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# THE VORTEX FREE ENERGY IN THE SCREENING PHASE OF THE Z(2) HIGGS MODEL

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The vortex free energy was proposed to distinguish between the confinement and the Higgs phase (in the sense of 't Hooft) in lattice gauge theory, when matter fields are present that transform according to an arbitrary representation of the gauge group In this paper I consider the Z(2) Higgs model and calculate the vortex free energy in the screening part of the confining/screening phase of Fradkin and Shenker The result does not agree with the expected behavior that corresponds to the structure of the phase diagram Therefore the vortex free energy is no longer a good indicator for confinement when matter fields transform non-trivially under the center of the gauge group (such as Z(2) Higgs scalars)

## 1. Introduction

The vortex free energy was established as a suitable order parameter for pure gauge theories or gauge theories with matter fields that transform trivially under the center of the gauge group [1]. It was supposed to remain suitable in the case when matter fields are present that transform non-trivially under the center, e.g. according to the fundamental representation of SU($N$). In [2] it was shown that it distinguishes between ranges of coupling constants in a $Z_2$ Higgs model where strong-coupling ($g_0^{-2} \ll 1$, $\kappa \ll 1$) and low-temperature ($g_0^{-2} \gg 1$, $\kappa \ll 1$) expansions are applicable. $g_0^{-2}$ is the gauge, $\kappa$ the matter coupling. The strong-coupling region is assumed to belong to the confinement phase; the low-temperature region should show deconfinement—it corresponds to the Higgs phase in the sense of 't Hooft [3]. These results were in agreement with the expectations from the phase diagram.

In this paper I investigate the vortex free energy for the Z(2) Higgs model in the region of $\kappa \gg 1$ and arbitrary $g_0^{-2}$. One calls this region the screening phase, because the limit models for large $g_0^{-2}$ show the usual Higgs mechanism, i.e. the global symmetry is spontaneously broken above some $\kappa_c$ for infinite $g_0^{-2}$, and Debye screening of charges becomes possible. The screening phase should not be confused

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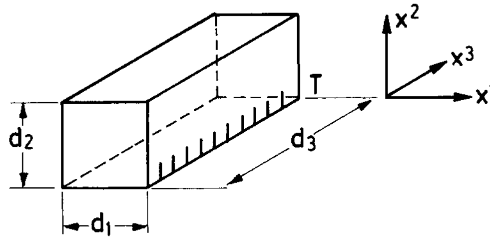


Fig 1. Boundary conditions on a three-dimensional container $\Lambda$. The action of the singular gauge transformation for gauge groups with center Z(2) is given by $\sigma(b) \to -\sigma(b)$ if $b \in \mathrm{T}$, $\sigma(b) \to \sigma(b)$ otherwise.

with the Higgs phase in 't Hooft's sense. The different regions of the phase diagram are shown in fig. 2.

We briefly recall the definition of the vortex free energy. One considers a system on a finite lattice $\Lambda$. For simplicity we restrict ourselves to three dimensions, the four-dimensional case can be treated analogously. A vortex free energy $\nu$ is the difference between the free energies of the container $\Lambda$, when the boundary conditions on $\Lambda$ are twisted for the gauge field by the action of a singular gauge transformation in the center of the gauge group, cf. fig. 1. One is interested in its limit per unit length, but in the explicit dependence on the breadth $d_1$ and the height $d_2$ of the container, see fig. 1:

$$\lim_{d_3 \to \infty} \frac{1}{d_3} \nu(\Lambda, \text{b.c.}) = \lim_{d_3 \to \infty} \frac{1}{d_3} \ln \frac{Z(\Lambda, \text{t.b.c.})}{Z(\Lambda, \text{unt.b.c.})}, \tag{1.1}$$

(t.b.c.) denote twisted, (unt.b.c.) untwisted boundary conditions. We choose periodic boundary conditions (p.b.c.) for the gauge field in directions 1 and 2, and free b.c. in direction 3. The action of the singular gauge transformation on the matter fields cannot be defined, therefore free b.c. for the matter fields in all directions are imposed.

It will be shown that in the region of large $\kappa$ and arbitrary $g_0^{-2}$ the vortex free energy approaches a constant value independent of $d_1$ and $d_2$. This behavior characterized the Higgs phase in the sense of 't Hooft with no vortex condensation. For further discussion of the results see sects. 2, 5.

## 2. The model

We consider a Z(2) gauge theory with Z(2) Higgs scalars (the so called Z(2) Higgs model or gauge-invariant Ising model). The variables are string bit variables $\sigma(b) \in Z(2) = \{\pm 1\}$ attached to links $b$ of the lattice, and matter fields $\tau(x) \in Z(2)$ attached to sites $x$. $\sigma(b) \to \sigma(b)^{-1} = \sigma(b)$ under reversal of the direction of the links

$b$. The action is given by

$$L(\sigma,\tau)=g_0^{-2}\sum_p\{\sigma(\partial p)-1\}+\kappa\sum_{b=\langle xy\rangle}\tau(x)\sigma(b)\tau(y), \tag{2.1a}$$

$$\sigma(\partial p)=\prod_{b\in\partial p}\sigma(b). \tag{2.1b}$$

$\partial p$ denotes the boundary of the plaquette $p$. The gauge part has the standard Wilson-Wegner form. $g_0^{-2}$ and $\kappa$ are the (bare) gauge and matter couplings, respectively. The sums run over all plaquettes $p$ and links $b$ of the lattice. Plaquettes on opposite sides of $\Lambda$ are summed only once, if the gauge field variables satisfy p.b.c. in the corresponding directions.

The Haar measures on Z(2) amount to summations over all configurations $\{\sigma(b)=\pm1\}$ and $\{\tau(x)=\pm1\}$:

$$\int\mathcal{D}\sigma\ldots=\int\prod_b\mathrm{d}\sigma(b)\ldots,\qquad\int\mathrm{d}\sigma(b)\ldots=\tfrac{1}{2}\sum_{\sigma(b)=\pm1}\ldots,$$

$$\int\mathcal{D}\tau\ldots=\int\prod_x\mathrm{d}\tau(x)\ldots,\qquad\int\mathrm{d}\tau(x)\ldots=\tfrac{1}{2}\sum_{\tau(x)=\pm1}\ldots. \tag{2.2}$$

For the phase diagram of this model, the rigorous results of Marra and Miracle Solé are available [4]. They establish analyticity of the free energy in the whole shaded region bounded by the dotted lines in fig. 2. The confining/screening region (I, III) was already investigated in the work of Osterwalder, Seiler [5], and Fradkin, Shenker [6]. Monte Carlo data confirm [7] that the regions (I, III) and (II) are separated by a line of phase transitions (full line in fig. 2.). For region (II) we have shown by low-temperature expansions [2] that the vortex free energy approaches a constant value independent of $d_1$ and $d_2$. This behavior corresponds to the definition of the Higgs phase in the sense of 't Hooft. High-temperature (i.e. strong-coupling)

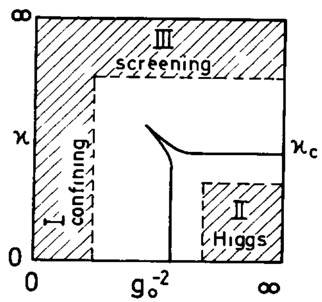


Fig 2. Phase diagram for the Z(2) Higgs model in three dimensions.

expansions in $g_0^{-2}$ and in $\kappa$ yielded a qualitatively different behavior of the vortex free energy in region (I). It decays exponentially with the perimeter $2 \cdot (d_1 + d_2)$ [2].

Region (I) and (III) is called the confining/screening phase. The name is justified by the limit models: $\kappa = 0$ gives the pure gauge theory where confinement was proved for $g_0^{-2} \ll 1$ [5]. (The vortex free energy decays with the cross section $d_1 d_2$ in this case.) $g_0^{-2} = \infty$ corresponds to the Ising ferromagnet for Z(2), where the global symmetry is spontaneously broken for $\kappa > \kappa_c$. This half-line $g_0^{-2} = \infty$, $\kappa > \kappa_c$ borders the region where a Higgs mechanism is said to occur. Since there is no phase transition in the whole region (I) and (III), confinement should be compatible with a kind of Higgs mechanism [8].

The vortex free energy is a non-local quantity. Therefore the absence of a phase transition between regions (I) and (III) does not imply the same $(d_1, d_2)$ dependence in these regions. However, if the behavior of the vortex free energy would reflect the confinement property throughout the whole region, it should be qualitatively the same in regions (I) and (III). It turns out that it behaves otherwise. This is shown by the calculation of the vortex free energy for arbitrary $g_0^{-2}$ but large $\kappa$ in the following sections.

## 3. Cluster expansion for the vortex free energy

Finally we turn to a cluster expansion of the quantity defined in (1.1). In order to exhibit the dependence on boundary conditions we define

$$Z_{\pm}(\Lambda) = Z(\Lambda, \text{unt.b.c.}) \pm Z(\Lambda, \text{t.b.c.}) . \tag{3.1}$$

In the limit of $d_3 \to \infty$ the second term becomes very small compared to the first. Therefore in this limit

$$Z(\Lambda, \text{t.b.c.})/Z(\Lambda, \text{unt.b.c.}) = \tfrac{1}{2}\ln\{Z_+(\Lambda)/Z_-(\Lambda)\} . \tag{3.2}$$

First we rewrite $Z_{\pm}$ as a polymer system with activities $\phi_{\pm}(\text{pol})$ defined in such a way that

$$Z_{\pm}(\Lambda) = 1 + \sum_{\Sigma \text{pol} \subseteq \Lambda} \prod_{\text{pol}} \phi_{\pm}(\text{pol}) . \tag{3.3}$$

The sum runs over all disjoint unions $\Sigma$ pol of polymers. Polymers are certain subsets of $\Lambda$. They will be specified in sect. 4. The meaning of "disjoint" will also be explained there. The activities $\phi_{\pm}(\text{pol})$ can depend on the boundary conditions. They are defined such that they are small in the range of coupling constants considered here. If they are small enough, the free energy $\ln Z_{\pm}$ admits a convergent cluster expansion of the form

$$\ln Z_{\pm}(\Lambda) = \sum_{\mathrm{Q}} a(\mathrm{Q}) \prod_{\mathrm{pol} \in \mathrm{Q}} \phi_{\pm}(\mathrm{pol}),$$

$$\mathrm{Q} = \{\mathrm{pol}_1^{n_1}, \ldots, \mathrm{pol}_N^{n_N}\}. \tag{3.4}$$

Here the sum runs over all sets of linked clusters. A linked cluster Q is a non-empty collection of not necessarily distinct polymers. It may contain polymers with multiplicity $n_i \geq 1$. It has to be linked in the following sense. Consider the abstract graph $\mathcal{G}_{\mathrm{Q}}$ whose vertices are the polymers in Q, and whose links are the pairs of polymers in Q which are not disjoint. $\mathcal{G}_{\mathrm{Q}}$ has to be connected. $a(\mathrm{Q})$ are some combinatorial factors.

## 4. Peierls expansions

Large $\kappa$ suggests to look for an expansion of $Z_{\pm}$ in sets $\mathbb{S}$ of negative links

$$\mathbb{S} = \{ b = \langle xy \rangle \in \Lambda / \tau(x) \sigma(b) \tau(y) = -1 \}. \tag{4.1}$$

This determines sets $\mathcal{P}$ of frustrated plaquettes

$$\mathcal{P} = \{ p \in \Lambda / \sigma(\partial p) = -1 \} = \hat{\partial} \mathbb{S}. \tag{4.2}$$

$\mathcal{P}$ is given by the coboundary $\hat{\partial}\mathbb{S}$ of $\mathbb{S}$ because of (2.1b). In terms of $\mathbb{S}$ and $\hat{\partial}\mathbb{S}$ $Z_{\pm}$ is reproduced by

$$Z_{\pm}(\Lambda) = \sum_{\mathbb{S} \subset \Lambda} (\pm 1)^{n(\hat{\partial}\mathbb{S})} \mathrm{e}^{-2\kappa|\mathbb{S}|} \mathrm{e}^{-2g_0^{-2}|\hat{\partial}\mathbb{S}|}, \tag{4.3}$$

if the sum is restricted to sets $\mathbb{S}$ that satisfy the following boundary condition: $\hat{\partial}\mathbb{S}$ is periodic in directions 1 and 2. It follows that

$$n(\hat{\partial}\mathbb{S}) := |\hat{\partial}\mathbb{S} \cap \Xi| \bmod 2, \tag{4.4}$$

(with $\Xi = (\text{plane } x^3 = \text{const}) \cap \Lambda$) is independent of $x^3$. ($n(\hat{\partial}\mathbb{S}) = 0$) corresponds to unt. b.c., ($n(\hat{\partial}\mathbb{S}) = 1$) to t.b.c. It counts the number of vortices that wind once (mod 2) through $\Lambda$ in direction 3.

Suppose that a certain configuration $\mathbb{S}$ is given and boundary conditions for $\hat{\partial}\mathbb{S}$ as specified above are fulfilled. Let us verify that to any given $\mathbb{S}$ there exists a configuration $(\sigma, \tau)$ such that the equality (4.1) holds, and the implied b.c. for $\tau$ and $\sigma$ are gauge equivalent to the originally chosen b.c., i.e. p.b.c. (t.b.c.) for the gauge field $\sigma$ in directions 1,2, free b.c. for $\sigma$ in direction 3, and free b.c. for $\tau$ in all directions. $\mathbb{S}$ fixes $\hat{\partial}\mathbb{S}$ uniquely.

Conversely $\hat{\partial}\mathbb{S}$ fixes the set of links $b$ with $\sigma(b) = -1$ up to the action of a local gauge transformation on gauge fields. Choose a gauge transformation such that $\sigma$ satisfies p.b.c. or t.b.c. in directions 1,2. This is possible, because $n(\hat{\partial}\mathbb{S})$ is independent of $x^3$ as a consequence of the periodicity of $\hat{\partial}\mathbb{S}$. This still leaves the freedom of gauge transformations that satisfy p.b.c. (t.b.c.) in directions 1,2. Then $\mathbb{S}$ fixes those $x$ for which $\tau(x) = -1$, up to the action of a global reflection $\tau(x) \to -\tau(x)$. Therefore $\mathbb{S}$ fixes the configuration in terms of $\sigma$ and $\tau$ with specified b.c. (periodic, twisted or free, respectively) up to a gauge transformation with p.b.c. (t.b.c.) for $\sigma$, and up to the reflection $\tau(x) \to -\tau(x)$. Since the action is locally gauge invariant and the Haar measure on Z(2) was normalized, the summation over gauge equivalent configurations yields a factor 1.

*Definition of polymers.* Consider the lattice as a cell complex made of sites, links, plaquettes and cubes. We associate graphs G to subsets $\mathbb{S}$ of links with $\mathbb{S}$ as in (4.3), i.e. $\hat{\partial}\mathbb{S}$ satisfies p.b.c. in directions 1,2 (hence $n(\hat{\partial}\mathbb{S})$ is independent of $x^3$). The vertices $v_i$ of the graphs are the links $b \in \mathbb{S}$. Two vertices $v_1$ and $v_2$ are connected if the coboundaries $\hat{\partial}b_i$ $(i = 1,2)$ of the links $b_1$ and $b_2$ share a common plaquette $p$

$$(\hat{\partial}b_1 \cap \hat{\partial}b_2) \neq \varnothing . \tag{4.5}$$

For plaquettes $p \in (\cup_i \hat{\partial}b_i \cap \partial\Lambda)$ note that those on opposite sides in directions 1,2 are identified.

Polymers are those sets $\mathbb{S}$ of links whose corresponding graph G($\mathbb{S}$) is connected. Two polymers $\text{pol}_1$ and $\text{pol}_2$ are disjoint if their graphs are not connected. Products of winding numbers factorize over disjoint polymers $\text{pol}_1$ and $\text{pol}_2$

$$\begin{aligned} n(\hat{\partial}\text{pol}_1 \dot{\cup} \hat{\partial}\text{pol}_2) &:= |(\hat{\partial}\text{pol}_1 \dot{\cup} \hat{\partial}\text{pol}_2) \cap \Xi| \bmod 2 \\ &= |(\hat{\partial}\text{pol}_1 \cap \Xi) \dot{\cup} (\hat{\partial}\text{pol}_2 \cap \Xi)| \bmod 2 \\ &= |n(\hat{\partial}\text{pol}_1) + n(\hat{\partial}\text{pol}_2)| \bmod 2 = n(\hat{\partial}\text{pol}_1)\, n(\hat{\partial}\text{pol}_2) . \end{aligned}$$

Then the polymer representation of $Z_\pm(\Lambda)$ reads

$$Z_\pm(\Lambda) = 1 + \sum_{n \geq 1} \sum_{\Sigma_{i=1}^n \text{pol}_i} \prod_i \phi_\pm(\text{pol}\, i) . \tag{4.6}$$

$\Sigma_{i=1}^n \text{pol}_i$ are unions of $n$ disjoint polymers. An overcounting of polymers in (4.6) is excluded, because disjoint unions of polymers are no longer polymers themselves.

The activities $\phi_\pm(\text{pol}_i)$ are defined as

$$\phi_\pm(\text{pol}) := (\pm 1)^{n(\hat{\partial}\text{pol})} e^{-2g_0^{-2}|\hat{\partial}\text{pol}|} e^{-2\kappa|\text{pol}|} . \tag{4.7}$$

The cluster expansion is given by (3.4) combined with (4.7).

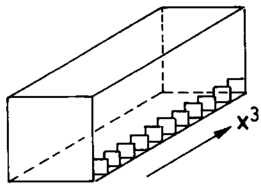


Fig 3 Polymer that makes the leading contribution

*The leading term.* The leading term is made by a polymer $\mathrm{pol}_\ell$ such that $\hat{\partial}\mathrm{pol}_\ell$ is a vortex that winds once through $\Lambda$ in direction 3 ($n(\hat{\partial}\mathrm{pol}_\ell) = +1$). In order to minimize the number of negative links, $\mathrm{pol}_\ell$ should lie in the boundary of $\Lambda$, cf. fig. 3. A possible choice of $\mathrm{pol}_\ell$ is the set T, the minimal set of links in $\partial\Lambda$ that admits the introduction of the twist. There are $2(d_1 + d_2)$ possible positions for $\mathrm{pol}_\ell$ on $\partial\Lambda$. Therefore we find

$$\ln Z_+ - \ln Z_- = \tfrac{1}{2}(d_1 + d_2)\left\{e^{-d_3(2g_0^{-2}+2\kappa)} + O\left(d_3 e^{-2(g_0^{-2}+\kappa)}\right)\right\}. \tag{4.8}$$

Corrections are of order $(d_3 e^{-2(g_0^{-2}+\kappa)})$, because the minimal decoration to the leading term consists of one negative link and one frustrated plaquette in addition. Then we get for the vortex free energy per unit length

$$\lim_{d_3\to\infty} \frac{1}{d_3}\{\ln Z(\Lambda, \mathrm{t.b.c.}) - \ln Z(\Lambda, \mathrm{p.b.c.})\}$$

$$= -2\left(g_0^{-2} + \kappa\right) + O\left(e^{-2(g_0^{-2}+\kappa)}\right), \tag{4.9}$$

where we have used (3.2) and (4.8).

In general, the activity of a polymer pol = S is at most of order $(e^{-2\kappa|S|}) \ll 1$ for $\kappa \gg 1$ and arbitrary $g_0^{-2}$. For small $g_0^{-2}$ the factor $e^{-2g_0^{-2}|\hat{\partial}S|}$ is of order 1. However, we use an expansion in sets S of negative links instead of sets of vortices $\hat{\partial}S$. Therefore suppression factors from the matter part are always present, the activities are small for large $\kappa$, and Peierls expansions are justified. Correction terms can be treated as in [2]. They sum up to an exponential independent of the cross section $d_1 d_2$ of the container.

## 5. Summary of results

In table 1 we have summarized the results that were obtained in regions (I), (II), and (III) of the phase diagram for different models. Similar results hold in region (I) for the Z(2) Higgs, the Z(2) quark, the SU(2) Higgs, and the SU(2) quark model.

TABLE 1
Vortex free energy per unit length for large $d_1$, $d_2$

| | Phases | Models | $\lim_{d_3\to\infty}(1/d_3)\nu(\Lambda,\text{b c})$ |
|---|---|---|---|
| (I) | $g_0^{-2} \ll 1$, $\kappa \ll 1$ (confinement) | Z(2) Higgs, quark; SU(2) Higgs, quark | $\propto e^{-\alpha(d_1+d_2)}$ |
| (II) | $g_0^{-2} \gg 1$, $\kappa \ll 1$ (deconfinement) | Z(2) Higgs, quark | $\text{const}_1 \neq 0$ |
| (III) | $g_0^{-2}$ arbitrary, $\kappa \gg 1$ (screening) | Z(2) Higgs | $\text{const}_2 \neq 0$ |

Also in region (II) they are qualitatively the same for Z(2) quarks as for Z(2) Higgs fields. From the behavior derived in sect. 4 for region (III) we conclude that the vortex free energy is not a good indicator of confinement when Higgs fields are present with non-trivial transformation properties under the center of the gauge group. Its behavior does not agree with the expectations from the phase diagram. It distinguishes between both regions where confinement is expected to occur, but not between phases (regions (III) and (II)) that are separated by a line of phase transitions. It seems that the vortex free energy tests for confinement only through vortex condensation. On the other hand the result is not surprising. In terms of gauge invariant variables $\tilde{\sigma}(b) := \tau(x)\sigma(b)\tau(y)$ we get an expansion in sets $\tilde{\mathbb{S}} := \{b \in \Lambda / \tilde{\sigma}(b) = -1\}$ of negative links and sets $\partial\tilde{\mathbb{S}}$ of frustrated plaquettes (vortices). At least $\frac{1}{4}|\tilde{\mathbb{S}}|$ negative links are necessary to create a vortex of length $|\partial\tilde{\mathbb{S}}|$ (in three dimensions). Therefore long vortices $\partial\tilde{\mathbb{S}}$ are suppressed by factors of order $(e^{-(\kappa/2)|\tilde{\mathbb{S}}|})$ at least, for large $\kappa$ and arbitrary $g_0^{-2}$.

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