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KAZAMA–YANG MONOPOLE-FERMION BOUND STATES (I). Analytic Results

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We present explicit, approximate, remarkably precise results for the Kazama-Yang monopole-fermion binding energies and wave functions. The results are valid for the states of lowest angular momentum and for the binding energy $M-E \ll M$. They agree very well with the numerically calculated values.

1. Introduction

The interaction of magnetic monopoles with fermions has been studied in considerable detail by Kazama, Yang and Goldhaber [1], who discussed scattering cross sections, and by Kazama and Yang [2] who showed that a fermion anomalous magnetic moment leads to the existence of bound states.

The bound-state spectrum is somewhat unfamiliar compared to ordinary atomic systems, in that there exist states of exactly zero total energy, i.e. with binding energy, E_B, equal to the fermion mass, M. The appearance of such states seems to be a rather general feature of topological objects. However, in this case of massive fermions, their presence is not dictated by the Atiyah–Singer index theorem [3].

Kazama and Yang [2] found that, provided that the anomalous magnetic moment, κ, is positive, a zero energy state exists for the lowest allowed angular momentum,

$$j = |q| - \tfrac{1}{2}, \tag{1.1}$$

where

$$q \equiv Zeg = \tfrac{1}{2}Zn, \tag{1.2}$$

with Ze and g the electric and magnetic charges of the fermion and the monopole,

respectively, and n an integer ≥ 1. Furthermore, for the higher angular momenta,

$$j \geq |q| + \tfrac{1}{2}, \tag{1.3}$$

such zero-energy states were found to exist [2] for any $\kappa \neq 0$ (positive or negative).

In addition, they found that, for $j = |q| - \frac{1}{2}$, there is an infinite sequence of excited bound states, provided

$$\kappa |q| > \tfrac{1}{4}. \tag{1.4}$$

These additional states are for moderate values of $\kappa|q|$ very loosely bound. For example, for $\kappa|q| = 2$, the three lowest-lying states have binding energies [2] $E_B/M \simeq 5\times10^{-3}$, 5×10^{-5}, and 4×10^{-7}.

Later, Yang [4] showed that such additional, loosely bound states exist for the higher angular momenta, $j \geq |q| + \frac{1}{2}$, provided the anomalous magnetic moment is large enough,

$$|\kappa q| > (j+\tfrac{1}{2})^2 - q^2 - \tfrac{1}{4}. \tag{1.5}$$

Comparing this restriction with eq. (1.1), one finds that, for fixed κ and q, only a limited number, N, of the higher angular momenta can exhibit excited bound states. This number, N, is the largest integer which satisfies the inequality

$$(|\kappa| - 2N)|q| \geq N^2 - \tfrac{1}{4}, \tag{1.6}$$

and the highest angular momentum for which excited bound states exist is

$$j_{\max} = |q| - \tfrac{1}{2} + N. \tag{1.7}$$

A problem of great current interest is the possible binding of nucleons or nuclei to the monopoles of grand unification. Since $\kappa = 1.79$ for the proton, and since SU(5) monopoles are only likely to be found with $g = 1/2e$ [5, 6], eq. (1.6) implies that the proton-monopole system can form loosely bound states of the lowest angular momentum only, $j = 0$. This is rather convenient from a theoretical point of view, since the bound state equations are simpler in this case.

The hamiltonian studied in refs. [1, 2, 4] corresponds to a Dirac monopole and a point-like fermion,

$$H = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - Ze\boldsymbol{A}) + \beta M - \kappa q \beta (\boldsymbol{\sigma} \cdot \boldsymbol{r})/(2Mr^3). \tag{1.8}$$

While the monopoles of grand unification also have an SU(3) colour magnetic charge [5], this is believed to be screened by the vacuum. For the considerations of bound states we shall therefore assume that (1.8) leads to an approximate description of the spectrum.

There are two reasons why the above hamiltonian will only give an approximate spectrum. First, nucleons and nuclei have finite extensions. At short distances the anomalous magnetic moment should be described by a finite distribution, thus taking into account the granular structure and the magnetic polarizability. Second, since

grand unification monopoles carry a colour magnetic charge, there will at short distances (inside the presumed screening radius) also be strong interactions present. However, hadrons are colour singlets, and these strong interactions would be entirely absent were the hadrons pointlike. Again, it is the finite extension which leads to a correction to the hamiltonian (1.8), related now to a hadronic colour charge distribution.

We assume that these finite-size effects yield corrections to the hamiltonian that fall off at large distances much faster than r^{-2}. Thus, the states having large radii will be very little affected.

For the lowest angular momentum state, there is only one possible eigensection [7] of angular momentum [1, 2]. With

$$\psi(\boldsymbol{r}) = \begin{bmatrix} f(r)\eta_{jj_z}(\hat{\boldsymbol{r}}) \\ g(r)\eta_{jj_z}(\hat{\boldsymbol{r}}) \end{bmatrix}, \tag{1.9}$$

the eigensection η_{jj_z} satisfies

$$\boldsymbol{J}^2\eta_{jj_z} = j(j+1)\eta_{jj_z} = (q^2-\tfrac{1}{4})\eta_{jj_z}, \tag{1.10a}$$

$$J_z\eta_{jj_z} = j_z\eta_{jj_z}, \tag{1.10b}$$

with

$$\boldsymbol{J} = \boldsymbol{L} + \tfrac{1}{2}\boldsymbol{\sigma}, \tag{1.11a}$$

$$\boldsymbol{L} = \boldsymbol{r}\times(\boldsymbol{p} - Ze\boldsymbol{A}) - q\boldsymbol{r}/r. \tag{1.11b}$$

Using concepts from the mathematics of fiber bundles, Wu and Yang [7] showed that the eigenfunctions of the angular momentum operator are free of discontinuities or cusps if thought of as "sections". As sections η_{jj_z} must be specified in different gauges in different regions of space, and will thus be defined by a different function in each region. However, using eqs. (1.10) and (1.11), this non-single-valuedness completely factors out from the eigenvalue problem

$$H\psi = E\psi, \tag{1.12}$$

which for $j = |q| - \frac{1}{2}$ may be reduced to a set of coupled radial equations [2],

$$\frac{\mathrm{d}G}{\mathrm{d}\rho} = \left(A - B - \frac{1}{\rho^2}\right)F, \tag{1.13a}$$

$$\frac{\mathrm{d}F}{\mathrm{d}\rho} = \left(A + B - \frac{1}{\rho^2}\right)G, \tag{1.13b}$$

where

$$\rho = \frac{2}{|\kappa q|}rM, \tag{1.14a}$$

$$A = \tfrac{1}{2}\kappa|q|, \qquad B = \tfrac{1}{2}\kappa|q|\frac{E}{M}, \tag{1.14b}$$

$$f=\frac{1}{r}\frac{\kappa q}{|\kappa q|}F, \qquad g=-\frac{i}{r}G. \tag{1.14c}$$

For the zero-energy state, $E=0$ (or $E_B=M$), the solutions to (1.13) are [2]

$$G(\rho)=-F(\rho)=\text{const}\times\exp[-A\rho-1/\rho]. \tag{1.15}$$

For the other states, however, the eigenvalues and the radial wave functions are so far known only numerically [2].

In the present paper we give approximate, analytic expressions for the remaining states of the lowest angular momentum, valid in the limit of weak binding. These solutions may be of practical value, e.g. for the evaluation of capture cross sections [8],

$$\text{monopole}+\text{nucleon (free)}\to\text{monopole}+\text{nucleon (bound)}+\text{photon},$$

and offer new insight into the physics of these states.

Also, it has just been shown [9] that the zero-energy bound states are absent for SU(5) monopoles. The lowest of the weakly bound states discussed here may then for SU(5) monopoles be the ground state of the monopole-fermion system.

2. The radial equations

We rewrite the radial equations (1.13) as

$$\begin{aligned}\frac{dG}{d\eta}&=a\left[\left(\frac{\varepsilon}{2-\varepsilon}\right)^{1/2}-\frac{1}{\eta^2}\right]F,\\ \frac{dF}{d\eta}&=a\left[\left(\frac{2-\varepsilon}{\varepsilon}\right)^{1/2}-\frac{1}{\eta^2}\right]G,\end{aligned} \tag{2.1}$$

where

$$\varepsilon=(M-E)/M\equiv E_B/M \tag{2.2}$$

is the fractional binding energy, i.e. our eigenvalue parameter. The quantities

$$\begin{aligned}a&=[A^2\varepsilon(2-\varepsilon)]^{1/4},\\ \eta&=a\rho=aMr/A,\end{aligned} \tag{2.3}$$

have been introduced for later convenience.

A local analysis of (2.1) shows that the equation has two irregular singular points; one at $\eta=0$, where the solutions behave like $\exp[\pm a/\eta]$, and one at $\eta=\infty$, where the solutions behave like $\exp[\pm a\eta]$.

It is possible to re-express (2.1) as a second-order equation for a single function without introducing additional singular points*. In terms of the combination

$$N=(F-G)\exp(a/\eta) \tag{2.4}$$

* We are grateful to Prof. T.T. Wu for demonstrating to us how this can be done.

this becomes

$$\frac{d^2N}{d\eta^2}+\frac{2a}{\eta^2}\frac{dN}{d\eta}-\left[a^2-\frac{2A}{\eta^2}\right]N=0\,. \tag{2.5}$$

But equations with two irregular singular points cannot generally be solved in terms of the known special functions. Thus, the prospects for solving (2.1) or (2.5) exactly are not very promising. However, eq. (2.1) has an inversion symmetry which we have found useful for constructing good approximate solutions.

3. Inversion symmetry

From equation (2.1) it follows that if a solution $\{F(\eta), G(\eta)\}$ is known, then the pair $\{([2-\varepsilon]/\varepsilon)^{1/4}G(1/\eta), (\varepsilon/[2-\varepsilon])^{1/4}F(1/\eta)\}$ is also a solution. This may be written as a transformation,

$$\begin{pmatrix}\tilde F(\eta)\\ \tilde G(\eta)\end{pmatrix}\equiv S\begin{pmatrix}F(1/\eta)\\ G(1/\eta)\end{pmatrix}=SI\begin{pmatrix}F(\eta)\\ G(\eta)\end{pmatrix}, \tag{3.1}$$

which maps solutions $F(\eta)$, $G(\eta)$ of (2.1) into (possibly new) solutions $\tilde F(\eta)$, $\tilde G(\eta)$. Here

$$S=\begin{pmatrix}0 & \left(\dfrac{2-\varepsilon}{\varepsilon}\right)^{1/4}\\ \left(\dfrac{\varepsilon}{2-\varepsilon}\right)^{1/4} & 0\end{pmatrix}, \tag{3.2}$$

and I is the inversion operator,

$$IF(\eta)=F\left(\frac{1}{\eta}\right). \tag{3.3}$$

We can easily see that a solution which is decaying both when $\eta\to 0^+$ and when $\eta\to\infty$ will be transformed into a solution with the same property. Thus, the bound-state wave functions are eigenstates of SI also. Since obviously $S^2=I^2=1$, the corresponding eigenvalues for SI must be ± 1. We shall denote them δ_I.

The transformation SI is slightly peculiar in that it depends explicitly upon the binding energy ε. In this respect it resembles the "accidental" O(4) symmetry of the hydrogen atom.

It is convenient to define a new basis H and K,

$$\begin{pmatrix}H\\ K\end{pmatrix}=\begin{pmatrix}1 & \left(\dfrac{2-\varepsilon}{\varepsilon}\right)^{1/4}\\ -1 & \left(\dfrac{2-\varepsilon}{\varepsilon}\right)^{1/4}\end{pmatrix}\begin{pmatrix}F\\ G\end{pmatrix}, \tag{3.4}$$

which diagonalizes the matrix S,

$$\begin{pmatrix}\tilde{H}(\eta)\\ \tilde{K}(\eta)\end{pmatrix} \equiv SI\begin{pmatrix}H(\eta)\\ K(\eta)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} I\begin{pmatrix}H(\eta)\\ K(\eta)\end{pmatrix}. \tag{3.5}$$

This implies that the bound-state wave functions, being eigenfunctions of SI also, must satisfy the boundary conditions,

$$\begin{aligned} H(1) &= 0\,, \quad \text{for} \quad \delta_I = -1\,,\\ K(1) &= 0, \quad \text{for} \quad \delta_I = +1\,. \end{aligned} \tag{3.6}$$

From the inversion symmetry, it is clear that we only need to solve the equations for $\eta > 1$, imposing the boundary conditions (3.6). This is useful because a weakly bound fermion will behave non-relativistically in this region, and can be treated by a non-relativistic approximation.

4. Approximate solutions for small ε

In the limit of weak binding, $\varepsilon \ll 1$, it is possible to solve the eigenvalue problem approximately. For cases of practical interest, the solutions turn out to be highly accurate.

We first eliminate G from eq. (2.1) and write the equation for F in the suggestive form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - a^2 + \frac{2A}{\eta^2}\right] F = \frac{2}{\sqrt{(2-\varepsilon)/\varepsilon}\,\eta^2 - 1}\,\frac{1}{\eta}\,\frac{\mathrm{d}F}{\mathrm{d}\eta} + \frac{a^2}{\eta^4}F. \tag{4.1}$$

An a priori estimate indicates that the right-hand side is of relative order $\sqrt{\varepsilon}$ or smaller when $\eta \geqslant 1$. Thus it can be set to zero in this region when $\varepsilon \to 0$. Note that this approximation is not valid for small values of η. This is why the symmetry (3.1) becomes helpful. It makes it possible to find the wave function for $\eta < 1$, by the inversion principle (3.1), when it is known for $\eta > 1$.

With the right-hand side of (4.1) set to zero, the solution with the proper exponentially damped behaviour as $\eta \to \infty$ is given by a Macdonald–Bessel function of imaginary order, $K_{i\beta}(a\eta)$,

$$\begin{aligned} F(\eta) &= \sqrt{a\eta}\,K_{i\beta}(a\eta)\,, \qquad \eta \geqslant 1\,,\\ &= -\sqrt{a\eta}\,\frac{\pi}{\sinh(\pi\beta)}\,\mathrm{Im}\,I_{i\beta}(a\eta)\,, \end{aligned} \tag{4.2}$$

where

$$\beta = \sqrt{2A - \tfrac{1}{4}}\,. \tag{4.3}$$

It is perhaps not obvious at this point that β has to be real, i.e. that we must require $A > \frac{1}{8}$. However, further analysis shows this to be the case, the reason being that we must require $F(\eta)$ to have some oscillatory behaviour when η is of order 1. This does not occur when $i\beta$ is real.

It is now trivial to find G from eq. (2.1). To the same accuracy as above we find

$$G(\eta) = \left(\frac{\varepsilon}{2-\varepsilon}\right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}(a\eta)} [\sqrt{a\eta}\, K_{i\beta}(a\eta)] . \tag{4.4}$$

And, still to the same accuracy,

$$\begin{Bmatrix} H(\eta) \\ K(\eta) \end{Bmatrix} = \left[\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \left(\frac{\varepsilon}{2-\varepsilon}\right)^{1/4} \frac{\mathrm{d}}{\mathrm{d}(a\eta)} \right] \sqrt{a\eta}\, K_{i\beta}(a\eta) . \tag{4.5}$$

Now we find [10] that for small values of $a\eta$,

$$K_{i\beta}^{(0)}(a\eta) \simeq C \sin[\beta \ln(\tfrac{1}{2}a\eta) - \phi] , \tag{4.6}$$

where

$$\phi(\beta) = \arg \Gamma(1 + i\beta) , \tag{4.7}$$

and $C = C(\beta)$ is a normalization constant, independent of η. Inserting (4.6) into (4.5) we find the zeroth-order result

$$\begin{Bmatrix} H^{(0)}(\eta) \\ K^{(0)}(\eta) \end{Bmatrix} = C\sqrt{a\eta} \left\{ \left[\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \cos(\psi) \frac{1}{\eta} \right] \sin[\beta \ln(\tfrac{1}{2}a\eta) - \phi] \right.$$
$$\left. + \sin(\psi) \frac{1}{\eta} \cos[\beta \ln(\tfrac{1}{2}a\eta) - \phi] \right\} , \tag{4.8}$$

where

$$\psi = \arccos \sqrt{\frac{1}{8A}} . \tag{4.9}$$

In deriving (4.8) from (4.5) and (4.6), we have made the approximation $\sqrt{1 - \frac{1}{2}\varepsilon} = 1$, which to this order is consistent.

Applying the quantization conditions (3.6) we find the bound-state spectrum to be, to lowest order in ε,

$$\varepsilon_n^{(0)} = \frac{8}{A^2} \exp\left\{ -\frac{4}{\sqrt{8A-1}} [n\pi + \psi - 2\phi(\beta)] \right\} , \tag{4.10}$$

$$n = 1, 2, \ldots .$$

with the inversion "parity" of the eigenstates being

$$\delta_I(n) = (-1)^{n+1} . \tag{4.11}$$

In the limit $A \to \infty$ we find that

$$\psi(\beta) - 2\phi(\beta) \sim 2\beta - 2\beta \ln \beta ,$$
$$\beta \sim \sqrt{2A} . \tag{4.12}$$

Inserted into (4.10) (for consistency we must then let $n \to \infty$ with $n/\sqrt{A} \gg 1$) we find that

$$\varepsilon_n \simeq 32\, \mathrm{e}^{-4} \exp\left(-\frac{2\pi n}{\sqrt{2A}}\right), \tag{4.13}$$

in this limit.

The obtained spectrum (4.10) is analogous to the non-relativistic spectrum for an attractive $1/r^2$ potential, which is what eq. (4.1) corresponds to in the present approximation. As shown by Case [11], for an attractive $1/r^2$ potential, the eigenvalues and the wave functions are only determined up to one parameter, which may be taken as a phase. In the notation of Case, the eigenvalues and the eigenfunctions are given by

$$\eta_n = \exp[(B - (n + \tfrac{1}{2})\pi)/\lambda'], \tag{4.14}$$

$$u \sim x^{1/2} \cos(\lambda' \ln x + B), \qquad \text{as } x \to 0, \quad x = r/r_0, \tag{4.15}$$

where λ' is related to the strength of the potential, and B is the undetermined phase.

For the present case, the inversion symmetry has made it possible to determine that parameter. Thus, we have obtained absolute values for the binding energies, not just their ratios.

In the limit of weak binding, the ratios between binding energies were given also by Kazama and Yang [2]. Eq. (4.10) is seen to be consistent with the asymptotic expressions (a) and (d) given in sect. 7 of their paper.

The absolute values of the binding energies, (4.10), are very close to those determined numerically from the exact equations, as will be shown in sect. 6. An additional benefit of our method is, of course, that we also get approximate results for the eigenfunctions.

One may wonder why we have used eq. (4.1) as a starting point for our approximation, instead of eq. (2.5), which looks neater. It turns out that (4.1) leads to a better approximation for the wave functions, with the corrections being of order $\sqrt{\varepsilon}$, while (2.5) leads to correction terms of order $\varepsilon^{1/4}$.

5. Normalized eigensections

Within the approximation introduced in sect. 4, the radial wave functions for the weakly bound states are given by eqs. (4.2) and (4.4) in the "external" region $\eta > 1$. Thus, for

$$\rho > \rho_0 \equiv \frac{1}{a} = [A^2 \varepsilon (2 - \varepsilon)]^{-1/4}, \tag{5.1}$$

$$F_{\rm ext}(\rho)=N\rho_0^{-1}\rho^{1/2}K_{i\beta}(\rho/\rho_0^2)\,,$$
$$G_{\rm ext}(\rho)=\tfrac{1}{2}N(\tfrac{1}{2}\varepsilon)^{1/2}\rho_0\rho^{-1/2}[K_{i\beta}(\rho/\rho_0^2)+2\frac{\rho}{\rho_0^2}K'_{i\beta}(\rho/\rho_0^2)]\,, \tag{5.2}$$

where the binding energies $\varepsilon_n=(M-E_n)/M$ are given by eq. (4.10),

$$\rho=r(2M/|\kappa q|)\,,$$
$$\beta=(2A-\tfrac{1}{4})^{1/2}\,,\qquad A=\tfrac{1}{2}\kappa|q|\,,$$

and the normalization constant N is given by eq. (5.9) below.

As discussed in sects. 3 and 4, the solution in the internal region $\rho<\rho_0$ can be obtained by the use of the inversion symmetry (cf. eqs. (3.1) and (4.11)),

$$F(\rho)=(-1)^{n+1}\left(\frac{2-\varepsilon}{\varepsilon}\right)^{1/4}G(\rho_0^2/\rho)\,,$$
$$G(\rho)=(-1)^{n+1}\left(\frac{\varepsilon}{2-\varepsilon}\right)^{1/4}F(\rho_0^2/\rho)\,. \tag{5.3}$$

Since $S^2=1$, the relative normalization follows from the inversion symmetry, eq. (3.1), up to a sign factor. That sign factor is determined by the inversion "parity" of each particular state.

From eqs. (5.2) and (5.3) we thus find

$$F_{\rm int}(\rho)=(-1)^{n+1}\tfrac{1}{2}N(\tfrac{1}{2}\varepsilon)^{1/4}\rho^{1/2}[K_{i\beta}(1/\rho)+\frac{2}{\rho}K'_{i\beta}(1/\rho)]\,,$$
$$G_{\rm int}(\rho)=(-1)^{n+1}N(\tfrac{1}{2}\varepsilon)^{1/4}\rho^{-1/2}K_{i\beta}(1/\rho)\,;\qquad n=1,2,\ldots\,. \tag{5.4}$$

As noted in sect. 4, and shown in detail in sect. 7, the correction terms to these wave functions are of relative order $\sqrt{\varepsilon}$. The matching of the solutions at $\rho=\rho_0$ is therefore slightly violated – to the same order $\sqrt{\varepsilon}$. This matching can be checked explicitly. Using the small-argument expression (4.6) for $K_{i\beta}$, which neglects terms of order $a^2\sim\sqrt{\varepsilon}$, we find

$$\frac{F_{\rm ext}(\rho_0)}{F_{\rm int}(\rho_0)}=\frac{(-1)^{n+1}2\sqrt{2A}}{1+2\beta\cot[\beta\ln(1/2\rho_0)-\phi(\beta)]}\,, \tag{5.5}$$

which is unity, when ρ_0 is expressed in terms of the eigenvalue ε_n of eq. (4.10).

The normalization condition, which for the original functions reads

$$\int_0^\infty r^2\,{\rm d}r\,\{|f(r)|^2+|g(r)|^2\}=1\,, \tag{5.6}$$

will in terms of $F(\rho)$ and $G(\rho)$ become

$$\frac{A}{M}\int_0^\infty {\rm d}\rho\{F^2(\rho)+G^2(\rho)\}=1\,. \tag{5.7}$$

This integral may be divided into internal ($0<\rho<\rho_0$) and external ($\rho_0<\rho<\infty$) parts. Using the inversion symmetry (5.3) and changing the variable for the internal part, we may then express the normalization integral as

$$\int_{\rho_0}^{\infty}[F^2(\rho)+G^2(\rho)]\,\mathrm{d}\rho+\int_{\rho_0}^{\infty}\left[\left(\frac{\varepsilon}{2-\varepsilon}\right)^{1/2}F^2(\rho)+\left(\frac{2-\varepsilon}{\varepsilon}\right)^{1/2}G^2(\rho)\right]\frac{\rho_0^2}{\rho^2}\,\mathrm{d}\rho=\frac{M}{A}, \tag{5.8}$$

where the second integral is the contribution from the internal region. We note that $G^2(\rho)=\mathrm{O}(\varepsilon^{1/2})F^2(\rho)$ for $\rho>\rho_0$, and that the second integral is of relative order $\sqrt{\varepsilon}$ compared to the first one.

Since the corrections to the wave functions (5.2) are also of order $\sqrt{\varepsilon}$, we are then, to lowest order, left with

$$\frac{M}{A}=\int_{\rho_0}^{\infty}F_{\mathrm{ext}}^2(\rho)\,\mathrm{d}\rho=N^2\rho_0^2\int_a^{\infty}xK_{i\beta}^2(x)\,\mathrm{d}x=\tfrac{1}{2}N^2\rho_0^2\frac{\pi\beta}{\sinh(\pi\beta)}.$$

In the last step, the lower limit $a=1/\rho_0\simeq(2A^2\varepsilon)^{1/4}$ has been replaced by zero, to the same accuracy as above. We thus find the normalization constant to be

$$N=(-1)^n\left[2M\sqrt{2\varepsilon}\frac{\sinh(\pi\beta)}{\pi\beta}\right]^{1/2}, \tag{5.9}$$

where the sign factor has been introduced in order to adopt the convention of ref. [2].

To summarize, we have for the weakly bound states (1.9) of lowest angular momentum, $j=|q|-\frac{1}{2}$, the following eigensections [2],

$$\psi(\boldsymbol{r})=\begin{bmatrix}f(r)\eta_{jj_z}(\hat{\boldsymbol{r}})\\ g(r)\eta_{jj_z}(\hat{\boldsymbol{r}})\end{bmatrix} \tag{5.10}$$

where the radial functions are given in terms of our solutions (5.2) and (5.4) by eqs. (1.14),

$$f(r)=\frac{1}{r}\frac{\kappa q}{|\kappa q|}F(r),\qquad g(r)=\frac{-i}{r}G(r), \tag{5.11}$$

and the angular eigensections η_{jj_z} are given in refs. [1, 7]. For $q=Zeg=\frac{1}{2}$, one has simply

$$\eta_{00}=\sqrt{\frac{1}{2}}\begin{pmatrix}-Y_{1/2,1/2,-1/2}\\ Y_{1/2,1/2,1/2}\end{pmatrix}, \tag{5.12}$$

where Y_{qlm} are the monopole harmonics introduced in ref. [7].

These are "sections", defined by different functions in two overlapping regions R_{a} and R_{b}, R_{a} containing the positive z-axis, and R_{b} containing the negative z-axis. In the overlap region $Y_{qlm}^{(\mathrm{a})}$ and $Y_{qlm}^{(\mathrm{b})}$ are connected by the gauge transformation

$$Y_{qlm}^{(\mathrm{b})}(\theta,\phi)=\mathrm{e}^{-2iq\phi}Y_{qlm}^{(\mathrm{a})}(\theta,\phi). \tag{5.13}$$

Explicitly, in region $R^{(a)}$:

$$Y^{(a)}_{1/2,1/2,-1/2}(\theta,\phi)=\left(\frac{1+\cos\theta}{4\pi}\right)^{1/2}$$

$$Y^{(a)}_{1/2,1/2,1/2}(\theta,\phi)=-e^{i\phi}\left(\frac{1-\cos\theta}{4\pi}\right)^{1/2}. \tag{5.14}$$

However, we can do without the explicit forms in many practical calculations, and only rely on the general group-theoretic properties of these objects.

Plots of some of our approximate radial density distributions and comparisons with the corresponding numerically determined ones are given in sect. 6.

6. Numerical results

In order to check the accuracy of formula (4.10) for the binding energies, one needs the exact values. These we determine following Kazama and Yang [2]. The values thus obtained are however a little different from theirs, so we present a brief discussion of how they are determined.

Kazama and Yang define an angle $\phi(\rho)$ through the equations

$$F(\rho)=R(\rho)\cos[-\tfrac{1}{4}\pi+\tfrac{1}{2}\phi(\rho)], \tag{6.1}$$

$$G(\rho)=R(\rho)\sin[-\tfrac{1}{4}\pi+\tfrac{1}{2}\phi(\rho)]. \tag{6.2}$$

The differential equations obeyed by $F(\rho)$ and $G(\rho)$ may then be transformed into the following equations for $\phi(\rho)$ and $R(\rho)$,

$$\frac{d\phi(\rho)}{d\rho}=-2B+2\left(A-\frac{1}{\rho^2}\right)\sin\phi(\rho), \tag{6.3}$$

$$R(\rho)=\exp\left[-\int^{\rho}\left(A-\frac{1}{x^2}\right)\cos\phi(x)\,dx\right]. \tag{6.4}$$

An analysis of the right-hand side of eq. (6.3) shows that $d\phi(\rho)/d\rho$ is negative in the ρ–ϕ plane ($\rho\geq 0$, $\phi\leq 0$) except for an infinite number of disconnected regions [2]. Half of these are band-like, starting at $\rho=(A-B)^{-1/2}$ and extending out to infinity. They represent traps for the solutions $\phi(\rho)$, since immediately above each band (i.e. for larger values of ϕ), $d\phi(\rho)/d\rho<0$.

Once a solution $\phi(\rho)$ has entered one of these bands, it can never escape, and the corresponding values of $\phi(\rho)$,

$$\phi(\rho)\xrightarrow[\rho\to\infty]{}(\pi-\gamma)-2n\pi, \tag{6.5}$$

$$\gamma=\arcsin(B/A), \tag{6.6}$$

are such that $dR(\rho)/d\rho$ becomes positive. As $\rho\to\infty$, the traps thus lead to exponentially large values for $R(\rho)$. The eigenvalues, B_n, on the other hand, correspond to

solutions $\phi(\rho)$ that for large ρ lie immediately below the traps (for details see ref. [2]),

$$\phi(\rho) \xrightarrow[\rho\to\infty]{} \phi_n \equiv \gamma - 2n\pi . \tag{6.7}$$

To determine a certain energy level B_n, or, equivalently

$$\varepsilon_n = 1 - B_n/A = 1 - E_n/M , \tag{6.8}$$

we thus pick a trial value B and integrate eq. (6.3) numerically from some small value $\rho = \rho_c$ out to some large value ρ where either

(i) $$\frac{d\phi(\rho)}{d\rho} > 0 \text{ (a trap)}, \qquad \text{then } B < B_n .$$

or

(ii) $$\phi(\rho) < \phi_n , \qquad \text{then } B > B_n .$$

Thus B_n can be bounded from below and from above to an arbitrary precision.

Initial values may be obtained from the following asymptotic expansion for $\phi(\rho)$, valid for small ρ:

$$\phi(\rho) = -B\rho^2 + B\rho^3 - B(A+\tfrac{3}{2})\rho^4 + 3B(A+1)\rho^5 + \cdots \tag{6.9}$$

This procedure requires some care. The asymptotic expansion (6.9) is only valid for very small values $\rho \leqslant \rho_c \ll 1$. Further, because of the singular behaviour of $d\phi(\rho)/d\rho$ for small ρ, the step length $\Delta\rho$ used in the numerical integration has to be chosen such that $\Delta\rho \ll \rho$. Our results appear stable for $\rho_c \leqslant 0.3$ and (initially) $\Delta\rho \leqslant 0.02\rho_c$. (A variable step length is employed.) For $A = 0.4475$ (this value corresponds to the proton anomalous magnetic moment, $\kappa = 1.79$, and $|q| = \frac{1}{2}$) a plot of $\phi(\rho)$ versus $\log\rho$ is given in fig. 1. The facts that (i) there are regions where $\phi(\rho)$ falls rapidly, and (ii) it has to be integrated out to quite large values of ρ for the weakly bound states, are indicative of the numerical problems involved.

Exact binding energies are given for a few values of A in table 1, for $n = 1$, 2, and 3. The values obtained by Kazama and Yang [2] are also quoted. They differ somewhat from ours for the most weakly bound states. Presumably, this deviation is related to numerical problems of the kind mentioned previously.

Table 1 also contains the approximate values for ε_n, determined from eq. (4.10). These are seen to agree very well with those determined numerically. As expected, the agreement improves as the binding gets weaker.

In figs. 2–4 we present plots of $|F(\rho)|^2$ and $|G(\rho)|^2$ for the lowest monopole-proton states. The numerically determined eigenfunctions (from eqs. (6.1)–(6.4)) are on these plots not distinguishable from the approximate ones, given by eqs. (5.2) and (5.4). With both scales logarithmic, the inversion symmetry is evident: apart from a shift along the vertical axis, $|G(\rho)|^2$ is seen to be the mirror image of $|F(\rho)|^2$. The shift is given by $|G_{\text{int}}(\rho_0)|^2/|F_{\text{ext}}(\rho_0)|^2 = \sqrt{\frac{1}{2}\varepsilon}$.

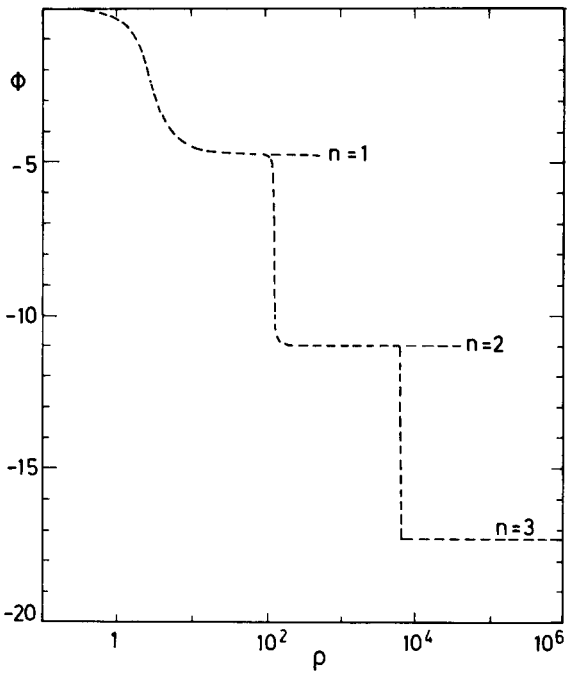

Fig. 1. The angle ϕ as defined by eqs. (6.1), (6.2) versus ρ. Three different levels are considered for the monopole-proton state, $n = 1$, 2, and 3. Note that the ρ-scale is logarithmic. At small ρ the three curves practically overlap since B is close to A (compare eq. (6.9)). Each "step" corresponds to a zero in $F(\rho)$.

As mentioned in sect. 5, there is a small discontinuity at the inversion point, expected to be of relative order $\sqrt{\varepsilon}$. This turns out to be an overestimate. For example, for the states with $n = 1$, from table 1 we determine the values of $\sqrt{\varepsilon}$ to be 7.4% and 15.4%, for $A = 1.0$ and 2.0, respectively. The actual discontinuities corresponding to these two states are only 1.7% and 6.2%, respectively, much smaller than the naive expectation.

Asymptotically, the wave functions behave like $\exp[-\sqrt{2\varepsilon}A\rho]$, so the radii roughly increase with n like

$$\langle \rho \rangle_n \sim \frac{1}{\sqrt{\varepsilon_n}} \sim \exp[2n\pi/\sqrt{8A-1}]\,. \tag{6.10}$$

For example, for $A = 1$, successive energy levels differ by roughly a factor 10^{-2} (see table 1), and the radii of successive states will increase by roughly a factor 10.

The quality of the approximation to the wave functions will in general depend on what the wave functions are to be used for. In order to study a simple example, and also for the purpose of checking the qualitative statement (6.10), we have evaluated the r.m.s. radii of the states considered in table 1. The results are given

TABLE 1

Binding energies ε_n in units of the fermion mass, M

A \ n	1	2	3
	ε (approx.) ε (num.) ε (ref. [2])	ε (approx.) ε (num.) ε (ref. [2])	ε (approx.) ε (num.) ε (ref. [2])
0.4475	$2.810 \cdot 10^{-4}$	$1.125 \cdot 10^{-7}$	$4.501 \cdot 10^{-11}$
(p)	$2.808 \cdot 10^{-4}$	$1.125 \cdot 10^{-7}$	$4.501 \cdot 10^{-11}$
	—	—	—
	$5.519 \cdot 10^{-3}$	$4.776 \cdot 10^{-5}$	$4.134 \cdot 10^{-7}$
1.0	$5.488 \cdot 10^{-3}$	$4.776 \cdot 10^{-5}$	$4.134 \cdot 10^{-7}$
	$5.75 \cdot 10^{-3}$	$5.12 \cdot 10^{-5}$	$4.47 \cdot 10^{-7}$
	$1.403 \cdot 10^{-2}$	$3.173 \cdot 10^{-4}$	$7.176 \cdot 10^{-6}$
1.5	$1.391 \cdot 10^{-2}$	$3.171 \cdot 10^{-4}$	$7.176 \cdot 10^{-6}$
	$1.41 \cdot 10^{-2}$	$3.28 \cdot 10^{-4}$	$7.53 \cdot 10^{-6}$
	$2.383 \cdot 10^{-2}$	$9.290 \cdot 10^{-4}$	$3.622 \cdot 10^{-5}$
2.0	$2.361 \cdot 10^{-2}$	$9.281 \cdot 10^{-4}$	$3.621 \cdot 10^{-5}$
	$2.35 \cdot 10^{-2}$	$9.45 \cdot 10^{-4}$	$3.74 \cdot 10^{-5}$
	$3.393 \cdot 10^{-2}$	$1.899 \cdot 10^{-3}$	$1.063 \cdot 10^{-4}$
2.5	$3.361 \cdot 10^{-2}$	$1.896 \cdot 10^{-3}$	$1.063 \cdot 10^{-4}$
	$3.37 \cdot 10^{-2}$	$1.92 \cdot 10^{-3}$	$1.08 \cdot 10^{-4}$

The parameter A is essentially the anomalous magnetic moment κ, $A = \frac{1}{2}|q|\kappa$, with $q = eg$ the Dirac quantum number. The value $A = 0.4475$ corresponds to a proton-monopole system, for $q = \frac{1}{2}$. For each A, the upper entries are obtained from the explicit formula, eq. (4.10), the middle ones are the numerically determined exact values, whereas the lower ones are from ref. [2].

in table 2 for the exact wave functions as well as for the approximate ones. The agreement is again very good. Also, the qualitative statement (6.10) is confirmed, the weakly bound states have very large radii.

7. Systematic corrections

Using the first-order binding energy (4.10) as a formal expansion parameter, we may proceed to compute systematic corrections to the previous results.

With the dependence upon the eigenvalue parameter ε written explicitly, the exact quantization conditions (3.6) are

$$H(1; \varepsilon_n) = 0\,, \qquad \text{respectively } K(1; \varepsilon_n) = 0\,. \tag{7.1}$$

We may construct a systematic expansion for H and K,

$$H(1; \varepsilon) = H^{(0)}(1; \varepsilon) + H^{(1)}(1; \varepsilon) + \cdots, \tag{7.2}$$

and for the solutions, ε_n, of (7.1),

$$\varepsilon_n = \varepsilon_n^{(0)} + \varepsilon_n^{(1)} + \cdots \tag{7.3}$$

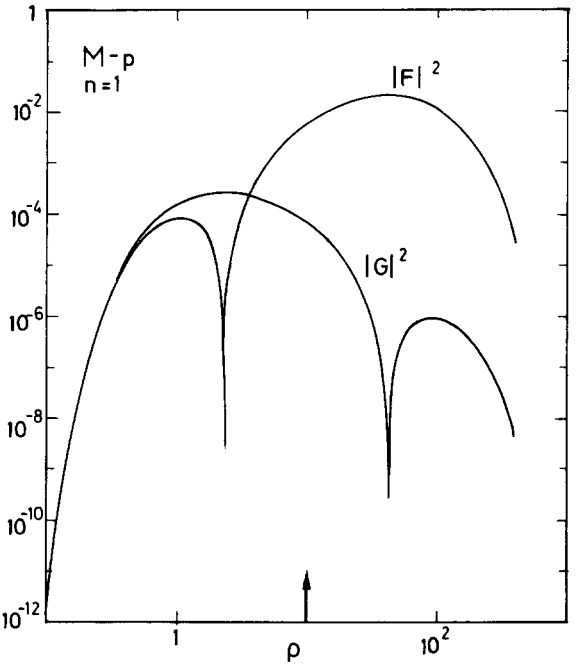

Fig. 2. Squared moduli of the radial wave functions, $|F(\rho)|^2$ and $|G(\rho)|^2$ versus ρ, for the monopole-proton state with $n=1$. Both scales are logarithmic. The arrow indicates the "inversion point", $\rho=\rho_0=9.71$.

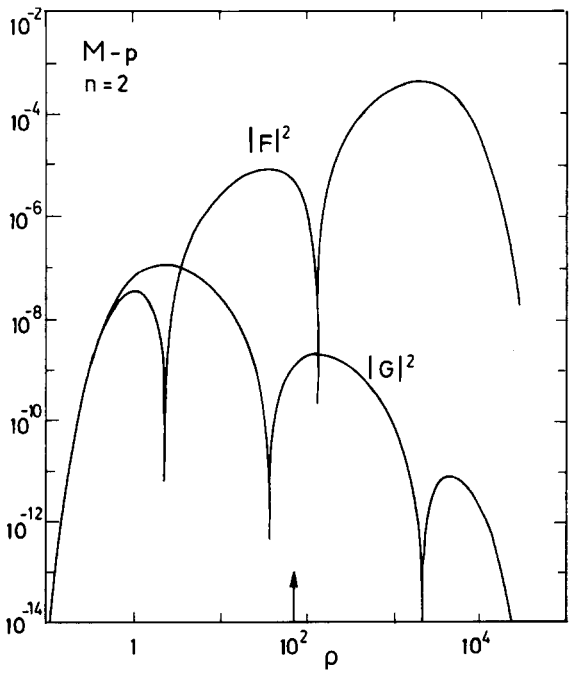

Fig. 3. As fig. 2 for $n=2$. Here $\rho_0=68.6$.

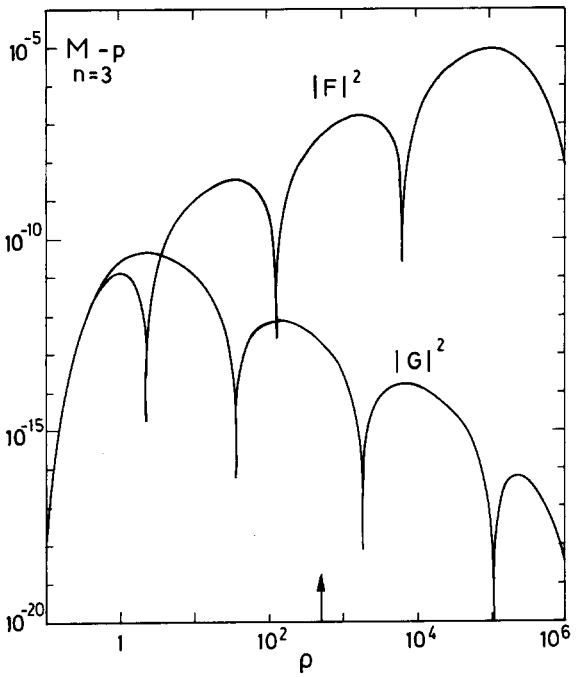

Fig. 4. As fig. 2 for $n = 3$. Here $\rho_0 = 485$.

TABLE 2

Root-mean-square radii in units of the Compton wave length of the fermion, $1/M$

A \ n	1	2	3
	$r^{\text{approx}}_{\text{r.m.s.}}$ $r^{\text{num}}_{\text{r.m.s.}}$	$r^{\text{approx}}_{\text{r.m.s.}}$ $r^{\text{num}}_{\text{r.m.s.}}$	$r^{\text{approx}}_{\text{r.m.s.}}$ $r^{\text{num}}_{\text{r.m.s.}}$
0.4475	$4.42 \cdot 10^1$	$2.20 \cdot 10^3$	$1.10 \cdot 10^5$
(p)	$4.42 \cdot 10^1$	$2.20 \cdot 10^3$	$1.10 \cdot 10^5$
1.0	$1.29 \cdot 10^1$	$1.38 \cdot 10^2$	$1.49 \cdot 10^3$
	$1.29 \cdot 10^1$	$1.39 \cdot 10^2$	$1.49 \cdot 10^3$
1.5	9.50	$6.27 \cdot 10^1$	$4.17 \cdot 10^2$
	9.42	$6.27 \cdot 10^1$	$4.17 \cdot 10^2$
2.0	8.25	$4.13 \cdot 10^1$	$2.09 \cdot 10^2$
	8.12	$4.13 \cdot 10^1$	$2.09 \cdot 10^2$
2.5	7.65	$3.18 \cdot 10^1$	$1.34 \cdot 10^2$
	7.50	$3.18 \cdot 10^1$	$1.34 \cdot 10^2$

The states are the same as those considered in table 1. The upper entries are obtained from the approximate wave functions, the lower ones are based on the numerically determined wave functions.

Consider for a moment the first of eqs. (7.1). Working to first order in the small quantity $\varepsilon_n^{(1)}$, and using the fact that $H^{(0)}(1;\varepsilon_n^{(0)})=0$, we find

$$\varepsilon_n^{(1)} = -H^{(1)}(1;\varepsilon_n^{(0)}) \Big/ \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon} H^{(0)}(1;\varepsilon)\right]_{\varepsilon=\varepsilon_n^{(0)}}. \tag{7.4}$$

$H^{(1)}$ receives contributions from many sources. One of them is the first (small argument) correction to the approximation (4.6) for $K_{i\beta}(a\eta)$. This correction is of order $(a\eta)^2$ relative to the leading term (which is itself of order $\sqrt{a}$). There are other corrections to $H^{(1)}$ of the same order, but none larger. Thus, remembering that $a=\mathrm{O}(\varepsilon^{1/4})$, we expect

$$H^{(1)}(1;\varepsilon)=\mathrm{O}(\varepsilon^{5/8}). \tag{7.5}$$

Likewise, from (4.8) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} H^{(0)}(1;\varepsilon)=\mathrm{O}(\varepsilon^{-7/8}). \tag{7.6}$$

Inserting these estimates into (7.4) we find $\varepsilon_n^{(1)}=\mathrm{O}(\varepsilon_n^{3/2})$, i.e.

$$\varepsilon_n=\varepsilon_n^{(0)}[1+c_1\sqrt{\varepsilon_n^{(0)}}+\cdots]. \tag{7.7}$$

Thus, from these order-of-magnitude estimates it appears that the relative corrections to the results (4.10) should be rather large. This is a bit surprising in view of the excellent agreement between (4.10) and the numerical results. In order to shed light on this question we shall proceed to compute the coefficient c_1 in eq. (7.7). Indeed, it will turn out to be zero.

First, we find the $\mathrm{O}((a\eta)^2)$ correction to the approximation (4.6) for $K_{i\beta}$,

$$K_{i\beta}^{(1)}(x)=\frac{Cx^2}{4+4\beta^2}[\sin(\beta\ln\tfrac{1}{2}x-\phi)-\beta\cos(\beta\ln\tfrac{1}{2}x-\phi)], \tag{7.8}$$

where $x=a\eta$. Inserting this into (4.5), we find

$$\begin{aligned} H^{(1a)}(1;\varepsilon_n^{(0)}) &= (1+3\cos\psi+6\cos^2\psi)Z_H, \\ K^{(1a)}(1;\varepsilon_n^{(0)}) &= (1-3\cos\psi+6\cos^2\psi)Z_K, \end{aligned} \tag{7.9}$$

where

$$\{Z_H, Z_K\}=\pm C\{\sin\tfrac{1}{2}\psi,\cos\tfrac{1}{2}\psi\}\frac{\cos\psi}{1+3\cos^2\psi}(2A^2\varepsilon_n^{(0)})^{5/8}. \tag{7.10}$$

The sign factor is for Z_K given by $(-1)^{(n+1)/2}$ (n odd), and for Z_H by $(-1)^{(n+2)/2}$ (n even).

We must also improve on eq. (4.4), relating G to F for fixed F. To the next order, we find from eqs. (2.1) and (4.4) that

$$G^{(1b)}(\eta)=\tfrac{1}{2}\varepsilon\frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}(a\eta)}[\sqrt{a\eta}K_{i\beta}^{(0)}(a\eta)]. \tag{7.11}$$

This leads to corrections

$$H^{(1b)}(1;\varepsilon_n^{(0)}) = -(4\cos\psi + 12\cos^3\psi)Z_H\,,$$
$$K^{(1b)}(1;\varepsilon_n^{(0)}) = (4\cos\psi + 12\cos^3\psi)Z_K\,. \tag{7.12}$$

Finally, we must solve equation (4.1) to order $\sqrt{\varepsilon}$. This leads to a correction $F^{(1c)}(\eta) = \varepsilon f(a\eta)$, where $f(x)$ is a solution (exponentially small as $x\to\infty$) of the inhomogeneous equation,

$$\left[\frac{d^2}{dx^2} - 1 + \frac{2A}{x^2}\right] f(x) = \left[\frac{2A}{x^3}\frac{d}{dx} + \frac{2A^2}{x^4}\right]\sqrt{x}K_{i\beta}(x)\,. \tag{7.13}$$

By systematic search we find

$$\left[\frac{d^2}{dx^2} - 1 + \frac{2A}{x^2}\right]\begin{Bmatrix}\dfrac{1}{x^2}\\[2ex] \dfrac{d}{dx}\dfrac{1}{x}\end{Bmatrix}\sqrt{x}K_{i\beta}(x)$$
$$= \begin{pmatrix} 6 & -4 \\ 8A-6 & 6\end{pmatrix}\begin{Bmatrix}\dfrac{1}{x^4}\\[2ex] \dfrac{1}{x^3}\dfrac{d}{dx}\end{Bmatrix}\sqrt{x}K_{i\beta}(x) + \begin{Bmatrix}0\\ -2\end{Bmatrix}\frac{1}{x^2}\sqrt{x}K_{i\beta}(x)\,, \tag{7.14}$$

and

$$\left[\frac{d^2}{dx^2} - 1 + \frac{2A}{x^2}\right]\frac{1}{\beta}\frac{\partial}{\partial\beta}\sqrt{x}K_{i\beta}(x) = -\frac{2}{x^2}\sqrt{x}K_{i\beta}(x)\,. \tag{7.15}$$

This is sufficient to construct solutions of eq. (7.13). The most general solution, obeying the boundary condition at $x=\infty$, is

$$f(x) = f^{(c)}(x) + f^{(d)}(x) + f^{(e)}(x)\,, \tag{7.16}$$

where

$$f^{(c)}(x) = \frac{A}{3+8A}\left[(3+2A)\frac{1}{x}\frac{\partial}{\partial x} - 3A\frac{1}{x^2}\right]\sqrt{x}K_{i\beta}\,,$$
$$f^{(d)}(x) = -\frac{A}{3+8A}(3+2A)\frac{1}{\beta}\frac{\partial}{\partial\beta}\sqrt{x}K_{i\beta}\,, \tag{7.17}$$
$$f^{(e)}(x) = C^{(e)}\sqrt{x}K_{i\beta}\,,$$

with $C^{(e)}$ an arbitrary constant. This last term, $f^{(e)}$, reflects the fact that we may freely add solutions of the homogeneous equation. A unique value for $C^{(e)}$ will emerge from the normalization requirement. However, it will not influence the determination of the binding energies to the order we are considering, and shall be ignored in the following.

Only the term $f^{(c)}$ contributes to the coefficient c_1 in eq. (7.7). The term $f^{(d)}$ leads to a correction $\Delta\varepsilon_n/\varepsilon_n^{(0)}$ of order $\varepsilon_n^{(0)} \ln \varepsilon_n^{(0)}$. However, since c_1 turns out to be zero, this correction actually is the dominating one. We shall return to it shortly, but first proceed to calculate the $O(\varepsilon^{5/8})$ contributions to H and K from $f^{(c)}$. It involves straightforward, but lengthy algebra to obtain these terms by inserting eq. (7.17) into eq. (4.5). We find

$$\begin{aligned} H^{(1c)}(1;\varepsilon_n^{(0)}) &= (-1+\cos\psi-6\cos^2\psi+12\cos^3\psi)Z_H\,, \\ K^{(1c)}(1;\varepsilon_n^{(0)}) &= -(1+\cos\psi+6\cos^2\psi+12\cos^3\psi)Z_K\,. \end{aligned} \tag{7.18}$$

Adding the contributions from (7.9), (7.12) and (7.18) we find the promised results,

$$\begin{aligned} H^{(1a)}+H^{(1b)}+H^{(1c)} &= 0\,, \\ K^{(1a)}+K^{(1b)}+K^{(1c)} &= 0\,. \end{aligned} \tag{7.19}$$

Even in retrospect, we have not found any convincing arguments for why this cancellation should take place.

Thus, $H^{(1)}$ and $K^{(1)}$ actually are of order $\varepsilon^{9/8}\ln\varepsilon$, with the sole contributions coming from the term $f^{(d)}$ in equation (7.16). We work these out to be

$$\begin{Bmatrix} H^{(1d)}(1;\varepsilon_n^{(0)}) \\ K^{(1d)}(1;\varepsilon_n^{(0)}) \end{Bmatrix} = \pm C \begin{Bmatrix} \cos\frac{1}{2}\psi \\ \sin\frac{1}{2}\psi \end{Bmatrix} \frac{1}{16\sin(2\psi)} \frac{1+12\cos^2\psi}{1+3\cos^2\psi} (2A^2\varepsilon)^{1/8}\varepsilon\ln\varepsilon\,, \tag{7.20}$$

where we have kept the logarithmic terms only. Similarly, we find from eq. (4.8)

$$\left\{ \begin{matrix} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}H^{(0)}(1;\varepsilon) \\ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}K^{(0)}(1;\varepsilon) \end{matrix} \right\}_{\varepsilon=\varepsilon_n^{(0)}} = \mp C \begin{Bmatrix} \cos\frac{1}{2}\psi \\ \sin\frac{1}{2}\psi \end{Bmatrix} \frac{1}{4}\,\mathrm{tg}\,\psi(2A^2\varepsilon)^{1/8}\varepsilon^{-1}\,. \tag{7.21}$$

Inserted into (6.4) this gives

$$\varepsilon_n^{(1)} = \frac{1}{8}\frac{(1+12\cos^2\psi)}{\sin^2(\psi)(1+3\cos^2\psi)}\varepsilon_n^{(0)2}\ln\varepsilon_n^{(0)}\,. \tag{7.22}$$

We note that this correction is negative, in agreement with table 1. However, for the most strongly bound state that we have considered ($A=2.5$, $n=1$), it overestimates the actual correction by approximately a factor of two. Thus, the correction to $\Delta\varepsilon_n/\varepsilon_n^{(0)}$ of $O(\varepsilon)$ (which we have not evaluated) is in this range of parameters comparable to the one given above.

8. Concluding remarks

For the weakly bound monopole-fermion states of lowest angular momentum, we have found excellent analytic approximations to the binding energies and to the

wave functions. With ε the binding energy in units of the fermion mass, the corrections to our results for the binding energies and the wave functions are of relative order $\varepsilon \log \varepsilon$ and $\sqrt{\varepsilon}$, respectively. These wave functions are useful for estimating proton-monopole capture cross sections [8].

Makino, Maruyama and Miyamura [12] have recently studied the proton-monopole bound states using a non-relativistic effective hamiltonian. They found the lowest level* to be at 190 keV, with a linear dimension, $r_{\text{r.m.s.}} = 11$ fm. In comparison, we find for the corresponding level a binding energy of $2.808 \cdot 10^{-4} M =$ 263 keV, and a radius, $r_{\text{r.m.s.}} = 44.2/M = 9.3$ fm.

Sivers [13] and, more recently, Bracci and Fiorentini [14] have used a somewhat different approach to the fermion-monopole problem. They consider a Schrödinger equation with a magnetic moment interaction. As a short-distance regulator, a repulsive core is introduced, with a radius comparable to that of the nucleon or nucleus. The imposed boundary condition leading to quantization is that the wave function vanishes at the surface of that core.

Bracci and Fiorentini [14] thus find that the ratio between successive energy levels for the proton-monopole system is $4 \cdot 10^{-4}$, in agreement with the present analysis (see table 1). However, the scale is very different. For the lowest level they find [14] a binding energy of 15.1 keV, and a radius, $\langle r \rangle = 32$ fm. Sivers [13], on the other hand, has a value of 320 keV, in rough agreement with ours.

If the zero-energy state is absent for the monopole-proton system, as suggested by the analysis of ref. [9], then the lowest state would be the one at a binding energy of 263 keV. We note that its radius is fairly large compared with the size of the proton. Thus, strong-interaction effects are expected to be small, unless the monopole colour-screening radius is comparable with this radius, i.e. of the order of 10 fm. Even in that case, strong-interaction effects might well be small. Since the proton is a colour singlet, interactions with the monopole colour field would be due to variations in this field over distances comparable with the proton radius.

Also, we expect effects due to the proton electromagnetic structure to be small. Such effects were incorporated in the work of ref. [12], and a binding energy quite different from ours was obtained. However, they used a non-relativistic treatment from the outset. From a direct analysis of their final hamiltonian this appears to be internally consistent. However, we suspect their treatment of the form factors to be inconsistent, thereby leading to an incorrect non-relativistic approximation. A consistent, presumably better treatment would be to set $F_1 = 1$ and for F_2 use their eqs. (5), (6). A simple argument why proton structure effects ought to be small is that the binding energy for the $n = 1$ state is some three orders of magnitude smaller than the natural excitation energy of the proton, characterized by Λ_{QCD} or the pion mass.

* The (relativistic) zero-energy level is absent in their treatment.

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References

[1] Y. Kazama, C.N. Yang and A.S. Goldhaber, Phys. Rev. D15 (1977) 2287
[2] Y. Kazama and C.N. Yang, Phys. Rev. D15 (1977) 2300
[3] M. Atiyah and I. Singer, Ann. of Math. 87 (1968) 484
[4] C.N. Yang, in Monopoles in quantum field theory, Proc. of the monopole meeting, Trieste, Italy, December 1981, eds. N.S. Craigie, P. Goddard and W. Nahm (World Scientific, Singapore, 1982) p. 237
[5] C.P. Dokos and T.N. Tomaras, Phys. Rev. D21 (1980) 2940
[6] S. Coleman, preprint Harvard HUTP-82/A032, to appear in the Proc. 1981 Int. School of Subnuclear Physics, Ettore Majorana (Erice)
[7] T.T. Wu and C.N. Yang, Nucl. Phys. B107 (1976) 365
[8] K. Olaussen, H.A. Olsen, P. Osland, and I. Øverbø, to be published
[9] T.F. Walsh, P. Weisz, and T.T. Wu, Nucl. Phys. B229 (1983), to be published
[10] Handbook of mathematical functions, eds. M. Abramowitz and I.A. Stegun (National Bureau of Standards, Washington, 1964)
[11] K.M. Case, Phys. Rev. 80 (1950) 797
[12] J. Makino, M. Maruyama and O. Miyamura, Prog. Theor. Phys. 69 (1983) 1042
[13] D. Sivers, Phys. Rev. D2 (1970) 2048
[14] L. Bracci and G. Fiorentini, Phys. Lett. 124B (1983) 493; preprint IFUP TH-83/2 (Pisa, 1983)