Z. Phys. C - Particles and Fields 20, 243-245 (1983)

Zeitschrift für Physik C Particles and Fields

Lambda Scale for the Improved Lattice $O(N)$ Non-Linear Sigma Model

B. Berg
II. Institut für Theoretische Physik der Universität Hamburg, Luruper Chaussee 149 D-2000 Hamburg 50, Federal Republic of Germany

Received 27 April 1983

Abstract. Using the background field technique I compute $\Lambda_L^I/\Lambda_L^S = e^{(0.5928(N-1)/2+0.2044)/(N-2)}$ in agreement with a calculation by Symanzik. Implications for Monte Carlo simulations are also discussed.

I. Introduction

Monte Carlo (MC) simulations of lattice spin and gauge theories try to extract information about the quantum continuum limit. As the same universality class allows infinitely many equivalent actions, the art would be to use an action which allows MC simulations close to the continuum limit. To construct such an action, Wilson [1] suggested block spin MC renormalization group calculations but statistical noice seems to be a severe problem. A decisive advantage of Symanik's [2-4] improvement program is that perturbative calculations allow an educated guess of a good action. Recent MC simulations [5, 6] of the $2d$ $O(3)$ non-linear σ-model have shown that the one-loop order improved action (= improved action henceforth) gives a considerably improved scaling behaviour.

Let us consider $2d$ $O(N)$ non-linear sigma models and use the definition of Λ-scales

$$\Lambda = a^{-1}\left(\frac{2\pi\beta}{N-2}\right)^{1/(N-2)} \cdot e^{-2\pi\beta/(N-2)}(1+O(g^2)). \tag{1}$$

where $\beta = 1/g^2$ is the bare coupling of the corresponding (lattice) regularization scheme. Our mass gap result [5] for the $O(3)$ improved action now reads

$$m \approx 16\Lambda_L^I \quad (L = \text{lattice}, I = \text{improved}). \tag{2}$$

For the standard action our MC data were consistent with mass gap estimates of previous literature [7-9]

$$m \approx 100\Lambda_L^S \quad (S = \text{standard}). \tag{3}$$

In the continuum limit the ratio of Λ-scales can be calculated by a one-loop computation [10]. For the $O(N)$ models the ratio between Λ_L^S and Λ_{PV} (PV = Pauli-Villars) has been first calculated by Parisi [11]:

$$\Lambda_{PV}/\Lambda_L^S = 27.21 \quad (N=3). \tag{4}$$

Using a similar method Symanzik [4] obtains for the improved action

$$\Lambda_L^I/\Lambda_L^S = 2.219 \quad (N=3). \tag{5}$$

As the ratio (5) is rather catastrophic for the MC result (3)*, an independent calculation of it is desirable. Using the background field method [12] I obtain the same number.

The paper is organized as follows: Section II contains details of my calculation. This is also a pedagogical exercise for the calculation of Λ-scales by means of the background field method. For the $O(N)$ models the background field method has been used previously in [13, 14]. In the final Sect. III some consequences of the result (5) for MC simulations are discussed.

II. The Improved Lambda Scale

Let us consider the improved action [4] in the notation of [5]. The order g^{2**} coefficients $c_i (i=1,\ldots,6)$ will *not* contribute to the Λ-scale. Therefore it is for our purposes sufficient to calculate with the action

$$S = \sum_\mu \left\{ \tfrac{1}{2}\Delta_\mu\pi\Delta_\mu\pi + \tfrac{1}{2}\Delta_\mu\sigma\Delta_\mu\sigma - \frac{H}{2}\sigma \right.$$
$$\left. + c_{24}(\Delta_\mu\Delta_\mu\pi\Delta_\mu\Delta_\mu\pi + \Delta_\mu\Delta_\mu\sigma\Delta_\mu\Delta_\mu\sigma) \right\}. \tag{6}$$

Here $c_{24} = \frac{1}{24}$ for the improved action [15, 4, 5] and $c_{24} = 0$ for the standard action. Further:

* Assuming (2) to be the more reliable result

** Order g in the notation of [5]

$$\Delta_\mu f(x) = f(x+\hat{\mu}) - f(x), \quad \sigma = (1-\pi^2)^{1/2}$$
and $\pi = (\pi_1, \ldots, \pi_{N-1})$.

The Boltzmann factor is e^{-S/g^2} and ϕ of [5] is of course $\phi = (\pi, \sigma)$. A small magnetic field H has been introduced so that all the integrals encountered in perturbation calculations are infrared finite.

I now partly follow some unpublished notes of A. and P. Hasenfratz [13]. Let us write

$$\pi = W + g\alpha. \tag{7}$$

Here W is a smoothly varying classical background field and α parametrizes the quantum fluctuations. We like to calculate the one-loop renormalization of the classical background field (see e.g. [14]). This means we have to calculate the terms of order W^2, $g^2\alpha^2$ of the action (6). Using the rules

$$\Delta_\mu(f\cdot g) = \Delta_\mu f\cdot \Delta_\mu g + \Delta_\mu f\cdot g + f\cdot \Delta_\mu g \tag{8a}$$
and
$$\begin{aligned}\Delta_\mu\Delta_\mu(f\cdot g) &= \Delta_\mu\Delta_\mu f\cdot \Delta_\mu\Delta_\mu g\\ &+ 2(\Delta_\mu\Delta_\mu f\cdot \Delta_\mu g + \Delta_\mu f\cdot\Delta_\mu\Delta_\mu g)\\ &+ \Delta_\mu\Delta_\mu f\cdot g + f\cdot\Delta_\mu\Delta_\mu g + 2\Delta_\mu f\cdot \Delta_\mu g,\end{aligned} \tag{8b}$$

this is a straightforward algebraic calculation. Writing

$$S = S_{\rm cl} + S_0 + S_{\rm int} \tag{9}$$

the classical part of the action becomes

$$S_{\rm cl} = \sum_\mu \{\tfrac{1}{2}\Delta_\mu W \Delta_\mu W + \tfrac{1}{4}H W^2\} + c_{24}\cdot\text{higher derivatives} \tag{10a}$$

and the free part is

$$S_0 = g^2 \sum_\mu \{\tfrac{1}{2}\Delta_\mu\alpha\Delta_\mu\alpha + \tfrac{1}{4}H\alpha^2 + c_{24}\Delta_\mu\Delta_\mu\alpha\Delta_\mu\Delta_\mu\alpha\} \tag{10b}$$

The corresponding propagator reads

$$G(x) = \int_K \frac{e^{iK_1x_1 + iK_2x_2}}{4 - 2\cos K_1 - 2\cos K_2 + 8c_{24}((1-\cos K_1)^2 + (1-\cos K_2)^2)} \tag{11}$$

where

$$\int_K = \frac{1}{(2\pi)^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} dK_1\, dK_2.$$

Note that

$$G(n\hat{1}) = G(n\hat{2}) = \tfrac{1}{4}\int_K \frac{2\cos nK_1 + 2\cos nK_2}{4 - 2\cos K_1 - 2\cos K_2 + 8c_{24}((1-\cos K_1)^2 + (1-\cos K_2)^2)} \tag{12}$$

From the interaction part $S_{\rm int}$ we have to pick out quantum fluctuations with coefficients $\frac{1}{2}\Delta_\mu W\Delta_\mu W$ and $\frac{1}{2}H W^2$. Recognize that after summation over the lattice sites (periodic boundary conditions assumed) one has

$$\sum_x W\cdot\Delta_\mu W = \sum_x -\tfrac{1}{2}\Delta_\mu W\cdot\Delta_\mu W \tag{12a}$$

and

$$\sum_x W\cdot\Delta_\mu\Delta_\mu W = \sum_x (-\Delta_\mu W\cdot\Delta_\mu W + \Delta_\mu W\cdot\Delta_\mu\Delta_\mu W). \tag{12b}$$

Derivatives of order like $\Delta_\mu W\cdot\Delta_\mu\Delta_\mu W$ or higher do not contribute.

The quantum fluctuations are calculated by contracting the α^2 terms according to

$$\alpha^a(x + n\hat{\mu})\alpha^b(x) = \delta_{ab}\, G(n\hat{1}). \tag{13}$$

The results are

$$g^2\sum_x \tfrac{1}{2}\Delta_\mu W\cdot\Delta_\mu W\cdot\lambda \quad \text{with}$$
$$\lambda = G(1) + 8c_{24}(G(1) - G(2)), G(n) := G(n\hat{1}) \tag{14a}$$
and
$$g^2\tfrac{1}{2}H W^2\cdot K \quad \text{with} \quad K = \frac{N-1}{2}G(0). \tag{14b}$$

To K also $\frac{1}{2}W^2$-terms contributed via the identity

$$G(0) - G(1) + 2c_{24}(3G(0) - 4G(1) + G(2)) = \tfrac{1}{4} - \tfrac{1}{4}H\,G(0) + O(H^2). \tag{15}$$

The Λ-ratio is given by

$$\frac{\Lambda_L^I}{\Lambda_L^S} = e^{-(1/\beta_0)(1/g_{LI}^2 - 1/g_{LS}^2)} = e^{+(2\pi/(N-2))(2(K^S - K^I) - (\lambda^S - \lambda^I))} \tag{16}$$

We have used $\beta_0 = (N-2)/2\pi$ [16, 17] and $g_{LS}^2/g_{LI}^2 = Z_1^S/Z_1^I$ with $Z_1 = (-2g^2K + g^2\lambda)$. ($Z_1$ as defined in [17]). The index S corresponds to the standard propagator $G^S(x) = G(x)$ with $c_{24} = 0$, and the index I corresponds to the improved propagator $G^I(x) = G(x)$ with $c_{24} = \frac{1}{24}$. Numerical integration gives

$$2\pi(\lambda^S - \lambda^I) = -0.2044 \tag{17a}$$

and

$$4\pi(G^S(0) - G^I(0)) = 0.5928. \tag{17b}$$

Thus equation (16) yields the result of the abstract (equation (5) of the introduction for $N=3$).

III. Implications for Monte Carlo Simulations

In units of $\Lambda_{\rm PV}$ (4) various mass gap estimates are summarized in Table 1. The improved MC result [5] is even lower than Lüschers estimate [20]. To the extent that the improved MC calculation is reliable, the scaling curve for the standard action has to have a shape, which is qualitatively indicated in Fig. 1. Politzer [21] seems to like it. It should be emphasized that no clear scaling window for the standard action has been found. The scaling windows of [8, 9] are on

Table 1. Mass gap estimates in units Λ PV

m	Method
4.1 ± 0.2	MC, standard action, block spin renormalization group [7].
4.8 ± 0.2	MC, standard action [8].
3.5 ± 0.2	MC, standard action [9].
5.7 ± 0.5	MC, standard action [18].
≈ 3.4	Strong coupling, Hamiltonian [14].
≈ 3.2	Strong coupling, Euclidean [19].
≈ 1.7	Spin wave (continuum $\mathbb{R} \times S^1$) [20].
≈ 1.3	MC, improved action [5].

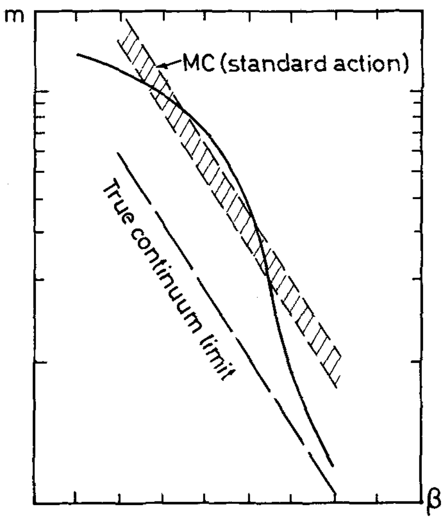

Fig. 1.

a very fine scan, and [18] now claims disagreement with scaling. Also there are large discrepancies in the final estimates.

A result of [6] is that the order g^2 coefficients c_i $(i = 1, \ldots, g)$ turn out to be important for the improved mass gap estimate [5]. In a MC simulation without these coefficients (they do not affect the asymptotic scale Λ_L^I) we obtain results more close to the standard action.

For the magnetic susceptibility clear scaling deviations were observed in MC investigations with the standard action [22–23]*. Nevertheless an estimate of the continuum limit was tried [9]:

$$c^S = 0.008\text{–}0.013. \tag{18a}$$

The constant c^S, c^I are defined by $\chi = c(2\pi\beta)^{-4} e^{4\pi\beta}$. The improved action exhibits clear scaling properties for the magnetic susceptibility and yields [5]

$$c^I \approx 0.3. \tag{18b}$$

The discrepancy similar as for the mass gap.

In summary the MC results [5, 6] for the improved $O(3)$ σ-model provide a severe warning against carelessly extracting numbers from lattice MC calculations. The magnitude of corrections may, however, be strongly model dependent. For instance the author would conjecture standard MC estimates in $O(4)$ and $O(5)$ σ-models [9] to be less effected. In $4d$ $SU(2)$ and $SU(3)$ lattice gauge theories scaling properties for the string tension [25] and the mass gap [26] are more clear than any "scaling" in the standard $O(3)$ σ-model. But there is *no* scaling [27] in lattice MC calculations of hadron masses. Fortunately the computation of the one-loop improved action for $4d$ $SU(N)$ lattice gauge theories is on the way [28].

* In [24] also non-standard actions were considered

Acknowledgements. I would like to thank K. Symanzik for useful discussions and P. Hasenfratz for unpublished notes [13].

Note added. After completing this paper I received a preprint by M. Falcioni, G. Martinelli, M.L. Paciello, G. Parisi and B. Taglienti [29], where the Λ-ratio (5) is also reported. Further the authors perform MC calculations with the tree-level improved action [15].

Note added in proof: Y. Iwasaki kindly informed me that the λ-ratio (16) was also calculated in [30]

References

1. K.G. Wilson: Cargèse lectures 1979. G. 't Hooft et al. eds. New York: Plenum Press 1980
2. K. Symanzik: in "Mathematical problems in theoretical physics." Eds. R. Schrader et al. In: Lecture Notes in Physics Vol. 153. Berlin, Heidelberg, New York: Springer 1982
3. K. Symanzik: Continuum limit and improved action in lattice theories I: Principles and ϕ^4 theory. Preprint, DESY 83-016
4. K. Symanzik: Continuum limit and improved action in lattice theories II: $O(N)$ non-linear sigma model in perturbation theory, Preprint, DESY 83-026
5. B. Berg, S. Meyer, I. Montvay, K. Symanzik: Phys. Lett. **126B**, 467 (1983)
6. B. Berg, S. Meyer, I. Montvay: in preparation
7. S.H. Shenker, J. Tobochnik: Phys. Rev. **B22**, 4462 (1980)
8. G. Fox, R. Gupta, O. Martin, S. Otto: Nucl. Phys. **B205** [FS5], 188 (1982)
9. M. Fukugita, Y. Oyanagi: Phys. Lett. **123B**, 71 (1983)
10. A. Hasenfratz, P. Hasenfratz: Phys. Lett. **93B**, 165 (1980)
11. G. Parisi: Phys. Lett. **92B**, 133 (1980)
12. R. Dashen, D.J. Gross: Phys. Rev. **D23**, 2340 (1981)
13. A. Hasenfratz, P. Hasenfratz: (unpublished)
14. J. Shigemitsu, J.B. Kogut: Nucl. Phys. **B190** [FS3], 365 (1981)
15. G. Martinelli, G. Parisi, R. Petronzio: Phys. Lett. **100B**, 485 (1981). The action of this reference does not improve the 4-point functions to one-loop order
16. M. Polyakov: Phys. Lett. **59B**, 79 (1975)
17. E. Brezin, J. Zinn-Justin: Phys. Rev. **D14**, 3110 (1976)
18. R. Gupta: Preprint, CALT-68-1010 (1983)
19. R. Musto, F. Nicodemi, R. Pettorino: Nucl. Phys. **B210** [FS6], 263 (1982). For previous work see: D.N. Lambeth, H.E. Stanley: Phys. Rev. **B12**, 5302 (1975)
20. M. Lüscher: Phys. Lett. **118B**, 391 (1982)
21. H.D. Politzer: Proceedings of the 21th international conference on high energy physics, Paris 1982. P. Petiau, M. Porneuf eds., p. c3—659
22. G. Martinelli, G. Parisi, R. Petronzio: Phys. Lett. **100B**, 485 (1981)
23. B. Berg, M. Lüscher: Nucl. Phys. **B190** [FS3], 412 (1981)
24. Y. Iwasaki, T. Yoshie: University of Tsukuba, Ibaraki preprint, UTHEP-105 (1982)
25. M. Creutz, K.J.M. Moriarty: Phys. Rev. **D26**, 43 (1983) and references given there
26. B. Berg, A. Billoire: Nucl. Phys. **B221**, 109 (1983)
27. A. Hasenfratz, P. Hasenfratz, Z. Kunszt, C.B. Lang: Phys. Lett. **110B**, 283 (1982)
28. P. Weisz: Nucl. Phys. **B212**, 1 (1983)
29. M. Falcioni, et al.: Preprint (1983)
30. Y. Iwasaki and T. Yoshie: University of Tsukuba, Ibaraki preprint, UTHEP-94 (1982)