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DIELECTRIC LATTICE GAUGE THEORY *

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Dielectric lattice gauge theory models are introduced They involve variables $\Phi(b) \in \mathcal{G}$ that are attached to the links $b = (x + e_\mu, x)$ of the lattice and take their values in the linear space $\mathcal{G}$ which consists of real linear combinations of matrices in the gauge group G The polar decomposition $\Phi(b) = U(b)\sigma_\mu(x)$ specifies an ordinary lattice gauge field $U(b)$ and a kind of dielectric field $\varepsilon_{ij} \propto \sigma_i \sigma_j^* \delta_{ij}$ A gauge invariant positive semidefinite kinetic term for the Φ-field is found, and it is shown how to incorporate Wilson fermions in a way which preserves Osterwalder-Schrader positivity Theories with G = SU(2) and without matter fields are studied in some detail It is proved that confinement holds, in the sense that Wilson-loop expectation values show an area law decay, if the euclidean action has certain qualitative features which imply that $\Phi = 0$ (i e dielectric field $\equiv 0$) is the unique maximum of the action

1. Introduction

The dielectric theory of quark confinement [1] is very appealing because it offers a physical picture which is easy to understand and predictive to some extent (fig. 1). It is therefore a challenging task to derive this theory from QCD and to develop it into a full theory from which one can compute "everything". This should also shed light on the dynamics of gauge theories in general. Some work in this direction was begun by Nielsen and Patkos [2].

In (classical) electrodynamics of polarizable media, the dielectric constant embodies sufficient information about the polarizable medium if attention is restricted to static, spatially homogeneous situations. In an effective dielectric theory based on QCD one restricts attention to long-distance properties only. The role of the "polarizable medium" is played by the high-frequency parts of the gauge fields (and possibly of the matter fields, when they are treated as dynamical fields), and information about it is embodied in a slowly varying dielectric field $\varepsilon(x)$, or related fields $\sigma(x)$. One seeks to describe the long-distance behavior of QCD by an effective (euclidean) action L which incorporates an ultraviolet (UV) cutoff M and depends

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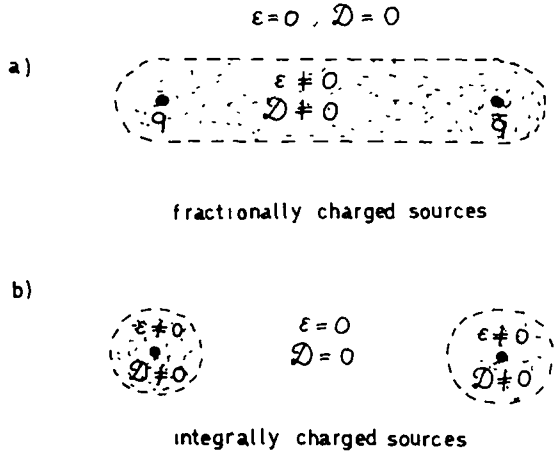

Fig 1 The dielectric theory of confinement [1]

on the variables $\sigma(x)$ in addition* to the gauge fields (and matter fields) that were present in the original QCD action.

It is expected that the effective action L will have a unique maximum at $\varepsilon = 0$ (i.e. $\sigma(x) \equiv 0$) when the UV-cutoff M is low enough (of the order of the ultimate physical mass scale). This is the basic hypothesis of the dielectric theory of confinement, and confinement is supposed to follow from it (see fig. 1). Coloured sources are the source of an electric induction $\mathcal{D}$ which is related to the colour electric field $\mathcal{E}$ by a dielectric constant (or rather field) ε, $\mathcal{D} = \varepsilon\mathcal{E}$. Therefore $\mathcal{D}$ can only be non-zero where $\varepsilon \neq 0$. If $\varepsilon = 0$ is the unique classical vacuum then $\varepsilon \neq 0$ costs energy and this prevents the $\mathcal{D}$-field from spreading. If the sources transform non-trivially under the center Γ of the gauge group then Gauss' law for the abelian group Γ forces the $\mathcal{D}$-field to extend to infinity or to another source, because the gauge field carries no Γ-charge itself [3]. As a result, a string will form whose energy is proportional to its length (fig. 1a). If the sources transform trivially under the center of the gauge group, the string can break and the source gets screened by the gauge field (fig. 1b).

Following Nielsen and Patkos one may imagine that block spins in a (pure) gauge theory are defined as superpositions of parallel transporters $U(\omega)$ along paths with fixed end points

$$\Phi(y, x) = \sum_{\omega \quad x \to y} \rho(\omega) U(\omega) . \tag{1.1}$$

Here, x and y are centers of neighbouring block cells of side length M^{-1}, $\omega : x \to y$ are paths from x to y on the original lattice (or continuum) and $\rho(\omega)$ is a

* This is in contrast with Adler's approach where the dielectric field is a function of the electromagnetic field strength [2]

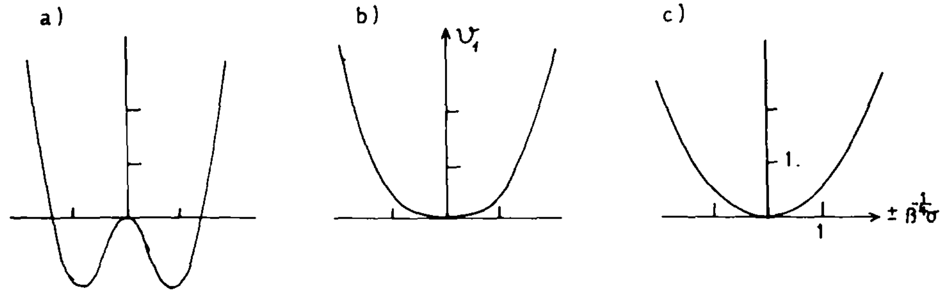

Fig 2 The potential $\mathcal{V}_1$ of eq (2.16)

non-negative real weight function. For instance one might choose

$$\Phi(y, x) = \frac{\partial^r}{\partial \mu^r}(-\Delta_U + \mu^2)^{-1}(y, x), \qquad \mu = O(M), \qquad r = d - 2, \quad (1.2)$$

in d dimensions. Δ_U is the covariant laplacian in the fundamental representation. Eq. (1.1) is obtained by a random walk expansion of the propagator in (1.2). On a lattice it gives

$$\rho(\omega) = a^2(2d + \mu^2 a^2)^{-|\omega|-1}, \qquad a = \text{lattice spacing},$$

if the path ω consists of $|\omega|$ links $\mathfrak{b}_1 \ldots \mathfrak{b}_{|\omega|}$ of the original lattice. The parallel transporter is $U(\omega) = U(\mathfrak{b}_{|\omega|}) \ldots U(\mathfrak{b}_1)$. [Such random walk representations were studied by Brydges and Federbush, Fröhlich and Durhuus, Aizenmann, and Brydges, Fröhlich and Spencer, following the pioneering work of Symanzik [4–6]. Similar expansions will be used extensively in the present paper.]

Evidently, $\Phi(b)$ will take their values in the linear space $\mathcal{G}$ which consists of real linear combinations of matrices in the gauge group G. Following Drouffe [7], one may write down a polar decomposition of $\Phi(b)$, this defines the variables $\sigma_\mu(x)$ and an ordinary lattice gauge field on the block lattice.

In the present paper I introduce models of effective actions $L(\Phi)$. This seems worthwhile since it will permit us to study the dielectric confinement mechanism in some detail. The models are local. A gauge invariant positive semidefinite kinetic term of the Φ-field is found. As a byproduct, a "natural" definition of field strengths in lattice gauge theory emerges. It is shown how to incorporate dynamical Wilson fermions in a way which preserves Osterwalder-Schrader positivity. Theories with G = SU(2) and without matter fields are studied in some detail. It is proved that confinement holds, in the sense that Wilson loop expectation values show an area law decay, if the action $L(\Phi)$ has certain qualitative features which imply that $\Phi = 0$ is the unique maximum of the action (theorem 1).

The models are described in sect. 2. The Wilson loop criterion for confinement [8] is also discussed there, theorem 1 is stated, and some qualitative expectations concerning the dependence of effective actions on the cutoff are formulated. Gauss' law and the hamiltonian limit are discussed in sect. 3. Sect. 4 spells out the dependence of the action on variables attached to links of a particular direction. In sect. 5 random walk representations of generalized covariant propagators are derived and estimated. In sect. 6, theorem 1 is proven, using the material of sects. 4, 5. Sect. 7 contains the proof of Osterwalder-Schrader positivity in the presence of fermions [9]. It is self-contained and does not use the material of sects. 3–6.

2. The models

The models live on hypercubic lattices Λ in d dimensions with sites x, links b, plaquettes p, etc. Unless indicated otherwise, the lattice spacing in all directions is the same and shall be set equal to 1. e_μ is the lattice vector in the μ-direction, $e_{-\mu} = -e_\mu$. If $b = (x + e_\mu, x)$ then $-b = (x, x + e_\mu)$ is the same link with reversed direction. We shall use a *symmetric* summation convention $v_\mu w_\mu \equiv \frac{1}{2}\Sigma_{\mu = \pm 1 \ldots \pm d} v_\mu w_\mu$.

The models shall possess local gauge invariance under a gauge group G, for instance G = SU(N) or U(N).

Let $\mathcal{G}$ be the linear space of all matrices Φ that admit a representation of the form $\Phi = \Sigma r_i U_i$, with $U_i \in$ G, r_i real. According to Drouffe [7], $\mathcal{G}$ consists of all complex $N \times N$ matrices if G = SU(N) or U(N) with $N \geqslant 3$, while it consists of all real multiples of elements of G if G = SU(2) or U(2).

The variables $\Phi(b) \equiv \Phi_\mu(x) \in \mathcal{G}$ of dielectric lattice gauge theory are attached to the links $b = (x + e_\mu, x)$ of the lattice. They take their values in the linear space $\mathcal{G}$, and

$$\Phi(-b) = \Phi(b)^*, \qquad \text{i.e. } \Phi_\mu(x)^* = \Phi_{-\mu}(x + e_\mu). \tag{2.0}$$

They will be integrated over using the Lebesgue measure dΦ on $\mathcal{G}$. They transform under gauge transformations in the standard way

$$\Phi_\mu(x) \to V(x + e_\mu)\Phi_\mu(x)V(x)^{-1}, \qquad V(z) \in \mathrm{G}. \tag{2.1a}$$

They admit a polar decomposition [7]

$$\Phi_\mu(x) = U(x + e_\mu, x)\sigma_\mu(x), \qquad \text{with } U(b) \in \mathrm{G}, \tag{2.2a}$$

$$\sigma_\mu(x) \geqslant 0, \qquad \text{real, if G = SU(2) or U(2)},$$

$$\sigma_\mu(x) = \text{a positive hermitean } N \times N \text{ matrix} \quad \text{for U}(N), N \geqslant 3,$$

$$\sigma_\mu(x) = \mathrm{e}^{i\theta} \cdot (\text{positive hermitean } N \times N \text{ matrix}) \quad \text{for SU}(N), N \geqslant 3, \tag{2.2b}$$

The lattice gauge field $U(b)$ transforms under gauge transformations in the same way (2.1a) as $\Phi(b)$. $\sigma_\mu(x)$ is gauge invariant if G = SU(2) or U(2), and transforms covariantly according to

$$\sigma_\mu(x) \to V(x)\sigma_\mu(x)V(x)^{-1}, \tag{2.1b}$$

in general. It is convenient to define generalized parallel transporters. If the path C consists of links $b_1 \dots b_n$ we set

$$\Phi(\mathrm{C}) = \Phi(b_n) \dots \Phi(b_1). \tag{2.4}$$

Note that $\Phi(-b) \neq \Phi(b)^{-1}$ in general; therefore the parallel transporter along a path $-\mathrm{C}\cdot\mathrm{C}$ which runs back and forth is not unity, and spikes in a path C effect Φ(C).

Given Φ, we define a kind of covariant derivative $\boldsymbol{D}_\mu$ which acts on $\mathcal{G}$-valued functions on links according to

$$\begin{aligned} \boldsymbol{D}_\mu \Psi_\nu(x) &= \Psi_\nu(x+e_\mu)\Phi_\mu(x) - \Phi_\mu(x+e_\nu)\Psi_\nu(x) \\ &\equiv \Psi(x+e_\mu+e_\nu, x+e_\mu)\Phi(x+e_\mu, x) \\ &\quad - \Phi(x+e_\nu+e_\mu, x+e_\nu)\Psi(x+e_\nu, x). \end{aligned} \tag{2.5}$$

Under gauge transformations, $\boldsymbol{D}_\mu\Psi_\nu$ does not transform like Ψ_ν itself, but rather

$$\boldsymbol{D}_\mu\Psi_\nu(x) \to V(x+e_\mu+e_\nu)\boldsymbol{D}_\mu\Psi_\nu(x)V(x)^{-1}. \tag{2.1c}$$

Evidently, in the special case $\Psi = \Phi$ we have antisymmetry

$$\mathcal{F}_{\mu\nu}(x) \equiv \boldsymbol{D}_\mu\Phi_\nu(x) = -\boldsymbol{D}_\nu\Phi_\mu(x). \tag{2.6}$$

$\mathcal{F}_{\mu\nu}$ is the difference of the parallel transporters along the two paths from x to $x+e_\mu+e_\nu$ shown in fig. 3: it is a generalized field strength. As a consequence of its definition it satisfies a Bianchi-identity (2nd Maxwell equation). Extend the definition (2.5) of $\boldsymbol{D}_\mu$ to

$$\boldsymbol{D}_\mu\Psi_{\nu_1 \dots \nu_k}(x) = \Psi_{\nu_1 \dots \nu_k}(x+e_\mu)\Phi_\mu(x) - \Phi_\mu(x+e_{\nu_1}+\cdots+e_{\nu_k})\Psi_{\nu_1 \dots \nu_k}(x). \tag{2.5'}$$

Then the Bianchi identity reads

$$\boldsymbol{D}_\lambda\mathcal{F}_{\mu\nu} + \boldsymbol{D}_\nu\mathcal{F}_{\lambda\mu} + \boldsymbol{D}_\mu\mathcal{F}_{\nu\lambda} = 0. \tag{2.7}$$

Square integrable functions $\Psi_{\nu_1 \dots \nu_k}$ form a Hilbert space $\mathcal{H}_k$, and covariant differ-

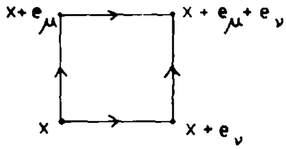

Fig 3 Field strength (see text)

entiation $\boldsymbol{D}$ specifies a map $\mathcal{H}_k \to \mathcal{H}_{k+1}$. Its adjoint $\boldsymbol{D}^*: \mathcal{H}_{k+1} \to \mathcal{H}_k$ takes $\Psi_{\mu\nu_1 \dots \nu_k}$ into $\boldsymbol{D}^*_\mu \Psi_{\mu\nu_1 \dots \nu_k}$. Explicit computation shows that

$$\boldsymbol{D}^*_\mu = \boldsymbol{D}_{-\mu}. \tag{2.5''}$$

These formulae can also be used for standard lattice gauge theory where $\Phi(b) = U(b) \in \mathrm{G}$. In particular, definition (2.6) of $\mathcal{F}_{\mu\nu}$ is a natural definition of a field strength on a lattice, in spite of the unusual transformations law (2.1c). In the limit of zero lattice spacing, $e_\mu \to 0$, and the standard transformation law in the continuum is recovered. The hermitean conjugate of $\mathcal{F}_{\mu\nu}$ is

$$\mathcal{F}^*_{\mu\nu}(x) = \mathcal{F}_{-\mu,-\nu}(x + e_\mu + e_\nu). \tag{2.9}$$

We shall use a norm $\|\cdot\|$ of $N \times N$ matrices A which is defined by

$$\|A\|^2 = \frac{1}{N} \operatorname{tr} A^* A. \tag{2.10}$$

Now we are ready to write down candidates for the dielectric lattice gauge theory actions. As a kinetic term for the dielectric gauge field Φ we take*

$$L_{\mathrm{kin}}(\Phi) = -\tfrac{1}{16} \sum_x \sum_{\substack{\mu,\nu = \pm 1 \dots \pm d \\ \mu \neq \pm \nu}} \|\mathcal{F}_{\mu\nu}(x)\|^2. \tag{2.11}$$

Evidently, $L_{\mathrm{kin}}(\Phi) \leqslant 0$. The absolute maximum $L_{\mathrm{kin}}(\Phi) = 0$ is assumed at $\Phi = 0$, but

* One may consider adding a term

$$L_{\mathrm{long}} = -\tfrac{1}{2}\gamma \sum_x \sum_{\mu=1}^{d} \|\boldsymbol{D}_{-\mu} \Phi_\mu(x)\|^2$$

In particular, inclusion of the term with $\mu = -\nu$ in the sum (2.11) would amount to that. The degeneracy (2.12b) would be removed by such a term. However, such a term wrecks the proof of Osterwalder-Schrader positivity (for reflections in planes with sites) because it is basically a next-to-nearest-neighbour interaction.

also for some more general field configurations:

$$L_{\text{kin}}(\Phi)=0 \text{ if } U(b)=\text{pure gauge, and}$$

$$\text{either } \sigma_\mu(x)=0 \text{ except possibly for a single value of } \mu, \tag{2.12a}$$

$$\text{or } \sigma_\mu(x) \text{ depends only on } x^\mu. \tag{2.12b}$$

This degeneracy is removed when mass terms are added. In the special case when Φ is a constant multiple of a standard lattice gauge field, $\Phi(b)=\beta^{1/4}U(b)$ with $U(b)\in \mathrm{G}$, the standard Wilson form of the lattice gauge theory action [8] is recovered

$$L_{\text{kin}}(\beta^{1/4}U)=\frac{\beta}{2N}\sum_p \operatorname{tr}[U(\partial p)-1]. \tag{2.13a}$$

As usual $U(\partial p)\equiv U(b_4)\dots U(b_1)$ where $b_1\dots b_4$ are the four links in the boundary of the oriented plaquette p.

$L_{\text{kin}}(\Phi)$ is invariant under gauge transformations in G. In the case that $\mathrm{G}=\mathrm{SU}(N)$, $N\geqslant 3$, its actual gauge symmetry is larger, $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$ (but *not** $\mathrm{GL}(N,C)$, since $\Phi(-b)=\Phi(b)^*\neq\Phi(b)^{-1}$). It could be broken down to $\mathrm{SU}(N)$ by adding terms involving $\det\Phi(b)$ to the action. If $N>4$ they have dimension >4 and the question arises whether such terms are irrelevant and whether this could lead to spontaneous creation of a $\mathrm{U}(N)$ symmetry**.

L_{kin} is a sum of products of four Φ's. We call it *biquadratic* because it is only quadratic in the fields Φ_μ that are attached to links of a particular direction μ. ($\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}^*$ involves two factors Φ_μ and two factors Φ_ν). This biquadratic character will be crucial in the proof of theorem 1.

We may add mass terms to the action. They can be either quadratic or biquadratic

$$\text{mass terms}=-\sum_x\Big\{\tfrac{1}{4}m^2\sum_{\mu=\pm1\ \ \pm d}\|\Phi_\mu(x)\|^2$$
$$+\tfrac{1}{8}\kappa^2\sum_{\mu\neq\pm\nu}\|\Phi_\mu(x)\|^2\|\Phi_\nu(x)\|^2\Big\}. \tag{2.11b}$$

If $m^2>0$ then the action L_{kin} + mass terms has $\Phi=0$ as its unique maximum.

* This is in contrast with Sharatchandras work [24]

** One speaks of spontaneous symmetry generation if the long-distance behavior of a theory shows a larger symmetry than its action. A celebrated example due to Frohlich and Spencer [10] is the sine-Gordon representation of a 2-dimensional Coulomb gas in its low-temperature (Kosterlitz-Thouless, or dipole) phase Its symmetry at long distances is **R** rather than **Z**

Finally we may add further local interactions, for instance

$$-V_1(\Phi) = -\sum_{x,\mu} \mathcal{V}_1(\sigma_\mu(x)),$$

$$-V_2(\Phi) = -\sum_{x,\mu,\nu} \mathcal{V}_2(\sigma_\mu(x), \sigma_\nu(x)), \qquad (1 \leqslant \mu < \nu \leqslant d). \tag{2.11c}$$

σ are determined by Φ through the polar decomposition (2.2). These expressions are automatically gauge invariant if G = SU(2) or U(2). If G = SU(N) or U(N) with $N \geqslant 3$ we require that $\mathcal{V}_1$ and $\mathcal{V}_2$ depend only on the eigenvalues of the matrices $\sigma(x)$.

One may wish to consider anisotropic lattices or anisotropic actions. In this case one may admit real functions $\mathcal{V}_{1\mu}, \mathcal{V}_{2\mu\nu}$ which depend on directions μ, ν in place of $\mathcal{V}_1$, $\mathcal{V}_2$, and similarly for the mass parameters m^2, κ.

Adding the terms (2.11a–c) one obtains an action

$$L(\Phi) = L_{\text{kin}}(\Phi) + \text{mass terms} - V_1(\Phi) - V_2(\Phi). \tag{2.14}$$

Partition functions and expectation values in the pure dielectric lattice gauge theory model with action (2.14) are defined as usual

$$Z = \int \prod_b \mathrm{d}\Phi(b) \mathrm{e}^{L(\Phi)},$$

$$\langle O \rangle = Z^{-1} \int \prod_b \mathrm{d}\Phi(b) O(\Phi) \mathrm{e}^{L(\Phi)}, \tag{2.15}$$

$\mathrm{d}\Phi$ is Lebesgue measure on the real-linear space $\mathcal{G}$, it is invariant under gauge transformations (2.1a). We assume that the potentials V are bounded below, so that the integrals are absolutely convergent (on a finite lattice). We adopt free boundary conditions unless otherwise indicated.

An interesting 2-parameter family of models of this kind is obtained by putting

$$m^2 = \kappa^2 = 0, \qquad \mathcal{V}_2 = 0,$$

$$\mathrm{e}^{-\mathcal{V}_1(\sigma_\mu)} = \text{const} \cdot \int_G \mathrm{d}U \exp\left[-\frac{1}{2\lambda} \|\beta^{-1/4}\Phi_\mu - U\|^2\right], \tag{2.16}$$

Fig. 2 demonstrates the expression (2.16) for G = SU(2):

(a) $\lambda = 0.1 < \lambda_c$, (b) $\lambda = 0.25 = \lambda_c$, (c) $\lambda = 0.5 > \lambda_c$.

$\mathrm{d}U$ is the Haar measure on G. $\mathcal{V}_1$ depends on $\Phi_\mu \in \mathcal{G}$ only through the factor σ_μ in its polar decomposition (2.2), because of invariance of the Haar measure $\mathrm{d}U$. If $\lambda \to 0$, $\mathcal{V}_1$ tends to a δ-function concentrated on $\beta^{1/4}G$. As a result, one obtains the standard lattice gauge theory with the Wilson action (2.13a) as a limiting case. By computing the second derivative at $\Phi_\mu = 0$ one sees that $\mathcal{V}_1$ has the qualitative shape shown in fig. 2a if $\lambda < \lambda_c$, or fig. 2b if $\lambda = \lambda_c$, and of fig. 2c if $\lambda > \lambda_c$. $\lambda_c = \frac{1}{4}$ if $G = SU(2)$. The main result of this paper, theorem 1 below, implies that static quarks are confined in these models, for $G = SU(2)$, if $\lambda > \lambda_c$, i.e. if the potential $\mathcal{V}_1$ has the qualitative shape of fig. 2c. (A quadratic mass term can be extracted from such a potential without affecting the validity of the hypotheses of theorem 1.)

Imagine now that the renormalization group transformations [11] map the two-parameter family of models with action (2.16) into itself, for a suitable choice of blockspins, and in such a way that an initial point $\lambda = 0$, β arbitrary, moves along a trajectory which reaches $\lambda > \lambda_c$ eventually. Then we could conclude that the standard lattice gauge theory model with Wilson action (= our model with $\lambda = 0$) shows confinement of static quarks for arbitrary β. Of course such a scenario is unrealistically simple (it could at the very best be approximately true) but it illustrates the general idea. Dielectric lattice gauge theory models are candidates for effective actions for Yang-Mills theory, and it is hoped that renormalization group transformations produce such an effective action with a single non-degenerate maximum at $\Phi = 0$ when the UV-cutoff is brought down far enough (to the order of the ultimate physical mass), in 4 or fewer dimensions, when asymptotic freedom is true in perturbation theory.

In contrast, in 5 or more dimensions, the model (2.16) is expected to undergo a deconfining phase transition as λ is lowered, if β is large enough.

Theorem 1 is for local actions. The exact effective action will be non-local but it ought to be possible to write it as a sum of a local action plus irrelevant terms which can hopefully be (neglected or) treated as a perturbation. The method of sects. 4–6 appears capable of generalization to include non-local terms, provided they are small and decay fast with distance. In place of the gaussian integration which leads to eq. (4.12) one would have to use cluster expansions as in refs. [12].

I will now state some general properties of the models with action (2.14). The models satisfy reflection positivity for reflections Θ in lattice planes through sites, assuming the boundary conditions are Θ-invariant (compare with sect. 7). As a result, the models admit a quantum mechanical interpretation. The Hilbert space $\mathcal{H}$ of physical states consists of gauge invariant square integrable wave functions $\Psi(\{\Phi(b)\}_{b\in\Sigma})$ which depend on variables $\Phi(b) \in \mathcal{G}$ attached to links b in the time $x_d = 0$ hyperplane Σ. The scalar product is defined by integration with the Lebesgue measure $\prod_{b\in\Sigma} \mathrm{d}\Phi(b)$. Furthermore, the transfer matrix T is hermitean and its square is therefore positive (semi) definite and can be used to define a hamiltonian [13, 14] so that $T^4 = e^{-2H}T^2$.

Next the Wilson criterium for confinement of static quarks [8] will be adapted. Consider a rectangular closed loop C composed of straight pieces C_1, C_2, C_3, C_4 of length T, L, T, L respectively, as in fig. 4, and define

$$W(C) = \mathrm{tr}(\Phi(C_4)U(C_3)\Phi(C_2)U(C_1)). \tag{2.17}$$

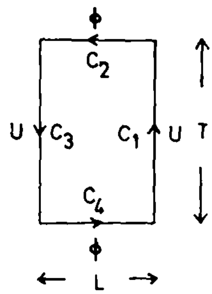

Fig 4. Wilson loop

The parallel transporter $\Phi(\mathrm{C})$ is defined by eq. (2.4), and $U(\mathrm{C})$ is defined by the same equation with U substituted for Φ. U is determined by Φ through the polar decomposition (2.2). (The set of field configurations Φ where either $U(\mathrm{C}_1)$ or $U(\mathrm{C}_3)$ is not well-defined has measure zero. This remains true in the infinite-volume limit, because of the Markov property [15].) In the limit $T \to \infty$

$$\langle W(\mathrm{C})\rangle = c(L)\mathrm{e}^{-TV(L)}, \tag{2.18}$$

and $V(L)$ can be interpreted as the potential energy of a pair of static quarks at a distance L. Therefore static quarks will be confined by a linearly rising potential $V(L)$ if $\langle W(C)\rangle$ shows an area law behavior.

Instead of $W(\mathrm{C})$ one can use $\operatorname{tr} U(\mathrm{C})$, but for our purposes $W(\mathrm{C})$ is more convenient.

The argument for the validity of this criterion is the standard one. One considers the Hilbert space $\mathcal{H}_{x_1,x_2}$ of states with static quarks (of opposite charge) at positions x_1 and x_2, and specifies a trial state $\Psi \in \mathcal{H}_{x_1,x_2}$. Then one considers matrix elements of powers of the transfer matrix $\boldsymbol{T}$.

$$(\Psi, \boldsymbol{T}^{2n}\Psi)/(\Psi,\Psi) = \mathrm{e}^{-2nV(x_1-x_2)}, \qquad \text{as } n \to \infty, \tag{2.19}$$

assuming Ψ does not happen to be orthogonal to the state of lowest energy (highest eigenvalue of $\boldsymbol{T}$) in $\mathcal{H}_{x_1,x_2}$. This is a standard assumption which is always needed to justify the Wilson criterion. As a trial state we take

$$\Psi(\{\Phi(b)\}_{b\in\Sigma}) = \Phi(\mathrm{C}_4)\int \prod_{b>0} \mathrm{d}\Phi(b)\mathrm{e}^{L_+(\Phi)}. \tag{2.20}$$

$b > 0$ are the links in the $x^d > 0$ half space, and $L_+(\Phi)$ is the sum of those terms in L which depend on some $\Phi(b)$ with $b > 0$, plus $\frac{1}{2}\cdot$ those which depend only on $\Phi(b)$ with $b \in \Sigma$. Then $(\Psi, \boldsymbol{T}^{2n}\Psi) = Z\langle W(\mathrm{C})\rangle$ and $(\Psi,\Psi) = Z\langle \operatorname{tr} \Phi(\mathrm{C}_4)\Phi^*(\mathrm{C}_4)\rangle$ if $\boldsymbol{T} = 2n$. This justifies eq. (2.18) with c independent of $\boldsymbol{T}$. (For simplicity we dropped some colour indices and sums over them.)

Finally the main result of this paper can be stated. I will only prove it for G = SU(2). It states that static quarks which transform according to the fundamental representation of SU(2) are confined by a linearly rising potential in dielectric lattice gauge theories with a local action L (of the form (2.16)) *provided* $\Phi = 0$ is the only maximum of $L(\Phi)$ and it is non-degenerate (i.e. $(\mathrm{d}^2/\mathrm{d}t^2)L(t\Phi)_{t=0} \neq 0$ for all $\Phi = \{\Phi(b) \in \mathcal{G}\}_{b \in \Lambda} \neq 0$). This is in agreement with the intuitive argument of fig. 1. The theorem is actually not quite as strong as that, though: It does not cover the case where the potential V has "valleys", such as $\mathcal{V}_2(\sigma_\mu, \sigma_\nu) = \lambda(\sigma_\mu^2 - \sigma_\nu^2)^2$ (which would favor isotropic σ).

Theorem 1. *Consider the dielectric lattice gauge theory model (without matter fields) in d dimensions with gauge group $G = SU(2)$ and action $L(\Phi)$ given by eqs. (2.14), (2.11a, b, c), with $m^2 > 0$, $\kappa^2 \geqslant 0$, $\mathcal{V}_1$ and $\mathcal{V}_2$ bounded below. If*

$$\mathcal{V}_1(\sigma_1) + \sum_{\nu=2}^{d} \mathcal{V}_2(\sigma_1, \sigma_\nu),$$

is a non-decreasing function of $\sigma_1 \in \mathbb{R}_+$, for arbitrary $\sigma_\nu \in \mathbb{R}_+$, $\nu = 2 \ldots d$, then the Wilson-loop expectation value obeys an area law

$$\langle W(\mathrm{C}) \rangle \leqslant c(L) \mathrm{e}^{-\alpha L T}, \qquad \text{with } \alpha > 0. \tag{2.21}$$

$\mathbb{R}_+$ is the set of non-negative real numbers and W(C) was defined in eq. (2.17) and fig. 4. The theorem remains valid for anisotropic lattices or actions, provided $m_1^2, \kappa_1^2, \mathcal{V}_{11}, \mathcal{V}_{21\nu}$ is substituted for $m^2, \kappa^2, \mathcal{V}_1, \mathcal{V}_2$ in its statement. Theorem 1 holds for arbitrary dimension d.

Finally I will discuss how one may put quark fields into the action. One could of course use one of the standard forms of the matter action in ordinary lattice gauge theory, with gauge field $U(b)$ that are obtained from $\Phi(b)$ by the polar decomposition (2.2a). This satisfies the requirement of physical positivity (Osterwalder-Schrader, or reflection positivity); but it is non-polynomial in the Φ's. A possible local matter action which is polynomial in the Φ's and in the quark fields ψ, ψ^+ is as follows (in 4 dimensions)

$$L_{\mathrm{matter}}(\psi, \psi^+, \Phi) = \sum_x \Big\{ -\bar{\psi}(x)\psi(x) + \sum_{\mu = \pm 1 \ldots \pm 4} \big[K_1 \bar{\psi}(x + e_\mu)(1 - \gamma_\mu)\Phi_\mu(x)\psi(x) - k\bar{\psi}(x)(1 - \gamma_\mu)\Phi_\mu^*(x)\Phi_\mu(x)\psi(x) \big] \Big\}. \tag{2.22}$$

Here γ_μ are euclidean Dirac matrices, and $\bar{\psi} = \psi^+ \gamma_4$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \text{for } \mu, \nu = 1 \ldots 4, \qquad \gamma_\mu^* = \gamma_\mu = -\gamma_{-\mu}.$$

For practical calculations we use the hermitean matrices (7.4). Colour and spinor indices α are suppressed in (2.22).

The computation of expectation values involves integrals over anticommuting variables $\psi^+(x)$, $\psi(x)$ [16]:

$$\langle O\rangle = Z^{-1}\int\prod_x\left[\prod_\mu \mathrm{d}\Phi_\mu(x)\prod_\alpha \mathrm{d}\psi^+_\alpha(x)\,\mathrm{d}\psi_\alpha(x)\right]$$

$$\times\, O(\psi,\psi^+,\Phi)\exp\left[L(\Phi)+L_{\mathrm{matter}}(\psi,\psi^+,\Phi)\right]. \qquad (2.22')$$

It will be shown in sect. 7 that physical positivity is satisfied provided

$$K_1^2<\tfrac{2}{3}k, \qquad \text{(sufficient condition)}. \qquad (2.23)$$

In the special case of an ordinary lattice gauge theory, where $\Phi_\mu(x)\in \mathrm{G}$ is unitary, the last term in (2.22) is proportional to $\bar\psi\psi$, because $\Phi^*\Phi=1$ and the terms with γ_μ and $\gamma_{-\mu}$ cancel. The matter action (2.22) reduces therefore to the standard Wilson action with hopping parameter $K=K_1(1+6k)^{-1}$ after suitable rescaling of the fields. For given K the constraint (2.23) can be satisfied by a suitable choice of K_1 if $K<\frac{1}{6}$. This reproduces Lüschers result for standard lattice gauge theory [14].

3. Hamiltonian limit and Gauss' law

The natural way to construct models in continuous time is to start with models on an anisotropic lattice, and to require that the variables $\sigma_d(x)$ attached to time-like links $(x+e_d, x)$ get frozen to a constant multiple $\sigma_0>0$ of $\mathbf{1}$ in the limit when the lattice spacing a_t in the time direction tends to zero. This can be achieved for instance by taking $\mathcal{V}_{1\mu}$ (which depends on the direction μ on an anisotropic lattice) to be of the form (2.16) for $\mu=d$ with $\lambda\to 0$ as $a_t\to 0$. Integration over the remaining variables $U(x+e_d,x)\in\mathrm{G}$ projects on the physical state space as in ordinary gauge theories. The limiting theory admits of a conventional hamiltonian description [18].

Let X^a, $a=1\ldots\dim\mathrm{G}$ be a complete set of antihermitean generators of the gauge group G, and let $B^a=\mathrm{tr}(BX^a)$ for $B\in\mathcal{G}$. Gauss' law takes the form (in 4 dimensions)

$$\left(D_l^*\mathcal{F}_{4l}(x)\right)^a=\rho^a(x), \qquad a=1\ldots\dim\mathrm{G}, \qquad (3.1a)$$

with

$$\rho^a=2K_1\psi^+(x)X^a\psi(x), \qquad (3.1b)$$

if the matter action has the form (2.22). The number of equations obtained in this way equals the number of generators of the gauge group, as it must be. Eq. (3.1) can be obtained, for instance, as the equation of motion that is obtained by varying U_d, cf. [19], and setting $U_d=1$ in the end.

Let the electric field $\mathcal{E}_i$ be defined like $\mathcal{F}_{4i}$, but with the U's in place of the Φ's. Inserting the definition and expanding to leading order in the lattice spacing a_s in the space direction gives, formally,

$$\sum_{i=1}^{3}\left((\partial_i - A_i)\sigma_i\sigma_i^*\sigma_0\mathcal{E}_i\right)^a + O(a_s) = \rho^a.$$

This suggests we identify

$$\varepsilon_{ij} \propto \sigma_i\sigma_i^*\delta_{ij}, \tag{3.2}$$

as a dielectric (tensor) field.

4. The Φ_1-dependent part of the action

We return to the consideration of models with discrete euclidean time. The action $L(\Phi)$ given by eqs. (2.14), (2.11) can be split into two pieces

$$L(\Phi) = L^{\parallel}(\Phi_1, \Phi^{\perp}) + L^{\perp}(\Phi^{\perp}). \tag{4.1}$$

The first piece contains all the dependence on the variables $\Phi_1(x)$ that are attached to links in the 1-direction, and the second piece depends therefore only on the remaining variables

$$\Phi^{\perp} = (\Phi_2, \ldots, \Phi_d). \tag{4.2}$$

If $\mathcal{G}$ consists of $N \times N$ matrices we set

$$(\Phi_1, \Psi_1) \equiv \sum_x \frac{1}{N} \operatorname{tr} \Phi_1^*(x)\Psi_1(x). \tag{4.3}$$

After a partial integration (summation), $L^{\parallel}$ takes the form

$$L^{\parallel}(\Phi) = -\tfrac{1}{2}\left(\Phi_1, [-\Delta^{\perp} + \kappa^2\mathcal{M} + m^2]\Phi_1\right) - \sum_x \mathcal{W}_x\left(\sigma_1(x)^2, \Phi^{\perp}\right). \tag{4.4}$$

$\mathcal{W}_x$ is given by

$$\mathcal{W}_x(\sigma_1^2, \Phi^{\perp}) = \mathcal{V}_1(\sigma_1) + \sum_{\nu=2}^{d} \mathcal{V}_2(\sigma_1, \sigma_\nu(x)). \tag{4.5}$$

The multiplication operator $\mathcal{M}$ and the "transverse covariant laplacian" $\Delta^{\perp}$ depend on $\Phi^{\perp}$ and are defined as follows

$$\begin{aligned}\mathcal{M}\Psi_1(x) &= \tfrac{1}{2}\sum_{\mu=\pm 2}^{\pm d}\left[\Phi_\mu(x+e_1)^*\Phi_\mu(x+e_1)\Psi_1(x) + \Psi_1(x)\Phi_\mu(x)^*\Phi_\mu(x)\right]\\ &\equiv \mathcal{M}(x)\Psi_1(x).\end{aligned} \tag{4.6}$$

If $G = SU(2)$, $\Phi_\mu^*\Phi_\mu(x)$ is a multiple of the unit matrix and commutes with matrices

$\Psi_1(x) \in \mathcal{G}$. Finally

$$-\Delta^{\perp} = \mathfrak{M} - \tilde{R} = \tfrac{1}{2} \sum_{\nu \neq \pm 1} D_\nu^* D_\nu . \tag{4.7}$$

$$\tilde{R}\Psi_1(x) = \sum_{\mu = \pm 2 \ \ldots \ \pm d} \Phi_\mu(x+e_1)^* \Psi_1(x+e_\mu)\Phi_\mu(x)$$

$$\equiv \sum_y \mathfrak{R}(x,y)\Psi_1(y). \tag{4.8}$$

$\mathfrak{R}(x,y)$ is a multiplication operator which is only non-zero if y is a nearest neighbour of x. $\Delta^{\perp}$ is a covariant generalization of the ordinary lattice laplacian which is defined by $\Delta f(x) = \Sigma_y [f(y) - f(x)]$, sum over nearest neighbour y of x.

Let us now consider the expectation value $\langle W(\mathrm{C})\rangle$ of a Wilson loop operator (2.17), with C positioned as in fig. 4, so that its "short legs" $\mathrm{C}_4, \mathrm{C}_2$ point in the 1-direction.

$$W(\mathrm{C}) = \mathrm{tr}(\Phi(\mathrm{C}_4)U(\mathrm{C}_3)\Phi(\mathrm{C}_2)U(\mathrm{C}_1)); \tag{4.9}$$

$U(\mathrm{C}_1)$ and $U(\mathrm{C}_3)$ are determined by $\Phi^{\perp}$.

We start with a finite lattice Λ and derive bounds which are uniform in the lattice size. They remain therefore valid in the infinite-volume limit.

We write $x^1 = s$, $x = (s, \mathrm{r})$, and denote the endpoints of the path C_3 by $(0, \mathrm{r}_1)$ and $(0, \mathrm{r}_2)$. Their distance is $|\mathrm{r}_1 - \mathrm{r}_2| = T$. Thus

$$\langle W(\mathrm{C})\rangle = \frac{1}{Z}\int \prod_x \mathrm{d}\Phi^{\perp}(x) \mathrm{e}^{L^{\perp}(\Phi^{\perp})} U(\mathrm{C}_3)_{\beta_0,\alpha_0} U(\mathrm{C}_1)_{\beta_L \alpha_L}$$

$$\times \int \prod_x \left[\mathrm{d}\Phi_1(x)\mathrm{e}^{-\mathfrak{W}_x(\sigma_1(x)^2, \Phi^{\perp})}\right] \exp\left\{-\tfrac{1}{2}\left(\Phi_1, [-\Delta^{\perp} + \kappa^2\mathfrak{M} + m^2]\Phi_1\right)\right\}$$

$$\times \prod_{s=1}^{L-1} \left[\Phi_1(s,\mathrm{r}_1)_{\alpha_{s+1}\alpha_s} \bar{\Phi}_1(s,\mathrm{r}_2)_{\beta_{s+1},\beta_s}\right]. \tag{4.10}$$

From now on we restrict attention to gauge group $\mathrm{G} = \mathrm{SU}(2)$ so that $\sigma_1(x)$ is real. We insert the Fourier-integral representations of $\mathrm{e}^{-\mathfrak{W}_x}$

$$\mathrm{e}^{-\mathfrak{W}_x(\sigma^2, \Phi^{\perp})} = \int_{-\infty}^{+\infty} \mathrm{d}a\, g_x(a, \Phi^{\perp})\mathrm{e}^{ia\sigma^2}, \qquad (\sigma \in \mathbf{R}_+). \tag{4.11}$$

If the hypotheses of theorem 1 are fulfilled, we may subtract a small fraction of the quadratic mass term $\propto m^2$ and add it to $\mathfrak{W}_x$. The monotonicity assumption guarantees then that the Fourier representation (4.11) exists. Since $\sigma_1(x)^2 = N^{-1}\mathrm{tr}\,\Phi_1(x)^*\Phi_1(x)$ we obtain, upon inserting (4.11)

$$\langle W(C)\rangle = \int \prod_x \mathrm{d}\Phi^\perp(x)\ (\text{as above}) \int \prod_x \left[\mathrm{d}a(x)\, g_x\left(a(x), \Phi^\perp\right) \mathrm{d}\Phi_1(x)\right]$$

$$\times \exp\left\{-\tfrac{1}{2}\left(\Phi_1, \left[-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia\right]\Phi_1\right)\right\}$$

$$\times \prod_{s=1}^{L-1}\left[\Phi_1(s, \mathrm{r}_1)_{\alpha_{s+1}\alpha_s} \bar{\Phi}_1(s, \mathrm{r}_2)_{\beta_{s+1}\beta_s}\right].$$

The operator a is multiplication with $a(x) \in \mathbb{R}$.

The Φ_1-integration is now gaussian, and can be performed after interchange of the order of the integrations. As a result

$$\langle W(C)\rangle = \frac{1}{Z}\int \prod_x \mathrm{d}\Phi^\perp(x) \mathrm{e}^{L^\perp(\Phi^\perp)} U(C_3)_{\beta_0\alpha_0} U(C_1)_{\beta_l\alpha_l}$$

$$\times \int \prod_x \left[\mathrm{d}a(x) g_x\left(a(x), \Phi^\perp\right)\right] \det\left(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia\right)^{-1/2}$$

$$\times \prod_{s=0}^{L-1}\left(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia\right)^{-1}_{\alpha_{s+1}\alpha_s, \beta_{s+1}\beta_s}(s, \mathrm{r}_1; s, \mathrm{r}_2), \tag{4.12}$$

for G = SU(2).

The inverse of $(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1}$ exists if $m^2 > 0$, because $-\Delta^\perp$ and $\mathfrak{M}$ are positive (see below). It is an integral operator whose kernel is the propagator $(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1}(x; y)$. The decay properties of such propagators will be studied in sect. 5.

The partition function Z is represented by the same integral as appears in (4.12), without the last factor $\Pi_s(\ldots)$ in the integrand, and without the factors $U(C_i)$.

5. Random walk representations

Random walk representations for propagators are well-known [4–6] and were studied in detail by Brydges, Fröhlich and Spencer [5]. In this section I will present a self-contained derivation of the random walk representation for the propagators that appeared at the end of sect. 4, and derive some elementary bounds on them, for $\kappa^2 > 0$.

We restrict attention to the gauge group G = SU(2) in the sequel. In this case, $\mathfrak{M}(x)$ is a multiplication with a $\Phi^\perp$-dependent non-negative real number, see eq. (4.6). We assume that $\mathfrak{M}(x) > 0$ to begin with. This restriction can be dropped at the end by a limiting argument; $m^2 > 0$ is always assumed.

$-\Delta^\perp$ is a positive operator because $-\Delta^\perp = \frac{1}{2}\Sigma_{\nu \neq \pm 1} D_\nu^* D_\nu \geqslant 0$. Writing $-\Delta^\perp = \mathfrak{M} - \tilde{R}$ as in (4.7) it follows that $\tilde{R} \leqslant \mathfrak{M}$. Call a site x odd if $\Sigma_{\mu=1}^d x^\mu$ is odd, and even otherwise. Define a unitary operator $\mathfrak{U}$ by $\mathfrak{U}\Psi_1(x) = \pm \Psi_1(x)$ with $(-)$ for odd sites x and $(+)$ for even sites. Then $\mathfrak{U}\mathfrak{M}\mathfrak{U}^* = \mathfrak{M}$ and $\mathfrak{U}\tilde{R}\mathfrak{U}^* = -\tilde{R}$ because the nearest neighbours $x + e_\mu$ of an even site x are odd and vice versa. Of course, $-\mathfrak{U}\Delta^\perp\mathfrak{U}^*$ is also positive. Therefore $\mathfrak{M} + \tilde{R} \geqslant 0$, i.e. $\tilde{R} \geqslant -\mathfrak{M}$. Both inequalities together imply that

$$\tilde{R} = \mathfrak{M}R, \qquad \text{with } |||R||| \leqslant 1. \tag{5.1}$$

Here $|||\cdot|||$ is the operator norm in the Hilbert space with scalar product (4.3).

Consider now the propagators that enter into eq. (4.12). Inserting $-\Delta^\perp = \mathfrak{M}(1 - R)$ we have

$$(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1} = \left(1 - \left[1 + \kappa^2 + (m^2 - 2ia)\mathfrak{M}^{-1}\right]^{-1} R\right)^{-1}$$
$$\times \left(m^2 - 2ia + [1 + \kappa^2]\mathfrak{M}\right)^{-1}, \tag{5.2}$$

a and $\mathfrak{M}$ are operators of multiplication with real numbers $a(x)$ and $\mathfrak{M}(x) > 0$. Therefore

$$\|\left[1 + \kappa^2 + (m^2 - 2ia)\mathfrak{M}^{-1}\right]^{-1}\| = \sup_x \left|\left[1 + \kappa^2 + (m^2 - 2ia(x))\mathfrak{M}(x)^{-1}\right]^{-1}\right|$$
$$\leqslant (1 + \kappa^2)^{-1}. \tag{5.3}$$

As a consequence of the bounds (5.1), (5.3) we have that

$$A \equiv \left[1 + \kappa^2 + (m^2 - 2ia)\mathfrak{M}^{-1}\right]^{-1} R, \qquad \text{has norm} \qquad |||A||| \leqslant (1 + \kappa^2)^{-1}, \tag{5.4}$$

and the first factor on the right-hand side of eq. (5.2) may be expanded into a convergent Neumann series:

$$(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1} = \sum_{n=0,1,}^{\infty} A^n\left(m^2 - 2ia + (1 + \kappa^2)\mathfrak{M}\right)^{-1} \tag{5.5a}$$

$$= \sum_{n=0}^{l-1} A^n\left(m^2 - 2ia + (1 + \kappa^2)\mathfrak{M}\right)^{-1}$$
$$+ A^l(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1}. \tag{5.5b}$$

Because of eq. (4.8) we have

$$A\Psi_1(x) = \left[(1+\kappa^2)\mathfrak{M} + m^2 - 2ia\right]^{-1}\tilde{R}\Psi_1(x)$$
$$= \sum_z \left[(1+\kappa^2)\mathfrak{M}(x) + m^2 - 2ia(x)\right]^{-1}\mathfrak{R}(x,z)\Psi_1(z). \qquad (5.6)$$

$\mathfrak{R}(x,z)$ is only non-zero if x and z are nearest neighbours and $x^1 = z^1$. Therefore, eq. (5.5a) produces a random walk representation for the kernel. If $y^1 = x^1 = s$ we obtain

$$(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1}(y;x)$$
$$= \sum_{\omega_s\ x\to y}\left\{\prod_{(z,z')\in\omega_s}\left[m^2 - 2ia(z) + (1+\kappa^2)\mathfrak{M}(z)\right]^{-1}\mathfrak{R}(z,z')\right\}$$
$$\times\left[m^2 - 2ia(x) + (1+\kappa^2)\mathfrak{M}(x)\right]^{-1}. \qquad (5.6')$$

Summation is over all paths ω_s from x to y which consist of links (z_1, z_2) in the $z^1 = s$ hyperplane. If $y^1 \neq x^1$ the kernel is 0. Upon inserting the explicit expression (4.8) for $\mathfrak{R}$, we obtain the formula

$$(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2 - 2ia)^{-1}_{\gamma\delta,\alpha\beta}(y,x)$$
$$= \sum_{\omega_s\ x\to y} \Phi(\omega_s + e_1)^*_{\alpha\beta}\Phi(\omega_s)_{\delta\beta}\prod_{z\in\omega_s}\left[m^2 - 2ia(z) + (1+\kappa^2)\mathfrak{M}(z)\right]^{-1}$$
$$(5.7)$$

for $x^1 = y^1 = s$.

If the path ω_s consists of links $(z_0, z_1), (z_1, z_2)\dots(z_{n-1}, z_n)$ the product over $z \in \omega_s$ is to be read as a product over z_i, $i = 0\dots n$. The *number of visits* $n_{\omega_s}(z)$ of the path ω_s to site z equals the number of values of i, $i = 0\dots n$, with $z_i = z$; it can be > 1. $\omega_s + e_1$ is the path ω_s shifted by one lattice spacing in the 1-direction. The parallel transporters $\Phi(\omega)$ were defined in eq. (2.4).

From eq. (5.5a) we will obtain a bound on the propagator (for $a(x) \equiv 0$)

$$(-\Delta^\perp + \kappa^2\mathfrak{M} + m^2)^{-1}_{\gamma\delta,\alpha\beta}(y,x) = \left(f_{\delta\gamma,y}, (-\Delta^\perp + \kappa^2\mathfrak{M} + m^2)^{-1}f_{\alpha\beta,x}\right), \qquad (5.8)$$

where $f_{\alpha\beta,x}$ is a $\mathcal{G}$-valued function defined by $[f_{\alpha\beta,x}(z)]_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{xz}\sqrt{N}$. The scalar product (4.3) is in use. Let us use a distance $|x-y|$ = length of the shortest path from x to y,

$$|x-y| = \sum_{\mu=1}^{d}|x^\mu - y^\mu|.$$

According to (5.6), the operator A makes only "single steps". Therefore, the first term in eq. (5.5b) makes no contribution to expression (5.8) if $l \leqslant |x-y|$. Consequently the left-hand side (l.h.s.) of (5.8) obeys

$$|\text{l.h.s. of (5.8)}| \leqslant |||f_{\delta\gamma,y}|||\,|||f_{\alpha\beta,x}|||\,|||A|||^{l}|||(-\Delta^{\perp}+\kappa^2\mathfrak{M}+m^2)^{-1}|||$$

$$\leqslant (1+\kappa^2)^{-l} m^{-2},$$

by (5.4) and the positivity of the operators $-\Delta^{\perp}$, $\mathfrak{M}$. Thus finally

$$|(-\Delta^{\perp}+\kappa^2\mathfrak{M}+m^2)^{-1}_{\gamma\delta,\alpha\beta}(y,x)| \leqslant m^{-2}(1+\kappa^2)^{-|x-y|}. \tag{5.9}$$

This is valid for arbitrary $\Phi^{\perp}$. It shows that the propagator decays exponentially if $\kappa^2>0$.

6. Generalized Fröhlich-Durhuus (random surface) representation and proof of theorem 1

We start from expression (4.12) for the Wilson loop expectation value. We reinsert the integral representation for the determinant

$$\det(-\Delta^{\perp}+\kappa^2\mathfrak{M}+m^2-2ia)^{-1/2}$$
$$=\int\prod_x \mathrm{d}\Phi_1(x)\exp\{i(\Phi_1,a\Phi_1)-\tfrac{1}{2}(\Phi_1,[-\Delta^{\perp}+\kappa^2\mathfrak{M}+m^2]\Phi_1)\}. \tag{6.1}$$

We also insert the random walk representation (5.7) for the propagators in the last factor of the integrand in (4.12). As a result there will appear sums over L-tuples of paths $\omega=(\omega_0,\dots,\omega_{L-1})$, where ω_s is a path from (s,r_2) to (s,r_1) in the $z^1=s$ hyperplane. Let

$$n_\omega(z)=n_{\omega_s}(z),\qquad \text{if } z^1=s$$
$$=\text{number of visits of a path } \omega_s \text{ to } z. \tag{6.2}$$

Then the result takes the form

$$\langle W(\mathrm{C})\rangle=\frac{1}{Z}\int\prod_{x,\mu}\mathrm{d}\Phi_\mu(x)\exp\{-\tfrac{1}{2}(\Phi_1,[-\Delta^{\perp}+\kappa^2\mathfrak{M}+m^2]\Phi_1)+L^{\perp}(\Phi^{\perp})\}$$
$$\times\sum_\omega\prod_z\{\mathrm{d}a(z)g_z(a(z),\Phi^{\perp})[\eta(z)-2ia(z)]^{-n_\omega(z)}\mathrm{e}^{ia(z)\sigma_1(z)^2}\}$$
$$\times\prod_{s=0}^{L-1}\Phi(\omega_s+e_1)^{*}_{\beta_{s+1}\alpha_{s+1}}\Phi(\omega_s)_{\alpha_s\beta_s}U(\mathrm{C}_3)_{\beta_0\alpha_0}U(\mathrm{C}_1)_{\beta_l\alpha_L}, \tag{6.3}$$

with

$$\eta(x) = m^2 + (1+\kappa^2)\mathfrak{M}(x) > 0. \tag{6.3'}$$

Now the a integrations can be done again. Upon inserting the integral representation

$$(\eta - 2ia)^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty d\xi\, \xi^{n-1} e^{-(\eta - 2ia)\xi},$$

it follows from the definition of $g_x(a, \Phi^\perp)$ (after a variable transformation $\eta\xi \to \xi$) that

$$\int_{-\infty}^{+\infty} da\, g_x(a, \Phi^\perp) e^{ia\sigma_1^2}[\eta - 2ia]^{-n} = \frac{\eta^{-n}}{\Gamma(n)} \int_0^\infty d\xi\, \xi^{n-1} e^{-\xi} e^{-\mathfrak{W}_x(\sigma_1^2 + 2\xi\eta^{-1}, \Phi^\perp)}. \tag{6.4}$$

Set

$$d\nu_n(\xi) = \frac{1}{\Gamma(n)} d\xi\, \xi^{n-1} e^{-\xi}, \qquad \text{if } n \geqslant 1, \qquad d\nu_0(\xi) = \delta(\xi)\, d\xi. \tag{6.5}$$

This is a probability measure on the positive real line, i.e. $\int_0^\infty d\nu_n(\xi) = 1$. The result of the a integrations is

$$\begin{aligned}
\langle W(C)\rangle = \frac{1}{Z} \int \prod d\Phi_\mu(x) \exp\left\{ -\tfrac{1}{2}\left(\Phi_1, [-\Delta^\perp + \kappa^2\mathfrak{M} + m^2]\Phi_1\right) + L^\perp(\Phi^\perp)\right\} \\
\times \sum_\omega \int \prod_z \Big(\eta(z)^{-n_\omega(z)} d\nu_{n_\omega(z)}(\xi(z)) \\
\times \exp\left\{ -\mathfrak{W}_z\left(\sigma_1(z)^2 + 2\xi(z)\eta(z)^{-1}, \Phi^\perp\right)\right\}\Big) \\
\times \prod_{s=0}^{L-1} \Phi(\omega_s + e_1)^*_{\beta_{s+1}\alpha_{s+1}} \Phi(\omega_s)_{\alpha_s\beta_s} U(C_3)_{\beta_0\alpha_0} U(C_1)_{\beta_L\alpha_L}.
\end{aligned} \tag{6.6}$$

Given $\Phi^\perp = (\Phi_\nu(x) = U(x+e_\nu, x)\sigma_\nu(x))_{\nu=2\ldots d}$ define $\hat{\Phi}^\perp = (\hat{\Phi}_\nu(x))$ by

$$\hat{\Phi}_\nu(x) = \sigma_\nu(x)\mathbb{1}. \tag{6.7}$$

Then

$$\Phi(\omega_s) = U(\omega_s)\hat{\Phi}(\omega_s), \tag{6.8}$$

and similarly for $\Phi(\omega_s + e_1)^*$.

The L-tuples of paths $\omega = (\omega_s)$ can be looked upon as defining a surface with boundary C. Eq. (6.6) exhibits the Wilson-loop expectation value as a sum of contributions from (random) surfaces ω. For the standard SU(2) lattice gauge theory model, this representation had been derived by Fröhlich and Durhuus [4]. In the standard lattice gauge theory, an area law decay is supposed to come about through destructive interference when the phase factors $U(\omega_s)$ from different paths ω_s are added up. In the present approach, part of this effect (at the level of the fundamental action) is supposed to have been taken care of in the course of the computation of the effective action $L(\Phi)$, when the block spin variables $\Phi(b)$ were constructed as linear superpositions of parallel transporters along different paths. Destructive interference among these will lead to favoring variables $\Phi(b)$ with small modulus $\sigma_\mu(x)$.

In accordance with this physical intuition, we will now proceed to the inequalities which are obtained from eq. (6.6) when all phase factors $U(\cdot)$ downstairs are dropped. This amounts to replacing Φ by $\hat{\Phi}$ in the last factor.

Since $\mathcal{W}_x$ is a non-decreasing function of σ_1^2 by hypothesis, and $\mathrm{d}\nu_n$ is a probability measure on the positive real axis (i.e. "it averages"), we have the inequality

$$\int \mathrm{d}\nu_n(\xi) \mathrm{e}^{-\mathcal{W}_x(\sigma_1^2 + 2\xi\eta^{-1}, \Phi^\perp)} \leqslant \mathrm{e}^{-\mathcal{W}_x(\sigma_1^2, \Phi^\perp)}. \tag{6.9}$$

As a result

$$|\langle W(\mathrm{C})\rangle| \leqslant \frac{1}{Z} \int \prod_{x,\mu} \mathrm{d}\Phi_\mu(x) \mathrm{e}^{L(\Phi)} \delta_{\beta_0\alpha_0} \delta_{\beta_l\alpha_l}$$
$$\times \sum_\omega \left\{ \prod_z \left[m^2 + (1+\kappa^2)\mathcal{M}(z) \right]^{-n_\omega(z)} \right.$$
$$\left. \times \prod_{s=0}^{L-1} \hat{\Phi}(\omega_s + e_1)_{\beta_{s+1}\alpha_{s+1}} \hat{\Phi}(\omega_s)_{\alpha_s\beta_s} \right\}. \tag{6.10}$$

When $\hat{\Phi}^\perp$ is substituted for $\Phi^\perp$, $\mathcal{M}$ remains unchanged whereas $-\Delta^\perp$ gets replaced by $-\hat{\Delta}^\perp$. We can therefore now go backwards and use the random walk representation (5.7) again to do the summations over L-tuples of paths ω in (6.10). In order to handle the special case $\kappa^2 = 0$ later on, we extract a factor first. We split

$$m^2 + (1+\kappa^2)\mathcal{M}(x) = \rho(x)\left[\tfrac{1}{2}m^2 + \left(1 + \tfrac{1}{2}\kappa^2\right)\mathcal{M}(x)\right],$$

with

$$\rho(x) = \frac{1 + \frac{1}{2}\kappa^2 + \frac{1}{2}m^2 \mathcal{M}(x)^{-1}}{1 + \kappa^2 + m^2 \mathcal{M}(x)^{-1}}. \tag{6.11}$$

We extract a factor

$$\sup_{\omega} \prod_{x \in \omega} \rho(x),$$

from the sum over ω in (6.10). After that we do the resummation with the result

$$|\langle W(C)\rangle| \leqslant \frac{1}{Z} \int \prod_{x,\mu} \mathrm{d}\Phi_\mu(x) \mathrm{e}^{L(\Phi)} \delta_{\beta_0 \alpha_0} \delta_{\beta_L \alpha_l} \Big\{ \sup_{\omega} \prod_{x \in \omega} \rho(x) \Big\}$$

$$\times \prod_{s=0}^{L-1} \left(-\hat{\Delta}^{\perp} + \tfrac{1}{2}\kappa^2 \mathcal{M} + \tfrac{1}{2} m^2\right)^{-1}_{\alpha_{s+1}\alpha_s, \beta_{s+1}\beta_s} (s, r_1; s, r_2).$$

Only terms with $\alpha_s = \beta_s$ contribute because $\hat{\Phi}(\omega_s)$ are multiples of $\mathbf{1}$. Now we insert the bound (5.9) on the propagators. It is valid for arbitrary $\Phi^{\perp}$ hence in particular for $\hat{\Phi}^{\perp}$. This gives

$$|\langle W(C)\rangle| \leqslant 2(4m^{-2})^L (1 + \tfrac{1}{2}\kappa^2)^{-LT} \Big\langle \sup_{\omega} \prod_{x \in \omega} \rho(x) \Big\rangle. \tag{6.12}$$

In case $\kappa^2 > 0$, the area law follows from this right away. Since $\mathcal{M}(z) \geqslant 0$, it follows that $0 \leqslant \rho(z) \leqslant (1 + \frac{1}{2}\kappa^2)/(1 + \kappa^2)$. Therefore the expectation value is $\leqslant [(1 + \frac{1}{2}\kappa^2)/(1 + \kappa^2)]^{LT}$. Therefore

$$|\langle W(C)\rangle| \leqslant 2(4m^{-2})^L \mathrm{e}^{-\alpha LT}, \tag{6.13}$$

with $\alpha = \ln(1 + \kappa^2) > 0$.

The case $\kappa^2 = 0$ is more complicated. Inequality (6.12) tells us that

$$|\langle W(C)\rangle| \leqslant 2(4m^{-2})^L \left\langle \sup_{\omega} \prod_{x \in \omega} \left(\frac{1 + \frac{1}{2} m^2 \mathcal{M}(x)^{-1}}{1 + m^2 \mathcal{M}(x)^{-1}} \right) \right\rangle. \tag{6.14}$$

Since

$$\mathcal{M}(x) = \tfrac{1}{2} \sum_{\mu=2}^{d} \left[\sigma_\mu(x)^2 + \sigma_\mu(x + e_1)^2\right] \geqslant 0,$$

$\mathcal{M}(x)^{-1}$ is non-negative but it can become arbitrarily small. This requires, however, that $\sigma_\mu(x)^2$ or $\sigma_\mu(x + e_1)^2$ is very large for some μ. Because of the hypothesis of theorem 1, the probability that this is the case is very small. The potential is bounded below and the quadratic mass term suppresses large σ_μ^2 by a factor $\exp[-\frac{1}{2}m^2\sigma_\mu^2]$. By standard procedure one can prove that the probability

$$\operatorname{prob}\left(\mathcal{M}(x_1)^{-1} < \varepsilon, \dots, \mathcal{M}(x_n)^{-1} < \varepsilon\right) < \mathrm{e}^{-n\lambda/\varepsilon}, \tag{6.15}$$

for some $\lambda > 0$, if $\varepsilon > 0$ is small enough and $x_1 \dots x_n$ are n *distinct* points. This inequality can be derived by using either chessboard estimates [20] (for a similar application see e.g. sect. 8 of ref. [21], or superstability [22].

The sum over ω is over all L-tuples of paths ω_s, $s = 0 \dots L-1$, with fixed initial and final points. These paths visit at least $T+1$ distinct points. Each point $x \in \omega_s$ visited by ω_s contributes a factor <1 to $\prod_{x \in \omega}(\dots)$ in (6.14). Therefore the sup can only be increased if we abandon the requirement that the end points be fixed and strip the path ω_s down to a self-avoiding walk of length precisely T by removing loops and cutting away a piece at the end if necessary. Therefore inequality (6.14) remains valid when the sup is read as a sup over L-tuples ω of self-avoiding walks ω_s of length T with a prescribed initial point. In the following we only consider ω's with these properties, and sups and sums over ω are restricted to these.

Set $n = L(T+1)$ = numbers of sites $x \in \omega$. All of these sites are distinct because ω_s are self-avoiding. Then

$$\left\langle \sup_{\omega} \prod_{x \in \omega} \frac{1 + \frac{1}{2} m^2 \mathfrak{M}(x)^{-1}}{1 + m^2 \mathfrak{M}(x)^{-1}} \right\rangle$$

$$\leqslant \left(\frac{1 + \frac{1}{2}\varepsilon m^2}{1 + \varepsilon m^2} \right)^{-(1/2)n} + \sum_{\substack{k \\ n \geqslant k > \frac{1}{2}n}} \left(\frac{1 + \frac{1}{2}\varepsilon m^2}{1 + \varepsilon m^2} \right)^{-n+k} \operatorname{prob} \left\{ \begin{matrix} \text{there exists } \omega \text{ such} \\ \text{that } \mathfrak{M}(x) > \varepsilon^{-1} \text{ for} \\ k \text{(distinct) sites } x \in \omega \end{matrix} \right\}$$

$$\leqslant \left(\frac{1 + \frac{1}{2}\varepsilon m^2}{1 + \varepsilon m^2} \right)^{-(1/2)n} + \sum_{n \geqslant k > \frac{1}{2}n} \left(\frac{1 + \frac{1}{2}\varepsilon m^2}{1 + \varepsilon m^2} \right)^{-n+k} \sum_{\omega} \operatorname{prob} \left\{ \begin{matrix} \mathfrak{M}(x) > \varepsilon^{-1} \text{ for} \\ k \text{ sites on } \omega \end{matrix} \right\}.$$

The number of all sets I of k sites x on ω is $\binom{n}{k}$, and the number of all paths ω_s of length T with given initial point is $\leqslant (2d)^T$. Together with the bound (6.15) this implies that

$$\sum_{\omega} \operatorname{prob} \left\{ \begin{matrix} \mathfrak{M}(x) > \varepsilon^{-1} \text{ for} \\ k \text{ sites on } \omega \end{matrix} \right\} \leqslant \sum_{\omega} \sum_{\substack{\mathrm{I} \subset \omega \\ |\mathrm{I}| = k}} \operatorname{prob}\{ \mathfrak{M}(x) > \varepsilon^{-1} \text{ for all sites } x \in \mathrm{I} \}$$

$$\leqslant (2d)^n \binom{n}{k} \mathrm{e}^{-k\lambda\varepsilon^{-1}}.$$

Setting $k = \frac{1}{2}n + l$ gives

$$\langle \sup \Pi - \rangle \leqslant \left(\frac{1 + \frac{1}{2}\varepsilon m^2}{1 + \varepsilon m^2} \right)^{-(1/2)n}$$

$$\times \left\{ 1 + (2d)^n \mathrm{e}^{-(1/2)n\lambda\varepsilon^{-1}} \sum_{\frac{1}{2}n \geqslant l > 0} \binom{n}{l + \frac{1}{2}n} \left[(1 + \varepsilon m^2) \mathrm{e}^{-\lambda\varepsilon^{-1}} \right]^l \right\}.$$

Now we choose ε so small that

$$(1+\varepsilon m^2)e^{-\lambda\varepsilon^{-1}} < 1, \qquad e^{-\lambda\varepsilon^{-1}} < (8d)^{-1}.$$

The sum over l is then bounded by $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. As a result

$$\langle \sup\Pi \dotdiv \rangle \leqslant \left(\frac{1+\frac{1}{2}\varepsilon m^2}{1+\varepsilon m^2}\right)^{-(1/2)n} [1+2^{-n}].$$

We insert this into inequality (6.14) and remember that $n = L(T+1)$. In this way we obtain the desired area law (2.21) with $\alpha = \frac{1}{2}\ln[(1+\varepsilon m^2)/(1+\frac{1}{2}\varepsilon m^2)] > 0$.

7. Physical positivity

Physical positivity can be expressed in terms of euclidean expectation values (2.22′). This is the celebrated Osterwalder-Schrader positivity condition [9]. It states that if O is any polynomial of positive time ($x^4 \geqslant 1$) fields we should find

$$\langle \Theta(O^+)O\rangle \geqslant 0. \tag{7.1}$$

Here, Θ denotes euclidean time reflection and O^+ is the complex conjugate of O (e.g. $\bar{\psi}^+ = \gamma_4\psi$). In 4 dimensions

$$\Theta\psi(x) = \psi(\Theta x), \qquad \text{etc.},$$

$$(\Theta x)^4 = -x^4, \qquad (\Theta x)^i = x^i, \qquad (i=1,2,3). \tag{7.2}$$

We assume that our lattice is positioned so that $x^4 = 0$ is a lattice plane containing sites, and that the boundary conditions are Θ-invariant. It will be shown that the expectation values of a theory with action $L(\Phi) + L_{\text{matter}}(\psi, \psi^+, \Phi) \equiv L_{\text{tot}}$ as defined by eqs. (2.11), (2.14), (2.22) satisfy the positivity condition (7.1) if the condition (2.23), viz. $K_1^2 \leqslant \frac{2}{3}k$, is fulfilled by the parameters in the matter action.

With any O as described above a wave function Ψ_O will be associated. A positive (semi) definite scalar product (,) of such wave functions will be defined so that

$$(\Psi_{O_1}, \Psi_{O_2}) = \langle \Theta(O_1^+)O_2\rangle. \tag{7.3}$$

Inequality (7.1) follows from positive semidefiniteness of the scalar product. The scalar product makes the space of all wave functions with finite norm $(\Psi,\Psi)^{1/2}$ into a Hilbert space $\mathcal{H}$ (division by the subspace of all null vectors and completion is understood). $\mathcal{H}$ is the quantum mechanical Hilbert space of states and Ψ are its states in the "Schrödinger representation" [23]. It is convenient to use the following hermitean representation of γ-matrices ($\gamma_\mu = \gamma_\mu^* = -\gamma_{-\mu}$)

$$\gamma_4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma_j = i\begin{pmatrix} 0 & \tau_j \\ -\tau_j & 0 \end{pmatrix} \qquad (\tau_j\text{: Pauli matrices}, j = 1,2,3). \quad (7.4)$$

We decompose the Dirac spinors into upper and lower components

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \qquad \psi^+ = (\psi_+^+, \psi_-^+). \quad (7.5)$$

For pedagogical reasons, I will consider the case of free boundary conditions first. Let Σ be the time zero hyperplane $x^4 = 0$ ("space"). In canonical quantum field theory, ψ and ψ^+ are conjugate variables. Therefore, in a Schrödinger representation the wave function should only depend on half of these variables. (In quantum mechanics, position Q and momentum P are conjugate variables, and the Schrödinger wave function depends only on Q.)

Our wave functions $\boldsymbol{\Psi}$ will be functions of the variables $\Phi(\boldsymbol{b}), \psi_+^+(\boldsymbol{x}), \psi_-(\boldsymbol{x})$ attached to spacelike links $\boldsymbol{b}$ and sites $\boldsymbol{x}$ in space Σ.

We make a split of the total action so that

$$L_{\text{tot}}(\psi, \psi^+, \Phi) = L_+(\psi, \psi^+, \Phi) + L_-(\psi, \psi^+, \Phi) + \mathscr{L}, \quad (7.6)$$

with

$$\mathscr{L} = \sum_{x \in \Sigma} \Big\{ -\bar{\psi}(x)\psi(x) + \sum_{i=-1,\ldots,+3} \times \big[K_1 \bar{\psi}(x+e_i)\Phi_i(x)\psi(x) - k\bar{\psi}(x)\Phi_i(x)^*\Phi_i(x)\psi(x) \big] \Big\}, \quad (7.6')$$

and L_+ depends only on variables $\Phi(b), \psi(x), \psi^+(x)$ attached to links b and sites x in the positive time halfspace (excluding those in Σ, but including links $(x+e_4, x)$ with $x \in \Sigma$), and on variables $\Phi(\boldsymbol{b}), \psi_+^+(\boldsymbol{x}), \psi_-(\boldsymbol{x})$ attached to links $\boldsymbol{b}$ and sites $\boldsymbol{x}$ in Σ. Furthermore it is required that

$$L_- = \Theta(L_+)^+. \quad (7.7)$$

The possibility of a split (7.6) with these properties is the crucial feature which will allow us to define wave functions $\boldsymbol{\Psi}$, by eq. (7.9) below, which depend only on the variables listed above. There are three things to be checked to see that such a split is possible – the rest is obvious.

(i) The contribution to L_{matter} from the links $(x+e_i, x)$ with $x \in \Sigma$ should admit the split. Referring to eq. (2.22) it is seen that those contributions which do not contain matrices γ_j make up the term $\mathscr{C}$ in (7.6). Because of the off-diagonal form of the matrices γ_i; the remaining terms take the form

$$\sum_x \sum_{j=\pm 1, \ldots \pm 3} \left\{ -iK_1 \psi_+^+(x+e_j)\tau_j \Phi_j(x)\psi_-(x) + ik\psi_+^-(x)\tau_j\Phi_j(x)^*\Phi_j(x)\psi_-(x) \right.$$

$$\left. + iK_1\psi_-^-(x+e_j)\tau_j\Phi_j(x)\psi_+(x) - ik\psi^-(x)\tau_j\Phi_j(x)^*\Phi_j(x)\psi_+(x) \right\}. \tag{7.8}$$

The first term depends only on Φ_j, ψ_-^- and ψ_-, and is incorporated into L_-, and the second one in L_-.

(ii) The contribution from links $(x+e_4, x)$ with $x \in \Sigma$ should not depend on $\psi^+(x)$ or $\psi_+(x)$. The presence of the projection operator $(1-\gamma_4)$ assures this. This is the reason why the projection operator $(1-\gamma_\mu)$ in the last term in (2.22) was included.

(iii) Eq. (7.7) needs to be checked. The $(+)$ operation is an antiautomorphism of the Grassmann algebra which takes c-numbers into their complex conjugate, ψ_α into ψ_α^+, ψ_α^+ into ψ_α, and $(AB)^+ = B^+A^+$. Using these properties, the required equality is straightforward to check. For instance, the second line in (7.8) is the $(+)$-conjugate of the first (when summed over x, i). To see this one needs to use that $\Phi_{-j}(x) = \Phi_j(x-e_j)^*$ by (2.0).

Now we are ready to define the wave functions

$$\Psi_O(\{\Phi_i(x), \psi_+^+(x), \psi_-(x)\}_{x\in\Sigma}) = Z^{-(1/2)} \int \prod_{x,\mu,\alpha}{}' \left[d\Phi_\mu(x)\, d\psi_\alpha^+(x)\, d\psi_\alpha(x) \right]$$

$$\times O(\Phi, \psi^+, \psi) e^{L(\psi, \psi^+, \Phi)}. \tag{7.9}$$

The product $\prod'$ runs over x, μ with $x^4 > 0$, $\mu = 1 \ldots 4$ or $x^4 = 0$, $\mu = 4$, and over Dirac and colour indices α. The variables $\psi_+^+(x)$, $\psi_-(x)$ for $x \in \Sigma$ are not integrated, instead the wave function Ψ_O depends on them.

Next, the scalar product of two wave functions will be defined. We introduce a new notation

$$\psi_-(x) = \chi_-^-(x)\varepsilon, \qquad \psi_-^+(x) = \varepsilon^{-1}\chi_-(x), \tag{7.10}$$

ε = antisymmetric tensor in 2 dimensions. (It acts on spinor indices and converts a column vector into a row vector in a rotation covariant way). With this notation, the wave functions Ψ depend on ψ_+^+, χ_-^+, and their adjoint Ψ^+ are functions of ψ_+ and

χ_-. Expression (7.6′) for $\mathscr{Q}$ takes the form

$$\mathscr{Q} = -\sum_x \left[\Psi_+^+(x)(A\Psi_+)(x) + \chi_-^+(x)(\bar{A}\chi_-)(x) \right], \tag{7.11a}$$

with

$$(A\Psi_+)(x) = \left[1 + k \sum_{i=\pm 1 \quad \pm 3} \Phi_i^*(x)\Phi_i(x) \right] \Psi_+(x)$$
$$- K_1 \sum_{i=\pm 1 \quad \pm 3} \Phi_i(x - e_i)\Psi_+(x - e_i). \tag{7.11b}$$

The entries of the matrix $\bar{A}$ are the complex conjugate of those of A. The scalar product of two wave functions is defined by

$$(\Psi_1, \Psi_2) = \frac{1}{Z} \int \prod_{b \in \Sigma} \mathrm{d}\Phi(b) \prod_{x \in \Sigma} [\mathrm{d}\psi_+^+(x)\mathrm{d}\psi_+(x)\mathrm{d}\chi^+(x)\mathrm{d}\chi_-(x)]$$
$$\times \mathrm{e}^{\mathscr{Q}} (\Psi_1(\{\Phi_i(x), \psi_-^+(x), \chi_-^+(x)\}))^+ \Psi_2(\{\Phi_i(x), \Psi_-^+(x), \chi_-^+(x)\}), \tag{7.12}$$

with $\mathscr{Q}$ given by eq. (7.11).

With these definitions, the relation (7.3) between scalar products of wave functions and expectation values is satisfied. To see this one needs only insert the definition (7.9) of Ψ_O, relabel the variables of integration $\psi(x) \to \psi(\Theta x)$ in the integral representation for $\Psi_{O_1}^+$, and refer to (7.6).

It remains for us to examine whether the scalar product (7.12) is positive definite. An answer is provided by the following well-known lemma [16, 14, 19].

Lemma. Let the Grassmann algebra $\mathfrak{a}$ be generated by the totally anticommuting objects $a_1, a_1^+, \dots, a_n, a_n^+$, and consider functions Ψ on the subalgebra $\mathfrak{a}^+$ which is generated by $a_1^+ \dots a_n^+$. The scalar product

$$(\Psi_1, \Psi_2) = \int \mathrm{d}a_n^+ \, \mathrm{d}a_n \dots \mathrm{d}a_1^+ \, \mathrm{d}a_1 \, \mathrm{e}^{-\Sigma a_i^+ A_{ij} a_j} (\Psi_1(a^+))^+ \Psi_2(a^+),$$

is positive definite if the matrix $A = (A_{ij})$ is positive definite.

(Integrals over Grassmann variables admit linear changes of variables. Therefore it suffices to have the validity of the lemma for $\Sigma a_i^+ A_{ij} a_j = \Sigma a_j^+ a_j$. In this case its validity follows from the well-known isomorphism of $\mathfrak{a}^+$ with a Fock space.) The matrix A given by eq. (7.11) is positive (semi-) definite if $K_1^2 \leqslant \frac{3}{2}k$. This is seen as follows. Define the covariant shift operator S_i by

$$S_i \psi_+(x) = \Phi_i(x - e_i)\psi_+(x - e_i).$$

Its adjoint $S_i^* = S_{-i}$ because of (2.0), and

$$A = \sum_{i=\pm 1 \quad \pm 3} \left[\tfrac{1}{6}\left(1 - 3K_1 S_i^*\right)\left(1 - 3K_1 S_i\right) + \left(k - \tfrac{3}{2}K_1^2\right) S_i^* S_i\right].$$

This is manifestly positive if $K_1^2 \leqslant \frac{2}{3}k$. This completes the proof of Osterwalder-Schrader positivity, for free boundary conditions.

Osterwalder-Schrader positivity holds also for antiperiodic boundary conditions in the time direction and either periodic or antiperiodic boundary conditions in the space directions. In this case, reflection Θ leaves a pair $\Sigma = (\Sigma_+, \Sigma_-)$ of hyperplanes $x^4 = \text{const.}$ invariant, see fig. 5.

The Schrödinger wave function Ψ depends on $\psi_+^{\dagger}$, ψ_- on Σ_+ and on ψ^+, ψ_+, on Σ . We make the substitution

$$\psi_- = \chi_-^+ \varepsilon, \qquad \psi_-^{\dagger} = \varepsilon^{-1} \chi_-, \qquad \text{on } \Sigma_+ ,$$

$$\psi_- = \chi_-^+ \varepsilon, \qquad \psi_+^{\dagger} = \varepsilon^{-1} \chi_+, \qquad \text{on } \Sigma \ .$$

For antiperiodic boundary conditions in time direction, eq. (7.3) with (7.12) gets replaced by

$$\begin{aligned}\langle \Theta(O^-)O \rangle = \frac{1}{Z} \int \prod_{b \in \Sigma} \mathrm{d}\Phi(b) \prod_{x \in \Sigma_+} &\left[\mathrm{d}\psi_+^{\dagger}(x)\, \mathrm{d}\psi_+(x)\, \mathrm{d}\chi^+(x)\, \mathrm{d}\chi_-(x)\right] \\ &\times \prod_{x \in \Sigma} \left[\mathrm{d}\psi_-^{+}(x)\, \mathrm{d}\psi_-(x)\, \mathrm{d}\chi_+^{\dagger}(x)\, \mathrm{d}\chi_+(x)\right] \mathrm{e}^{\mathscr{A}} \\ &\times \Psi\left(\{\Phi, \psi_+^+, \chi^-\}_{\Sigma_+}, \{\Phi, -\psi_-^{\dagger}, -\chi_-^{\pm}\}_{\Sigma_-}\right)^+ \\ &\times \Psi\left(\{\Phi, \psi_+^{\dagger}, \chi_-^+\}_{\Sigma_-}, \{\Phi, \psi_-^{\dagger}, \chi_-^+\}_{\Sigma_-}\right),\end{aligned} \tag{7.13}$$

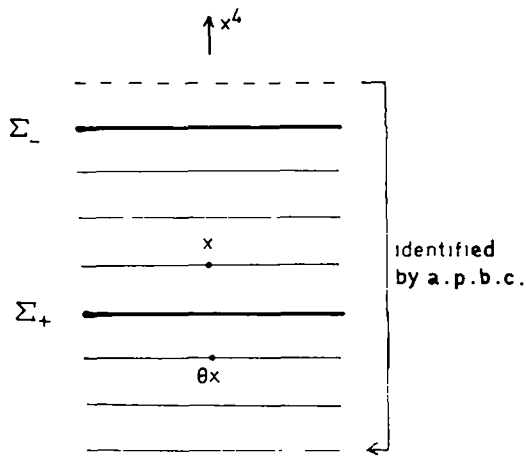

Fig 5 Reflections in a lattice with antiperiodic boundary conditions

in obvious notation, with

$$-\mathscr{Q} = \sum_{x\in\Sigma_+} \left[\psi_-^+(x)(A\psi_+)(x) + \chi_-^+(x)(\bar{A}\chi_-)(x)\right]$$

$$- \sum_{x\in\Sigma_-} \left[\psi_-^+(x)('\bar{A}\psi_-)(x) + \chi_+^+(x)('A\chi_+)(x)\right].$$

The lemma can be applied again. Since ψ, ψ^+ are independent integration variables, they can be transformed independently. By a variable transformation $\psi_- \to -\psi_-, \chi_- \to -\chi_-$ the minus sign in the second term of $-\mathscr{Q}$ can be cancelled against the $(-)$ sign in the argument $(-\psi_-, -\chi_-)_\Sigma$ of the wave function Ψ^+ which came from the use of antiperiodic boundary conditions. As a result, the positivity of expression (7.13) is obtained. This completes the proof of Osterwalder-Schrader positivity in the case of antiperiodic boundary conditions in the time direction.

We note, finally, that one can also write down a hermitean transfer matrix. It is given by a formula which involves an integration over variables $\Phi_4(x)$ attached to timelike links.

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