A_{1\,\mathrm{inv}}^{1\,\mathrm{loop}}$ rather than A_0, are O(a^4) ones and negligible in the two-loop order under consideration, but they contribute to O(a^0) and O(a^2) terms in three-loop order. The O(a^0) terms in that order are again transformed into O(N) invariant form, and the new $a^{2-3\varepsilon}$ terms treated as before etc.

In each loop-order the freedom of choosing coefficients according to the invariances (4.5) arises anew such that in (4.3a) e.g. one pair $(\bar{c}_i, \bar{c}_j)$ from the following pairs of coefficient functions can be made zero to all orders: one member from $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$, one member from $\bar{c}_1, \bar{c}_2, \bar{c}_5, \bar{c}_7$, except for the pair $(\bar{c}_1, \bar{c}_2)$. That the identities (3.9) and (3.11) are invariant also in the "higher"-loop terms prevents inconsistencies.

Finally, we consider the normal product in the $\bar{c}_3$ term in (4.3a), which differs, by perturbative derivation in one-loop order as discussed in subsect. 3.3, from the ordinary product by the perturbative contraction. In two dimensions in the absence of a symmetry-breaking external field (e.g. $\hat{J}_N \equiv h = \mathrm{const}$) only O($N$) invariant linear combinations of GFs are perturbatively IR finite [2], and this only if the action itself is O(N) invariant. This would be violated unless in the lattice action, to be used in two dimensions, all components of $\phi \equiv (\pi, g^{-1/2}\sigma)$ are treated (e.g. contracted) symmetrically. The perturbation theoretical contraction yields

$$(\hat{J}\cdot\phi)^2 - :(\hat{J}\cdot\phi)^2: = g^{-1}\hat{J}_N^2 + \text{higher orders} \rightarrow (gN)^{-1}\hat{J}^2 + \text{higher orders},$$

upon symmetrization. The higher-order terms must again be local ones, and can thus be absorbed in the higher-$\mathfrak{L}$ terms in (4.3) if they could affect GFs to the $O(a^2)$ accuracy of interest. Thus, the form (4.3a) is sufficiently general even without the double dots in the $\bar{c}_3$ term, if in two dimensions we consider only $O(N)$ invariant GFs. Outside of perturbation theory, there are for $N \geq 3$ no IR problems due to a spontaneous mass gap. (See also the $1/N$ expansion analysis in the following paper [12] of this series.) Perturbation theory is unsatisfactory for $N = 2$, and unavailable for $N = 1$.

5. Concluding remarks

5.1 EXTENSION OF PROCEDURE

The improvement detailed in sects. 2 and 4 can be extended to include $O(a^4)$ terms, $O(a^6)$ terms etc. as described in subsect. 5.1 of (I). Perturbation theory would yield (after transforming terms of lower loop-order and lower a^2-order into manifestly $O(N)$ invariant form) as improvement terms higher-dimensional operators that correspond to local solutions of Ward identities [4]. The general form of their solutions was determined by Heidenreich and Kluberg-Stern [13]. The corresponding higher-dimensional operators are in general not manifestly $O(N)$ invariant but can be replaced by invariant ones with the help of identities analogous to (3.4) and (3.6) derivable from (3.1). The result is that the improved action, analogous to (5.1) of (I), can be chosen to have only manifestly $O(N)$ invariant terms, with remaining arbitrariness due to the existence of $O(N)$ invariant identities as discussed in subsect. 4.3 for $O(a^2)$ order.

5.2 NON-PERTURBATIVE DETERMINATION OF IMPROVEMENT COEFFICIENTS

Improvement in the sense of sect. 1 of (I) requires us to determine the coefficients in (4.3a) to all loop-orders "exactly". As for the ϕ^4 model (subsect. 5.2 of (I)), this is possible only by Monte Carlo checks of improvement itself, of the simplest quantities on the finite lattice. These are, as in ϕ^4 theory, "generalized susceptibilities", i e. Green functions with all momenta zero, or some momenta at the lowest discrete values available on the finite lattice with periodic boundary conditions. These would, if made normalization independent, have to obey eq. (1.1) of (I) with decreased RHS. That even one-loop-order improvement terms give improved scaling behavior is shown in refs. [14]. (In not unlimited space-"time", perturbative calculations for vanishing IR regularizing field require care, since the limits $g \searrow 0$, i.e. weak-coupling expansion, and $H \searrow 0$ are in general not interchangeable there, not even for $O(N)$ invariant quantities [15]. Available then is the "gauge choice" of David [16] and an entirely $O(N)$ invariant but only implicit method of Luscher [17].)

5.3 OTHER MODELS, AND LIMITATIONS OF IMPROVEMENT

In order to obtain a workable improved action for the model of this paper, it was essential to exploit the availability of alternative improved actions since an action of e.g. the direct perturbative form (2.18) is unsuitable for Monte Carlo simulation since σ can vanish. The present technique would be applicable to a large class of models with a global symmetry and a non-linear constraint, e.g. to the CP^{N-1} model that has instantons for all N and "confinement" as non-perturbative effects. Such effects, leading to corrections to scaling, are expected also for the non-linear sigma model (e.g. refs. [18]). If such corrections, quantitatively not yet well understood, would not disappear faster than proportional to a^2, they could not be removed by use of an action with local improvement terms only.

The author is indebted to M. Luscher for numerous discussions and communications on the non-linear sigma model, and to I. Montvay for evaluating the numerical integrals in appendices A and B on the DESY IBM 3081 computer.

Appendix A

CALCULATION OF RENORMALIZATION COEFFICIENTS

The graph fig. 2a gives

$$\int_k N(k)^{-1}N(k+p) \equiv G(p) = a^{-2-\varepsilon} + a^{-\varepsilon}d_1(\varepsilon)p^2 + a^{2-\varepsilon}d_2(\varepsilon)\sum p_\mu^4 + O(p^6), \tag{A.1}$$

where

$$\int_k \equiv \prod_{\mu=1}^{2+\varepsilon}(2\pi)^{-1}\int_{-\pi/a}^{\pi/a} dk_\mu,$$

and (see (2 4) and formula (4.1) of (I))

$$N(k) \equiv \sum_{\mu=1}^{2+\varepsilon} a^{-2}\left[2(1-\cos k_\mu a) + \tfrac{1}{3}(1-\cos k_\mu a)^2\right].$$

Thus,

$$d_1(\varepsilon) = \tfrac{1}{2}a^\varepsilon \int_k N(k)^{-1}N''(k_1) = (2\pi\varepsilon)^{-1} - 0.0971525 + O(\varepsilon). \tag{A.2}$$

The graph fig. 2b has the value

$$\int_k N(k)^{-1} \equiv a^{-\varepsilon}e(\varepsilon), \tag{A.3}$$

with

$$e(\varepsilon)=d_1(\varepsilon)+\tfrac{2}{3}a^{\varepsilon}\int_k N(k)^{-1}(1-\cos k_1 a)^2=(2\pi\varepsilon)^{-1}+0.0731455+\mathrm{O}(\varepsilon). \tag{A.4}$$

With an unimproved propagator, we obtain instead

$$d_1(\varepsilon)\to d_{1\,\mathrm{stan}}(\varepsilon)=(2\pi\varepsilon)^{-1}+(4\pi)^{-1}(-\psi(1)+3\ln 2-\ln\pi)-\tfrac{1}{4}+\mathrm{O}(\varepsilon)$$

$$=(2\pi\varepsilon)^{-1}-0.129685+\mathrm{O}(\varepsilon), \tag{A.5}$$

$$e(\varepsilon)\to e_{\mathrm{stan}}(\varepsilon)=d_{1\,\mathrm{stan}}(\varepsilon)+\tfrac{1}{2}(2+\varepsilon)^{-1}. \tag{A.6}$$

The "Λ-ratio" [19], which is up to a relative error of $\mathrm{O}(g)$ the ratio of the correlation lengths at the same value of g, is [20]

$$\Lambda_{\mathrm{imp}}/\Lambda_{\mathrm{stan}}=\exp\left\{-2\pi(N-2)^{-1}\left[(N-1)(e(\varepsilon)-e_{\mathrm{stan}}(\varepsilon))-d_1(\varepsilon)+d_{1\,\mathrm{stan}}(\varepsilon)\right]\right\}\Big|_{\varepsilon=0}=2.21923,\ \text{if } N=3. \tag{A.7}$$

Appendix B

CALCULATION OF IMPROVEMENT COEFFICIENTS

The graph fig. 2a gives, from (A.1), to fourth order

$$d_2(\varepsilon)=\tfrac{1}{24}\int_k N(k)^{-1}N''''(k_1)\equiv\tfrac{1}{24}c_1(\varepsilon), \tag{B.1}$$

regular for $\mathrm{Re}\,\varepsilon>-2$. The graph fig. 3a is, according to subsect. 4.2 of (I), to be computed for $\mathrm{Re}\,\varepsilon>2$ and analytically continued to $\varepsilon=0$. In analogy to (4.5) of (I) we have, for $-2<\mathrm{Re}\,\varepsilon<2$

$$a^{-2+\varepsilon}\cdot(\text{fig. 3a})=a^{-2+\varepsilon}\int_k\left[N(k)^{-2}-(k^2)^{-2}\right]-\left[\tfrac{1}{8}\pi^{-4}(\pi+2)+\mathrm{O}(\varepsilon)\right]\equiv c_5(\varepsilon). \tag{B.2}$$

The graph fig. 3b has the value $a^{-\varepsilon}e(\varepsilon)$ of (A.3) at $p=0$ and at $q=0$, such that

$$a^{-2+\varepsilon}\cdot(\text{fig. 3b})=a^{-2}e(\varepsilon)+f(\varepsilon)pq+\mathrm{O}(a^2),$$

with $f(\varepsilon)$ easiest by setting $p=-q$, such that for $-2<\operatorname{Re}\varepsilon<2$

$$f(\varepsilon)=-\tfrac{1}{2}a^{-2+\varepsilon}\int_k\left[N(k)^{-2}N''(k_1)-2(k^2)^{-2}\right]$$

$$+\left[\tfrac{1}{8}\pi^{-4}(\pi+2)+\mathrm{O}(\varepsilon)\right]\equiv-\tfrac{1}{2}c_4(\varepsilon)\,. \tag{B.3}$$

The graph fig. 3c reduces to fig. 2a upon setting p, q, r, or $s=-p-q-r$ zero. Thus

$$(\text{fig. 3c})=a^{-2-\varepsilon}+G(p+r)+a^{2-\varepsilon}F(pqrs)+\mathrm{O}(a^{4-\varepsilon})\,,$$

with $G(\ )$ defined in (A.1) and $F(pqrs)$ linear in all four momenta, such that it has the form

$$F(pqrs)=\alpha_1 p\ qr\ s+\alpha_2 p\ rq\ s+\alpha_3 p\ sq\ r+\alpha_0\sum_{\mu=1}^{2+\varepsilon}p_\mu q_\mu r_\mu s_\mu\,.$$

Setting $p+q=0$ yields $\alpha_2+\alpha_3=0$ and integrals for $\alpha_0+\alpha_1$ and α_1 Setting $p=t-q=-r$, with $t_\mu\sim\delta_{\mu_1}$, $q_\mu\sim\delta_{\mu_2}$ allows us to isolate easily $\alpha_2=\frac{1}{3}\alpha_1$. Using this expansion of the graph fig. 3c in sect. 2 4 leads us to define, for $-2<\operatorname{Re}\varepsilon<2$

$$c_2(\varepsilon)=a^{-2+\varepsilon}\int_k\left[N(k)^{-2}N''(k_1)N''(k_2)-4(k^2)^2\right]-\left[\tfrac{1}{2}\pi^{-4}(\pi+2)+\mathrm{O}(\varepsilon)\right], \tag{B.4}$$

and for $\operatorname{Re}\varepsilon>-2$

$$c_3(\varepsilon)\equiv a^{-2+\varepsilon}\int_k N(k)^{-2}\left[N''(k_1)^2-N''(k_1)N''(k_2)\right]. \tag{B.5}$$

At $\varepsilon=0$, the numerical values are

$$c_1(0)=-0.29702\,,\qquad c_2(0)=-0.23324\,,$$

$$c_3(0)=0.13998\,,\qquad c_4(0)=-0\ 069230\,,$$

$$c_5(0)=4.7976\cdot 10^{-3}\,. \tag{B.6}$$

Appendix C

EVALUATION OF CONTRACTIONS

The contractions needed in subsect. 3.3 are 2-point Green functions with coinciding arguments. In two dimensions, these are IR finite if at least one argument is (lattice) differentiated. Since in subsect. 3 3 it is argued that we can ignore contractions (with one exception, mentioned there), we will for simplicity take as a Green function the free one with an unimproved laplacian:

$$G(an0) = 2a^{-2}(2\pi)^{-2-\varepsilon}\prod_{\mu=1}^{2+\varepsilon}\left(\int \mathrm{d}k_\mu\right)\exp(-ian\cdot\boldsymbol{k})\left[\sum_{\mu=1}^{2+\varepsilon}(1-\cos k_\mu a)\right]^{-1}. \tag{C.1}$$

With use of

$$\partial_\mu - \partial_\mu^+ = a\partial_\mu\partial_\mu^+, \tag{C.2a}$$

$$\partial_\mu\partial_\mu^+(AB) = A\partial_\mu\partial_\mu^+B + B\partial_\mu\partial_\mu^+A + \partial_\mu A\partial_\mu B + \partial_\mu^+A\partial_\mu^+B, \tag{C.2b}$$

$$A\partial_\mu B = -B\partial_\mu^+A + \partial_\mu\left[B\left(1-a\partial_\mu^+\right)A\right], \tag{C.2c}$$

(for notation see (2.2) of (I), we do not use a summation convention) the following evaluations at coinciding arguments are straightforward (in (C.3b), $\mu \neq \nu$):

$$\langle\partial_\mu\pi_i\partial_\mu\pi_j\rangle = \langle\partial_\mu^+\pi_i\partial_\mu^+\pi_j\rangle = -\langle\pi_i\partial_\mu\partial_\mu^+\pi_j\rangle$$

$$= (2+\varepsilon)^{-1}\langle\pi_iK\pi_j\rangle = (2+\varepsilon)^{-1}a^{-2-\varepsilon}\delta_{ij}, \tag{C.3a}$$

$$\langle\partial_\mu\pi_i\partial_\nu\pi_j\rangle = \langle\partial_\mu^+\pi_i\partial_\nu^+\pi_j\rangle = -\langle\partial_\mu\pi_i\partial_\nu^+\pi_j\rangle$$

$$= -\langle\partial_\mu^+\pi_i\partial_\nu\pi_j\rangle = \tfrac{1}{2}a\langle\partial_\mu\pi_i\partial_\nu\partial_\nu^+\pi_j\rangle$$

$$= -\tfrac{1}{2}a(1+\varepsilon)^{-1}\langle\partial_\mu\pi_iK\pi_j\rangle - \tfrac{1}{2}\alpha(1+\varepsilon)^{-1}\langle\partial_\mu\pi_i\partial_\mu\partial_\mu^+\pi_j\rangle$$

$$= \tfrac{1}{2}(1+\varepsilon)^{-1}a^{-2-\varepsilon}(1-r(\varepsilon))\delta_{ij}, \qquad \mu\neq\nu, \tag{C.3b}$$

$$\langle\partial_\mu\pi_i\partial_\mu^+\pi_j\rangle = \langle\partial_\mu\pi_i\partial_\mu\pi_j\rangle - a\langle\partial_\mu\pi_i\partial_\mu\partial_\mu^+\pi_j\rangle$$

$$= a^{-2-\varepsilon}\left[(2+\varepsilon)^{-1} - r(\varepsilon)\right]\delta_{ij}, \tag{C.3c}$$

$$\langle\pi_i\partial_\mu\pi_j\rangle = -\langle\pi_i\partial_\mu^+\pi_j\rangle = \tfrac{1}{2}a\langle\pi_i\partial_\mu\partial_\mu^+\pi_j\rangle$$

$$= -\tfrac{1}{2}(2+\varepsilon)^{-1}a^{-1-\varepsilon}\delta_{ij}, \tag{C.3d}$$

where $r(\varepsilon) = 2\pi^{-1} + O(\varepsilon)$ from (C.1). The $\varepsilon\to 0$ singularity of the contraction (see (A.6))

$$\langle\pi_i\pi_j\rangle = \left[(2\pi\varepsilon)^{-1} + (4\pi)^{-1}(-\psi(1) + 3\ln 2 - \ln\pi) + O(\varepsilon)\right]\delta_{ij}, \tag{C.4}$$

is the familiar perturbative IR one of non-O(N)-invariant expectations [2] and would be avoided by use of an external field as intermediate IR regulator [4].

From (C.2b) we obtain with $\sigma = (1 - g\pi^2)^{1/2}$ in lowest order

$$\pi \cdot K\pi + g^{-1}\sigma K\sigma = \tfrac{1}{2}\left(\partial_\mu \pi \cdot \partial_\mu \pi + \partial_\mu^+ \pi \cdot \partial_\mu^+ \pi\right). \tag{C.5}$$

Using this and repeatedly (C.2a) and (C.3a–c) yields the contractions to lowest order, at $\varepsilon = 0$,

$$\begin{aligned}
&\left(\pi \cdot K\pi + g^{-1}\sigma K\sigma\right)^2 - :\left(\pi \cdot K\pi + g^{-1}\sigma K\sigma\right)^2: \\
&= a^{-2}\left(N + 1 - 4\pi^{-1}\right)\left(\pi \cdot K\pi + g^{-1}\sigma K\sigma\right) \\
&\quad + \left(-1 + 3\pi^{-1}\right)\sum_\mu \left(\partial_\mu \partial_\mu^+ \pi \cdot \partial_\mu \partial_\mu^+ \pi + g^{-1}\partial_\mu \partial_\mu^+ \sigma \partial_\mu \partial_\mu^+ \sigma\right) \\
&\quad + \tfrac{1}{2}\left(1 - 2\pi^{-1}\right)\left(K\pi \cdot K\pi + g^{-1}K\sigma K\sigma\right),
\end{aligned} \tag{C.6}$$

up to a c-number. In (C.6), the double dots have their usual meaning with respect to the π field as canonical one. The notation introduced in (2.18) might suggest different "contractions":

$$\begin{aligned}
(\phi \cdot K\phi)^2 - :(\phi \cdot K\phi)^2: &= \sum_{ij}\left[\langle \phi_i \phi_j \rangle K\phi_i K\phi_j + 2\langle \phi_i K\phi_i \rangle \phi_j K\phi_j\right. \\
&\quad \left. + 2\langle \phi_i K\phi_j \rangle \phi_j K\phi_i + \langle K\phi_i K\phi_j \rangle \phi_i \phi_j\right] + \text{c-number} \\
&= (gN)^{-1}(K\phi)^2 \\
&\quad + 2(1 + N^{-1})\langle \phi \cdot K\phi \rangle \phi \cdot K\phi + \text{c-number},
\end{aligned} \tag{C.7}$$

using that, due to symmetry restoration in two dimensions,

$$\langle \phi_i \phi_j \rangle = (gN)^{-1}\delta_{ij} \text{ and } \langle \phi_i K\phi_j \rangle \propto \delta_{ij}.$$

(C.7) differs from (C.6), the latter being the correct one in perturbation theory. (In ref. [14], contractions of the type (C.7) instead of the, in restriction to one-loop order ignorable, type (C.6) ones were used; the difference in the coefficients is small, however, and without consequences for the conclusions of ref. [14] since the improvement coefficients should really be optimized rather than taken from one-loop perturbation theory.)

Other lowest-order contractions at $\varepsilon = 0$, evaluated with (C 3a–d) and expressed in the notation introduced in (2.18), are the following ones:

$$\tfrac{1}{4}\sum_{\mu\nu}\left[(\partial_\mu+\partial_\mu^+)\phi\cdot(\partial_\nu+\partial_\nu^+)\phi\right]^2 - \text{:same expression:}$$

$$=(1-2\pi^{-1})(N+2)\left[a^{-2}\phi\cdot K\phi-\tfrac{1}{4}\sum_\mu \partial_\mu\partial_\mu^+\phi\cdot\partial_\mu\partial_\mu^+\phi\right],$$

$$\sum_\mu(\phi\cdot\partial_\mu\partial_\mu^+\phi)^2 - \text{:same expression:}$$

$$=(N+1-4\pi^{-1})a^{-2}\phi\cdot K\phi+\sum_\mu\partial_\mu\partial_\mu^+\phi\cdot\partial_\mu\partial_\mu^+\phi,$$

$$(\phi\cdot K\phi)\sigma^{-1}K\sigma - \text{:same expression:}$$

$$=\left(\tfrac{5}{2}-4\pi^{-1}\right)a^{-2}g\phi\cdot K\phi$$

$$+g(-1+3\pi^{-1})\sum_\mu\partial_\mu\partial_\mu^+\phi\cdot\partial_\mu\partial_\mu^+\phi$$

$$+g\left(\tfrac{3}{4}-\pi^{-1}\right)K\phi\cdot K\phi-(2N-3)a^{-2}\sigma^{-1}K\sigma,$$

$$J\cdot\phi\phi\cdot K\phi - \text{:same expression.} = \tfrac{1}{4}J\cdot K\phi+\tfrac{1}{4}g^{1/2}h\phi\cdot K\phi. \qquad \text{(C.8)}$$

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