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UNEXPECTED BEHAVIOR OF AN ORDER PARAMETER FOR LATTICE GAUGE THEORIES WITH MATTER FIELDS

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I consider a slightly modified definition of an order parameter that was recently suggested by DeTar and McLerran. It is supposed to test for confinement in lattice gauge theories when arbitrary matter fields are present, at finite physical temperature $\beta^{-1} > 0$. Its definition is quite directly related to confinement in the sense that no physical states with fractional baryon number can be observed. We test the parameter for different ranges of the coupling constants in the Z(2) Higgs model, whose phase structure is well known at zero temperature. It is found that the order parameter always shows the behavior characteristic of confinement, for all values of the coupling constants and arbitrary non-zero temperature.

1. Introduction

Lattice gauge theories can have different phases, since they are described as systems of classical statistical mechanics. In approximative models for QCD one is mainly interested in phase transitions which can influence the confinement property of the system, when parameters such as the physical temperature β^{-1} or the bare coupling constant g_0^{-2} are varied. Confinement is understood in the sense that there are no physical states with the flavor quantum numbers of single quarks (fractional baryon number in QCD). Suitable order parameters should distinguish between different phases such that confinement is implied if they behave in a specific way.

For pure gauge theories order parameters are well known that fulfill this task: at zero temperature it is the Wilson/Wegner loop [1] and at finite temperature the Wilson/Polyakov line [2].

Since they can no longer be used when matter fields are present that transform non-trivially under the center of the gauge group, DeTar and McLerran [3] proposed a quantity as a candidate for an order parameter that was supposed to test for confinement in the presence of quark fields, at finite temperature β^{-1}. The idea is to calculate the probability of finding in the Gibbs ensemble a quantum mechanical state that transforms non-trivially under the center $Z(N)$ of the gauge group $SU(N)$.

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It might seem natural to say that confinement occurs at zero temperature if it occurs at arbitrarily small temperature, and the order parameter should be able to test for that.

I have investigated a slightly different definition of the parameter which takes care of the necessity to impose proper boundary conditions. It exhibits the order parameter as a limit of a finite-volume quantity.

For the Z(2) Higgs model it will be shown that the definition leads to a negative result. It is found that the parameter exhibits the qualitative behavior that is supposed to be characteristic of confinement for all values of the two coupling constants and for arbitrarily small but finite temperature. The proof of this result is contained in sect. 3, following the precise definition of the parameter in sect. 2.

I will now discuss the meaning of this unexpected result. It will be argued that it is inappropriate to define confinement at zero temperature as confinement for arbitrarily small temperature. Indeed it is doubtful that the notion of confinement versus deconfinement has a well-defined meaning in theories with quark fields (i.e. matter fields that transform non-trivially under the center of the gauge group) at finite non-zero temperature at all. Recent results of DeGrand, DeTar and McLerran point in the same direction [4].

To speak of confinement one needs to consider a system in infinitely extended space. However, for a system in infinitely extended space it is *incorrect* to write expectation values of observables as $\langle O \rangle = \mathrm{Tr}\, \rho O$ with a trace that runs over the same Hilbert space of physical states that is relevant at zero temperature (ρ is the density operator). This point has been emphasized by Araki, Haag and Kastler (cf. the treatment of the thermodynamical limit in the C^* algebra formulation of statistical mechanics [5]). Roughly speaking, the states at zero temperature have a finite particle number (if there is a mass gap) hence zero particle density, while the relevant states at finite temperature have finite particle density. Therefore one cannot check whether there exist states at zero temperature with fractional baryon number (non-trivial transformation law under the center of the gauge group) by looking at what states contribute to the trace in $\mathrm{Tr}\, \rho O$ at finite temperature. Moreover, in a state with finite particle density there is no obvious way how the total baryon-number modulo 1 could be determined (e.g. as a limit of a finite-volume quantity). The result of the present calculations suggest that it cannot be done.

The Z(2) Higgs model is known to possess two distinct phases at zero temperature. It was shown by Marra and Miracle Solé [6] that there are two ranges (I), (II) of the coupling constants where strong-coupling expansions (I) respectively "low-temperature" expansions (II) converge. Fredenhagen and Marcu have shown [7] that charged states exist in region (II) which can be handled by low-temperature expansions, but not in the other regions. Monte Carlo work confirmed [8] that the two regions are separated by a line of phase transitions.

We have examined whether the result of Marra and Miracle Solé extends to small but finite temperature. If it did, one would have a candidate model with matter fields for a deconfining phase transition at finite temperature. It turns out, however, that the convergence of the low-temperature expansions breaks down for arbitrarily small but non-zero temperature. This is shown in sect. 4.

2. Definition of the parameter

According to the definition of quark confinement, in nature confinement holds if there are no physical states with fractional baryon number. To derive this property from QCD it is sufficient to show that all physical states transform trivially under global transformations in the center Z(3) of the gauge group SU(3) [9]. Then the probability of finding a state in the Gibbs ensemble with N-ality n, i.e. a state that transforms according to the nth irreducible representation of the center $Z(N)$, (for finite volume Λ) is given by

$$\frac{Z_n}{Z} = \frac{\mathrm{Tr}_{\mathcal{H}_n} e^{-\beta H_\Lambda}}{\mathrm{Tr}_{\mathcal{H}} e^{-\beta H_\Lambda}}. \tag{2.1}$$

H_Λ is the hamiltonian in a finite space volume Λ defined by the transfer matrix $T = e^{-H_\Lambda a_t}$ (a_t = lattice spacing in time direction). The traces $\mathrm{Tr}_{\mathcal{H}}$ and $\mathrm{Tr}_{\mathcal{H}_n}$ run over the Hilbert space of all physical states ψ and all physical states ψ_n with N-ality n, respectively. DeTar and McLerran proposed to use Z_n/Z as an order parameter [3].

Consider a system that is defined on a (3 + 1)-dimensional hypercubic lattice $\Lambda_\beta = \Lambda \times [0, \beta]$ with lattice spacing $a = 1$, extensions β in the time direction and d_i in spatial directions, $i = 1,2,3$. β and d_i are finite.

The path integral for the partition function Z has the general form

$$Z_{\Lambda_\beta}(\mathrm{b.c.}) = \int_{\mathrm{b.c.}} \mathcal{D}U \int_{\mathrm{b.c}} \mathcal{D}q \; e^{L(U,q)}. \tag{2.2}$$

It depends on the lattice Λ_β and the boundary conditions (b.c.) imposed on the gauge fields $U \in SU(N)$ and the matter fields q (q can be Higgs fields or fermion fields). The measures $\int \mathcal{D}U$ and $\int \mathcal{D}q$ and the action $L(U, q)$ will be specified later. We choose periodic boundary conditions (p.b.c.) for the gauge and matter fields in time and space directions, antiperiodic boundary conditions (a.p.b.c.) in the time direction for fermions.

Instead of (2.1) I have investigated a modified definition of the order parameter for reasons that will now be explained.

Consider the Fourier transform $\tilde{Z}_k$ of Z_n on the center $Z(N)$ of $SU(N)$, given by

$$\tilde{Z}_k = \sum_{n=0}^{N-1} e^{2\pi i k n/N} Z_n. \tag{2.3}$$

In the confinement phase Z_n should be zero for $n \neq 0$, corresponding to $\tilde{Z}_k$ independent of k. Define the twist γ^k in the representation D of $Z(N)$ on states $\psi_n \in \mathcal{H}_n$ by

$$D(\gamma^k)\psi_n := e^{2\pi i k n/N}\psi_n. \tag{2.4}$$

If we choose p.b.c. for $q(x)$ in Z (2.2), it follows that

$$\tilde{Z}_k = \mathrm{Tr}_{\mathcal{H}}\left(D(\gamma^k)e^{-\beta H_\Lambda}\right) = \int \underset{\text{p.b.c.}}{\mathcal{D}U}\ \underset{\text{t.b.c.}}{\mathcal{D}q}\ e^{L(U,q)}. \tag{2.5}$$

t.b.c. denote twisted boundary conditions for $q(x)$. The meaning of t.b.c. depends on k and is given by

$$q(\boldsymbol{x},\beta) = \gamma^k q(\boldsymbol{x},0), \tag{2.6}$$

($\boldsymbol{x}$ = space coordinate, β = time coordinate, $\boldsymbol{x} \in \Lambda$).

The action of the twist γ^k amounts to a change in the boundary conditions of $\tilde{Z}_k$ compared to Z. However, the k-dependence of $\tilde{Z}_k$ can be eliminated by a change of gauge field variables according to [10]:

$$\begin{aligned} U(b) &\to \gamma^k U(b), \qquad \text{if } b \in \mathrm{T}, \\ U(b) &\to U(b), \qquad \text{otherwise}. \end{aligned} \tag{2.7}$$

T is the set of timelike links b that touch the $t=0$ slice in a site x and point in the box Λ_β. Therefore $\tilde{Z}_k$ is not yet a suitable candidate for an order parameter. The possibility to eliminate the k-dependence of $\tilde{Z}_k$ follows from the Gauss law for abelian gauge fields. The Gauss law enforces N-ality zero inside Λ_β if we choose p.b.c. (t.b.c.) in Z and in $\tilde{Z}_k$ as in (2.5). That is why we restrict the twist to a spatial three-dimensional subset B of Λ in the $t=0$ hyperplane. A projection of B on two space dimensions is shown in fig. 1.

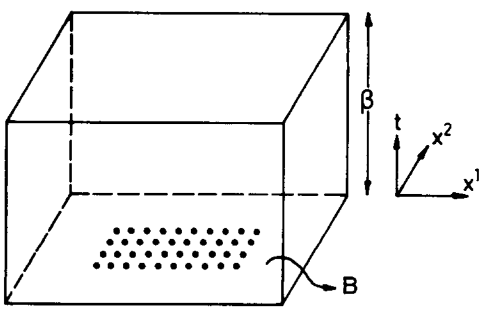

Fig. 1. Subset B of sites x in the $t=0$ hyperplane for two space dimensions.

The restricted twist γ_{B}^{k} acts on states according to

$$D(\gamma_{\mathrm{B}}^{k})\psi(U,q)=\psi(U,q'),$$

$$q'(x)=\mathrm{e}^{2\pi i k\chi_{\mathrm{B}}n/N}q(x), \tag{2.8}$$

$n=1$ for quarks, $k=0,1$ and $N=2$ for gauge groups SU(2) and Z(2).

$$\chi_{\mathrm{B}}(x)=\begin{cases}1, & \text{if } x\in\mathrm{B}\\ 0, & \text{otherwise}.\end{cases} \tag{2.9}$$

Instead of $\tilde{Z}_k$ consider now

$$\tilde{Z}_{\Lambda_\beta}(\mathrm{B},k)=\int \underset{\text{p.b.c.}}{\mathcal{D}U}\ \underset{\text{t.b.c.(B)}}{\mathcal{D}q}\ \mathrm{e}^{L(U,q)} \tag{2.10}$$

where t.b.c.(B) specify boundary conditions

$$q(\boldsymbol{x},\beta)=\begin{cases}\gamma^k q(\boldsymbol{x},0), & \text{if } x\in\mathrm{B}\\ q(\boldsymbol{x},0), & \text{otherwise},\end{cases} \tag{2.11}$$

if $q(x)$ are Higgs fields. For fermions substitute a.p.b.c. for p.b.c. in the time direction.

The order parameter is now defined in terms of the k-dependence of the free energy $\ln \tilde{Z}_{\Lambda_\beta}(\mathrm{B},k)$ by

$$\tilde{P}=\lim\frac{1}{|\mathrm{B}|}\tilde{P}_{\mathrm{B}},$$

$$\tilde{P}_{\mathrm{B}}=\lim_{\Lambda\to\infty}\left\{\ln\tilde{Z}_{\Lambda_\beta}(\mathrm{B},0)-\ln\tilde{Z}_{\Lambda_\beta}(\mathrm{B},1)\right\}, \tag{2.12}$$

for gauge groups SU(2) and Z(2). More generally one could look for different $|\mathrm{B}|$-dependence of $\tilde{P}_{\mathrm{B}}$. $|\mathrm{B}|$ is the volume of B. Confinement should correspond to $\tilde{Z}_{\Lambda_\beta}(\mathrm{B},k)\approx$ independent of k. More precisely, we expect

$$\tilde{P}=0,\qquad \text{in the confinement phase},$$

$$\tilde{P}\neq 0,\qquad \text{in the deconfinement phase}. \tag{2.13}$$

In the confinement phase only boundary terms should contribute to the difference of $\ln\tilde{Z}_{\Lambda_\beta}(\mathrm{B},1)$ and $\ln\tilde{Z}_{\Lambda_\beta}(\mathrm{B},0)$, e.g. contributions from mesons that lie partly inside partly outside B.

Let us illustrate the intuitive picture behind definition (2.12) in figs. 2a–d. We have drawn "mesons" with N-ality ±1 in small and large volumes $|\mathrm{B}|$. For small

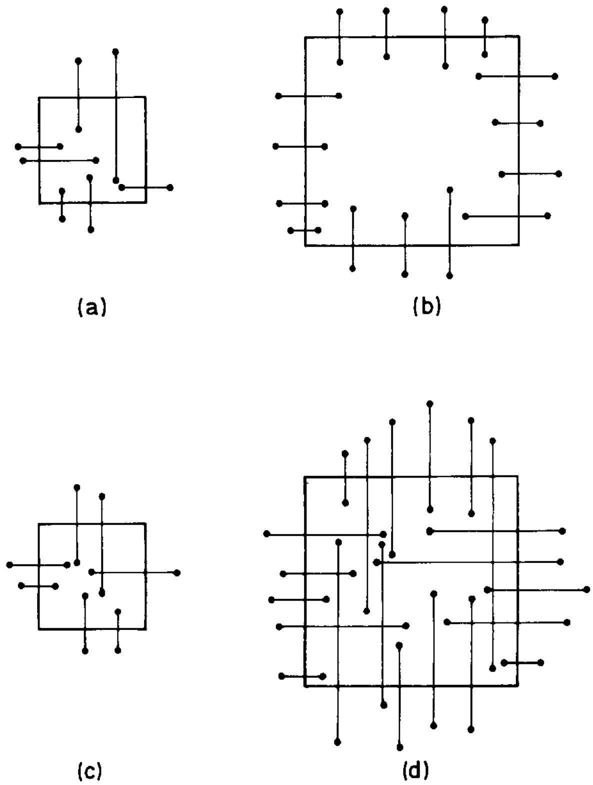

Fig. 2. Gas picture of the distribution of "mesons" for small and large volumes $|B|$ in the confinement phase (fig. 2a–b) and in the deconfinement phase (fig. 2c–d).

$|B|$ we do not expect a difference between the confinement and the deconfinement phase in the probability of finding such mesons, $|\partial B| \approx |B|$. For large volumes $|B|$ only short strings near the boundary of B are candidates for N-ality states ± 1 in the confinement phase, while strings can be spanned over the whole volume B (leading to a contribution proportional to $|B|$) in the deconfinement phase.

Remark. Consider a quark q_1 and an antiquark $\bar{q}_2$ as shown in fig. 3a, b. The decay law of the Wilson loop is determined by the energy which is necessary to separate a quark q_1 at a position 1 from an antiquark $\bar{q}_2$ at a position 2. This energy is finite even in the confinement phase when matter fields are present that transform non-trivially under the center of the gauge group. Therefore the Wilson loop does not distinguish between situations as shown in fig. 3a and b. It obeys a perimeter law in both phases.

Our parameter should be sensitive to the difference between (a) and (b). Instead of the broken string between q_1 and $\bar{q}_2$ of fig. 3a only the string between q_1 and $\bar{q}_3$

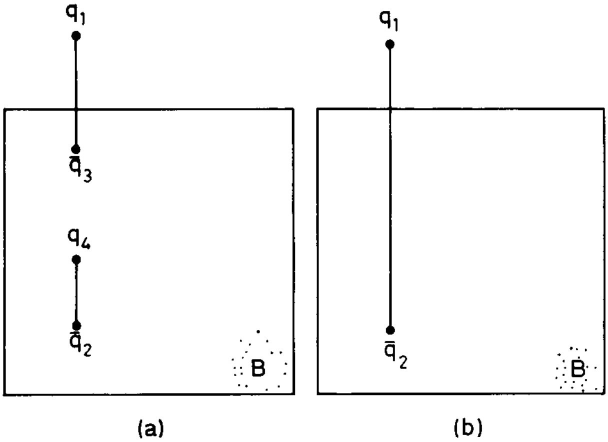

Fig. 3. (a) Pair creation of $\bar{q}_3 q_4$ leads to a broken string between a quark antiquark pair at positions 1 and 2 in the confinement phase. (b) Long string between the same quark antiquark pair at positions 1 and 2 in the deconfinement phase.

near the boundary of B will contribute to $\tilde{P}$. The second state $\bar{q}_2 q_4$ has N-ality 0 and drops out of $\tilde{P}$.

3. Bounds on the parameter in the Z(2) Higgs model

3.1. THE MODEL

We consider a Z(2) gauge theory with Z(2) Higgs scalars. The gauge fields $\sigma(b) = \pm 1$ are attached to links b, the matter fields are Higgs scalars $\tau(x) = \pm 1$ attached to sites $x = (\boldsymbol{x}, t)$. The action is given by

$$L(\sigma, \tau) = g_0^{-2} \sum_p \{\sigma(\partial p) - 1\} + \kappa \sum_{b = \langle xy \rangle} \tau(x) \sigma(b) \tau(y),$$

$$\sigma(\partial p) = \prod_{b \in \partial p} \sigma(b), \tag{3.1}$$

∂p denotes the boundary of a plaquette p. The gauge part has the standard Wilson/Wegner form. g_0^{-2} and κ are the gauge and matter couplings, respectively. The sum runs over all plaquettes p and links b of the lattice. Plaquettes on opposite sides are summed only once if the gauge field variables satisfy p.b.c. in the corresponding directions. The Haar measures reduce to discrete sums over all configurations $\{\sigma(b) = \pm 1\}$ and $\{\tau(x) = \pm 1\}$.

For the Z(2) Higgs model a line of phase transitions was established [8] between the shaded regions of the diagram (full line in fig. 4), at zero temperature β^{-1}. The model is expected to show confinement in the whole region (I) and (III), but deconfinement in region (II).

In order to establish $\tilde{P}$ as a candidate for a suitable order parameter we should derive a qualitatively different behavior of $\tilde{P}$ in these regions if we make in addition the following assumption: it is justified to take the limit $\beta \to \infty$ last (i.e. after the thermodynamic limit). Then $\tilde{P}$ is supposed to test for confinement also at $\beta^{-1} = 0$. Without this assumption qualitatively the same behavior in the whole range of couplings (as indicated in the diagram) would not contradict the well-known phase structure at zero temperature.

3.2. BOUNDS ON P

Configurations $\{\sigma(b) = \pm 1\}$ and $\{\tau(x) = \pm 1\}$ that are summed over in $\tilde{Z}_{\Lambda_\beta}(\mathrm{B}, k)$ depend on the twist γ_{B}^k, because they must be compatible with t.b.c. for $\tau(x)$. The action $L(\sigma, \tau)$ depends via τ implicitly on the twist. In order to exhibit the γ-dependence of L explicitly we make the following transformation of τ

$$\tau(x) \to \tau'(x) = \gamma_{\mathrm{B}}^k \tau(x) \begin{cases} -\tau(x) & \text{if } k = 1 \text{ and } x \in \mathrm{B} \\ \tau(x) & \text{otherwise.} \end{cases} \tag{3.2}$$

It follows that

$$\tilde{Z}_{\Lambda_\beta}(\mathrm{B}, k) = \int_{\text{p.b.c.}} \mathcal{D}\sigma \int_{\text{p.b.c.}} \mathcal{D}\tau' \exp\left\{ g_0^{-2} \sum_p \{\sigma(\partial p) - 1\} + \kappa \sum_{b = \langle xy \rangle} \tau'(x) \sigma(b) \tau'(y) \right\}, \tag{3.3}$$

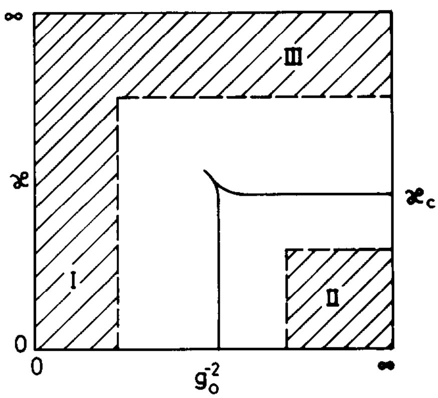

Fig. 4. Phase diagram for the Z(2) Higgs model at zero temperature. The full line corresponds to phase transitions between the confining/screening phase (region (I) and (III)) and the deconfinement phase (region (II)).

with τ' defined as in (3.2), or again in terms of τ

$$\tilde{Z}_{\Lambda_\beta}(\mathrm{B},k)=\int_{\text{p.b.c.}}\mathcal{D}\sigma\int_{\text{p.b.c.}}\mathcal{D}\tau\,\exp\left\{g_0^{-2}\sum_p\{\sigma(\partial p)-1\}+\kappa\sum_{b=\langle xy\rangle}\gamma_{\mathrm{B}}^k\tau(x)\sigma(b)\tau(y)\right\}. \tag{3.4}$$

Under the action of the twist γ_{B}^k, $\tilde{Z}_{\Lambda_\beta}(\mathrm{B},0)\to\tilde{Z}_{\Lambda_\beta}(\mathrm{B},1)$, the matter part of the exponent in (3.4) is changed by (-1) for all links $b\in\hat{\partial}\mathrm{B}$. The coboundary $\hat{\partial}\mathrm{B}$ of B is given by

$$\hat{\partial}\mathrm{B}=\{b=\langle xy\rangle\,|\,x\in\mathrm{B},y\notin\mathrm{B}\}, \tag{3.5}$$

$\hat{\partial}\mathrm{B}$ is shown in fig. 5. Denote by $(\hat{\partial}\mathrm{B})_{\mathrm{v}}$ the "vertical" part of $\hat{\partial}\mathrm{B}$, i.e. the set of timelike links

$$(\hat{\partial}\mathrm{B})_{\mathrm{v}}=\{b=\langle xy\rangle\in\hat{\partial}\mathrm{B}\,|\,x\in\mathrm{B},y=x+e_t\}, \tag{3.6}$$

where e_t is a lattice vector in time direction.

Write $(\hat{\partial}\mathrm{B})_{\mathrm{h}}$ for the "horizontal" part of $\hat{\partial}\mathrm{B}$, i.e. the set of spacelike links

$$(\hat{\partial}\mathrm{B})_{\mathrm{h}}=\{b=\langle xy\rangle\in\hat{\partial}\mathrm{B}\,|\,x\in B,y=x+e_i\}, \tag{3.7}$$

where e_i is a lattice vector in space direction i. Then

$$\hat{\partial}\mathrm{B}=(\hat{\partial}\mathrm{B})_{\mathrm{v}}\,\dot{\cup}\,(\hat{\partial}\mathrm{B})_{\mathrm{h}}. \tag{3.8}$$

We make a second transformation of field variables

$$\sigma(b)\to\sigma'(b)=\begin{cases}(-1)^k\sigma(b), & \text{for } b\in(\hat{\partial}\mathrm{B})_{\mathrm{v}}\\ \sigma(b), & \text{otherwise}.\end{cases} \tag{3.9}$$

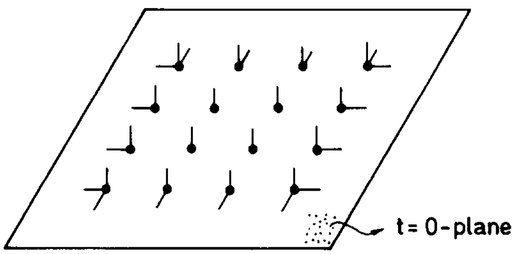

Fig. 5. Set $\hat{\partial}\mathrm{B}=\{b=\langle xy\rangle\,|\,x\in\mathrm{B},y\notin\mathrm{B}\}$.

Transformation (3.9) has the following properties:

(i) $\int \mathfrak{D}\sigma = \int \mathfrak{D}\sigma'$.

(ii) $\sigma'(b)$ still satisfies p.c.b. in all directions, especially in the time direction.

(iii) Since the new gauge field variables depend on k, the gauge part of the action in terms of $\sigma'[L_G(\sigma', k)]$ now depends on the twist too. For $k=1$ it gets a minus sign for those plaquettes that contain an odd number of links of $(\partial B)_v$. This set of plaquettes is given by $\hat{\partial}[(\partial B)_v]$. Denote by ∂B the set

$$\partial B = \{x \in B \,|\, \text{no. n.n.} < 4\ (6) \text{ in } 2\ (3) \text{ space dimensions}\}. \tag{3.10}$$

n.n. are the nearest neighbours of x in B, ∂B is the "perimeter" of B. Then

$$|\hat{\partial}[(\partial B)_v]| = |\partial B| = |(\partial B)_h|, \tag{3.11}$$

$\hat{\partial}[(\partial B)_v]$ is shown in fig. 6.

Note that

$$\sigma(\partial p) \lessgtr \begin{matrix} (-1)^k \sigma'(\partial p) & \text{if } p \in \hat{\partial}[(\hat{\partial} B)_v] \\ \sigma'(\partial p) & \text{otherwise,} \end{matrix} \tag{3.12}$$

for a fixed configuration $\sigma'(b)$ the gauge part changes according to

$$L_G(\sigma', k=1) - L_G(\sigma', k=0) = g_0^{-2} \sum_{p \in \hat{\partial}[(\hat{\partial} B)_v]} \{-\sigma'(\partial p) - \sigma'(\partial p)\}$$

$$= -2g_0^{-2} \sum_{p \in \hat{\partial}[(\hat{\partial} B)_v]} \sigma'(\partial p). \tag{3.13}$$

(iv) The change in the matter part of the action is given by

$$L_M(\sigma', \tau', k=1) - L_M(\sigma', \tau', k=0) = -2\kappa \sum_{b=\langle xy \rangle \in (\hat{\partial} B)_h} \tau'(x)\sigma'(b)\tau'(y). \tag{3.14}$$

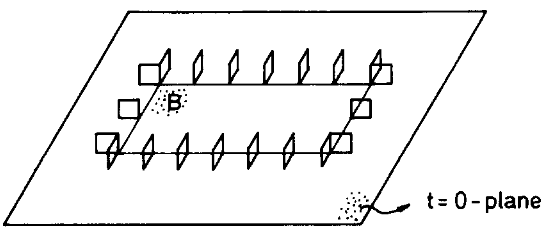

Fig. 6. Set $\hat{\partial}[(\hat{\partial} B)_v]$ of plaquettes p for which $\sigma'(\partial p) = (-1)^k \sigma(\partial p)$.

(If $b \in (\partial B)_v$, $x \in B$, $y = x + e_t$ the term $\tau(x)\sigma(b)\tau(y) = (-1)^k \tau'(x) \times (-1)^k \sigma'(b)\tau'(y)$ is independent of k and drops out in (3.14).

Consider now the quotient

$$\frac{\tilde{Z}_{\Lambda_\beta}(B,1)}{\tilde{Z}_{\Lambda_\beta}(B,0)} = \frac{\int_{\text{p.b.c.}} \mathcal{D}\sigma' \int_{\text{p.b.c.}} \mathcal{D}\tau' \exp\{L_G(\sigma', k=1) + L_M(\sigma', \tau', k=1)\}}{\int_{\text{p.b.c.}} \mathcal{D}\sigma' \int_{\text{p.b.c.}} \mathcal{D}\tau' \exp\{L_G(\sigma', k=0) + L_M(\sigma', \tau', k=0)\}}. \tag{3.15}$$

To get a bound on (3.15) note that

$$|\sigma'(\partial p)| \leqslant 1, \qquad \text{for all } p \in \Lambda_\beta,$$

$$|\tau'(x)\sigma'(b)\tau'(y)| \leqslant 1, \qquad \text{for all } b = \langle xy \rangle \in \Lambda_\beta. \tag{3.16}$$

Then it follows from (3.13), (3.14), (3.11), and (3.16) that

$$e^{-2[g_0^{-2}+\kappa]|\partial B|} \leqslant \frac{Z_{\Lambda_\beta}(B,1)}{Z_{\Lambda_\beta}(B,0)} \leqslant e^{2[g_0^{-2}+\kappa]|\partial B|}. \tag{3.17}$$

Taking $\lim_{|B|\to\infty}(1/|B|)\lim_{\Lambda\to\infty}$ in (3.17) finally yields

$$0 \leqslant \tilde{P} \leqslant 0, \qquad \text{or} \qquad \tilde{P} = 0. \tag{3.18}$$

According to (2.13) it means that the order parameter indicates confinement independently of the range of couplings κ and g_0^{-2} and independently of the finite temperature $\beta^{-1} > 0$.

This result indicates the failure of the parameter in the sense that it contradicts the phase diagram of fig. 4, at zero temperature if it is justified to take the limit $\beta^{-1} \to 0$ last, i.e. after the thermodynamic limit in (3.18). In the introduction we have already commented on the doubtfulness of this order of limits. At finite temperature $\beta^{-1} > 0$ if the phase diagram remains qualitatively the same as at zero temperature. The phase diagram at zero temperature is characterized by the existence of two phases, the so-called confining/screening and the deconfining phase, separated by a line of phase transitions. It has not been shown that the phase transitions persist at $\beta^{-1} > 0$.

In sect. 4 we show that the result of Marra and Miracle Solé cannot be extended to finite temperature. It is therefore quite possible that the phase transition which exists at zero temperature disappears for arbitrarily small finite temperature.

3.3. REMARK

The change of boundary conditions between $\tilde{Z}_{\Lambda_\beta}(B,0)$ and $\tilde{Z}_{\Lambda_\beta}(B,1)$ induced a change in the partition functions that became independent of the range of κ and g_0^{-2} and independent of $|B|$ in the limit of $\tilde{P}$. Let us compare with the vortex free energy

ν, the difference between the free energy of two vortex containers whose boundary conditions differ from each other by the action of a singular gauge transformation [11]. For gauge groups SU(2) and Z(2) the action of a singular gauge transformation enforces an odd number of quantums of magnetic flux through the container in a certain direction. Whether this number is small or large can depend on the range of coupling constants (e.g. $g_0^{-2} \ll 1$, $\kappa \ll 1$ or $g_0^{-2} \gg 1$, $\kappa \ll 1$). This is reflected by a different dependence of ν on the cross section of the container, see refs. [11].

4. Phase diagram of the Z(2) Higgs model at finite temperature $\beta^{-1} > 0$

In this section we want to show that the analysis of Marra and Miracle Solé to the Z(2) Higgs model [6] is no longer applicable at $\beta^{-1} > 0$.

Remember now the phase diagram of the Z(2) Higgs model at zero temperature. Analyticity of the free energy $\ln Z$ was proved for all the whole shaded regions, especially for region (II) ($g_0^{-2} \gg 1$, $\kappa \ll 1$) – the so-called deconfinement phase. It followed from the result obtained by Marra and Miracle Solé [6] that the cluster expansion of $\ln Z$ is absolutely convergent for sufficiently large g_0^{-2} and sufficiently small κ; the convergence is uniform in $|\Lambda_\beta|$. Their proof does not generalize to systems at $\beta^{-1} > 0$. The absolute convergence of the cluster expansion will be destroyed by a certain class of graphs which contribute to $\ln Z$ only because of the p.b.c. in the time direction for the matter fields.

Consider a (2 + 1)-dimensional box $\Lambda_\beta = \Lambda \times [0, \beta]$ (cf. fig. 7) whose size is d_1, d_2 in spatial directions 1,2 and $\beta < \infty$ in the time direction. The (3 + 1)-dimensional case could be treated in an analogous way. We impose free boundary conditions (f.b.c.) in directions 1,2 and p.b.c. in the time direction for both gauge and matter fields such that the top and bottom of the box can be identified. We take the thermodynamic limit $d_1, d_2 \to \infty$ in the end, but keep β finite corresponding to a temperature $\beta^{-1} > 0$.

Because of the identification of the top and bottom of Λ_β there exist closed paths which are not boundaries. If C is a closed path (more precisely a 1-chain with

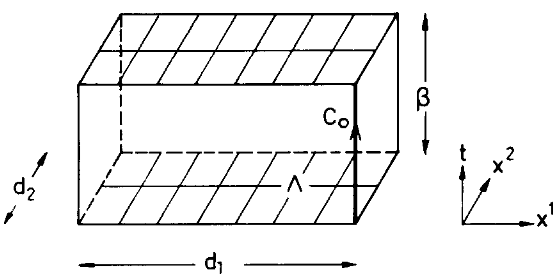

Fig. 7. (2 + 1)-dimensional box with free b.c. in directions 1, 2, and p.b.c. in the time direction for gauge and matter fields. C_0 plays the role of a "reference loop", for further explanation see the text.

coefficients in $\mathbb{F}_2 = \{0,1\}$), i.e. $\partial C = 0$ then

$$\text{either:}\quad C = \partial \Xi \quad (a), \quad \text{or} \quad C = C_0 + \partial \Xi \quad (b), \tag{4.1}$$

for some surface (2-chain) Ξ. The representative C_0 will be chosen on one of the corners of the box, see fig. 7.

4.1. REWRITING OF THE PARTITION FUNCTION

Given a gauge field configuration $\{\sigma(b)\}$, this determines uniquely a set S_σ of frustrated plaquettes

$$S_\sigma = \{ p | \sigma(\partial p) = -1 \}. \tag{4.2}$$

Given a set of frustrated plaquettes S

$$S = \{ p | \sigma(\partial p) = -1 \}, \tag{4.3}$$

the gauge field configuration $\{\sigma(b)\}$ is not uniquely determined, even not up to ordinary gauge transformations, because such gauge transformations have to respect periodicity in the time direction (otherwise they destroy p.b.c. for the matter fields). The partition function

$$Z_{\Lambda_\beta}(\text{b.c.}) = \int \underset{\text{b.c.}}{\mathcal{D}\sigma}\, \underset{\text{b.c.}}{\mathcal{D}\tau} \exp\left\{ + g_0^{-2} \sum_p (\sigma(\partial p) - 1) + \kappa \sum_{b = \langle xy \rangle} \tau(x)\sigma(b)\tau(y) \right\}, \tag{4.4}$$

can be written as

$$Z_{\Lambda_\beta}(\text{b.c.}) = \sum_{\{\sigma,\tau\}} e^{L(\tau,\sigma)}$$

$$= \sum_S e^{-2g_0^{-2}|S|} \sum_{\substack{\{\sigma,\tau\} \\ S_\sigma = S}} \prod_{\langle xy \rangle = b} \{1 + [\tanh\kappa]\tau(x)\sigma(b)\tau(y)\}, \tag{4.5}$$

for a suitable choice of the constant in L. The constraint $S_\sigma = S$ guarantees that $\{\sigma, \tau\}$ is compatible with S. S_σ is co-closed in Λ_β (i.e. its co-boundary in infinite space is contained in the boundary of Λ_β), therefore the sum over S needs only be taken over co-closed sets of plaquettes p with $\sigma(\partial p) = -1$. In the following we write Z for Z_{Λ_β}(b.c.).

A gauge field configuration σ is specified up to an ordinary gauge transformation on Λ_β (with top and bottom identified) by prescribing S *and* $\sigma(C_0)$. [$\sigma(C_0)$ can be changed into its negative without affecting S by taking $\sigma(b) \to -\sigma(b)$ for all timelike links $b = \langle xy \rangle$ with $x = (0, \mathbf{x})$, $y = x + e_t$, $\mathbf{x}$ varies over the points of a spacelike plane $t = \text{const.}$] Because of the periodic boundary conditions for the matter fields, the two classes of gauge fields with given S will give different contributions to Z. Therefore we write

$$Z=Z_{+}+Z_{-}, \qquad Z_{\pm}=\sum_{\sigma,\tau}\delta(\sigma(\mathrm{C}_0)\mp 1)\mathrm{e}^{L(\sigma,\tau)}. \tag{4.6}$$

The expression (4.5) is split into two contributions accordingly. Next the τ summations are carried out. The result is

$$Z_{\pm}=\sum_{\mathrm{S}}\mathrm{e}^{-2g_0^{-2}|\mathrm{S}|}\sum_{\substack{\sigma\\ \mathrm{S}_\sigma=\mathrm{S}}}\delta(\sigma(\mathrm{C}_0)\mp 1)\sum_{\mathrm{C}}(\tanh\kappa)^{|\mathrm{C}|}\sigma(\mathrm{C}). \tag{4.7}$$

Summation is over all closed 1-chains C, and $\sigma(\mathrm{C})=\prod_{b\in\mathrm{C}'}\sigma(b)$ if C′ is defined by $\mathrm{C}=:\sum_{b\in\mathrm{C}'}b$.

Finally we note that $\sigma(\mathrm{C})$ is determined by $\sigma(\mathrm{C}_0)$ and S. Suppose first that

$$\mathrm{C}=\partial\Xi, \tag{4.8}$$

then

$$\sigma(\mathrm{C})=\prod_{p\in\Xi}\sigma(\partial p)=(-1)^{|\Xi\cap\mathrm{S}|}. \tag{4.9}$$

Define the winding number

$$n_{\pm}(\mathrm{S},\mathrm{C})=(-1)^{|\Xi\cap\mathrm{S}|}, \qquad \text{if } \mathrm{C}=\partial\Xi. \tag{4.10}$$

This is independent of the choice of Ξ. Then

$$\sigma(\mathrm{C})=n_{\pm}(\mathrm{S},\mathrm{C}). \tag{4.11}$$

Next consider the case that $\mathrm{C}=\mathrm{C}_0+\partial\Xi$. Then

$$\sigma(\mathrm{C})=\sigma(\mathrm{C}_0)\prod_{p\in\Xi}\sigma(\partial p). \tag{4.12}$$

In this case we define

$$n_{\pm}(\mathrm{S},\mathrm{C})=\pm(-1)^{|\Xi\cap\mathrm{S}|}, \qquad \text{if } \mathrm{C}=\mathrm{C}_0+\partial\Xi. \tag{4.13}$$

The summation over σ is now trivial. Dropping factors of 2 which come from the freedom of gauge transformations one obtains

$$Z_{\pm}=\sum_{\mathrm{S}}\mathrm{e}^{-2g_0^{-2}|\mathrm{S}|}\sum_{\mathrm{C}}(\tanh\kappa)^{|\mathrm{C}|}n_{\pm}(\mathrm{S},\mathrm{C}). \tag{4.14}$$

Summation over S is over all co-closed sets of frustrated plaquettes in Λ_β and sum over C is over all closed paths (1-chains) in Λ_β (which do not have end points on $\partial\Lambda_\beta$ either; note that the top and bottom of Λ_β were identified).

4.2. REPRESENTATION AS A POLYMER SYSTEM

We want to represent $Z_\pm$ as partition functions of a polymer system. To do so we must specify a notion of disjointness of polymers. The obvious generalization of the prescription of Marra and Miracle Solé [6] is as follows.

Suppose a certain set (S, C) is given. First one decomposes C into connected components (non-intersecting closed curves) c_j and similarly S into connected components s_i. A graph Γ is associated to (S, C) according to the following rule. The vertices of the graph are in a one-to-one correspondence with the "vortices" s_i and the "loops" c_j. Two such vertices are joined by a line if one is a vortex s_i and the other a loop c_j, and

$$(-1)^{|\Xi_j\cap s_i|} = -1, \qquad \text{if } c_j = \partial\Xi_j \text{ or } c_j = C_0 + \partial\Xi_j. \tag{4.15}$$

(S, C) is called a polymer if Γ is connected. Every set (S, C) as described above admits of a unique decomposition into polymers $(s_1, \ldots, s_n, c_1, \ldots, c_m)$, and

$$n_\pm(S,C) = \prod_{i,j} n_\pm(s_i, c_j). \tag{4.16}$$

To see this note that c_j are closed by construction. Therefore $c_j = \partial\Xi_j$ or $c_j = \partial\Xi_j + C_0$ for some Ξ_j, and $\Xi = \sum\Xi_j$ obeys the relation $C = \partial\Xi$ respectively $C = C_0 + \partial\Xi$. Therefore $c_j = \partial\Xi_j + C_0$ for an odd number of polymers if $C = C_0 + \partial\Xi$, and an even number otherwise. Moreover, by the definition of polymers, $|\Xi_j \cap s_i| = 0 \pmod 2$ if $i \neq j$. Eq. (4.16) follows from these relations and the definition of $n_\pm(s_i, c_j)$.

To every polymer $(s_1, \ldots, s_n, c_1, \ldots, c_m)$ one assigns an activity

$$\phi_\pm(s_1, \ldots, s_n, c_1, \ldots, c_m) = \exp\left(-2g_0^{-2}\sum_{i=1}^{n}|s_i|\right)(\tanh\kappa)^{\sum_{j=1}^{m}|c_j|}\prod_{i,j} n_\pm(s_i, c_j). \tag{4.17}$$

In this way, Z_+ and Z_- are reexpressed as partition functions of a polymer system

$$Z_\pm = \sum_{\substack{(\mathrm{Pol}_1, \ldots, \mathrm{Pol}_n)\\ n\geq 0}} \prod_{i=1}^{n} \phi_\pm(\mathrm{Pol}_i). \tag{4.18}$$

Summation is over collections of mutually disjoint polymers.

The cluster expansion of the free energies can be written as

$$\ln Z_{\pm} = \sum_{Q} a(Q) \prod_{\mathrm{Pol} \in Q} \phi_{\pm}(\mathrm{Pol}). \tag{4.19}$$

Summation is over all linked clusters $Q = \{\mathrm{Pol}_1^{n_1}, \ldots, \mathrm{Pol}_N^{n_N}\}$. A linked cluster is a non-empty collection of not necessarily distinct polymers. It may contain a polymer Pol_i $n_i \geqslant 1$ times. It is linked in the sense that the clustergraph $\mathcal{V}_Q$ is connected. The vertices of the clustergraph are the polymers of Q, the links of the graph are pairs of polymers that are not disjoint. The coefficients $a(Q)$ are combinatorial factors. For further explanation see for instance [12].

In order to exhibit the divergence of $\ln Z_{\pm}$ in the limit $|\Lambda_\beta| \to \infty$, ($\beta$ finite) we consider clusters $Q_1 = \{\mathrm{Pol}_1\}$ with $\mathrm{Pol}_1 = (s_{L_1 L_2}, c_1 \ldots c_m)$, i.e. one-polymer clusters made of a vortex $s_{L_1 L_2}$ and m matter loops c_j. The vortex consists of L_1 plaquettes in direction 1 and L_2 plaquettes in direction 2; the loops c_j with $|c_j| = \beta$ wind once through Λ_β in the time direction. They intersect the surface Ω_0 "spanned between $s_{L_1 L_2}$" at a single point. Such a polymer is shown in fig. 8.

The abstract graph associated to Pol_1 is shown in fig. 9. Its activity is given by

$$\phi_{\pm}(s_{L_1 L_2}, c_1, \ldots, c_m) = e^{-2g_0^{-2} \cdot 2(L_1 + L_2)} (\tanh \kappa)^{\sum |c_j|} \prod_{j=1}^{m} n_{\pm}(s_{L_1 L_2}, c_j). \tag{4.20}$$

The product of winding numbers equals $(-)^m$.
For the cluster expansion we find

$$\sum_{Q} |a(Q)| \prod_{\mathrm{Pol} \in Q} |\phi_{\pm}(\mathrm{Pol})| \geqslant \sum_{Q_1 = \{(s_{L_1 L_2}, c_1, \ldots, c_m)\}} |a(Q_1) \phi(\mathrm{Pol}_1)|$$

$$= \sum_{\substack{L_1 \leqslant d_1 \\ L_2 \leqslant d_2}} \sum_{m \leqslant L_1 L_2} \left| a(Q_1) e^{-2g_0^{-2} \cdot 2(L_1 + L_2)} (\tanh \kappa)^{m\beta} \right|. \tag{4.21}$$

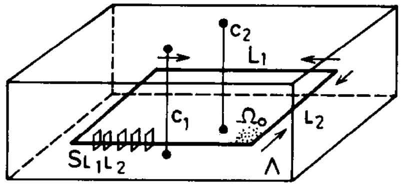

Fig. 8. One-polymer cluster $(s_{L_1 L_2}, c_1, c_2)$ for which $|s_{L_1 L_2}| = 2(L_1 + L_2)$, $|c_j| = \beta$, $j = 1, 2$. The vortex is represented by the corresponding closed loop on the dual lattice.

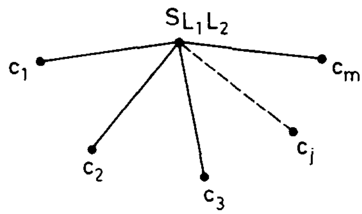

Fig. 9. Graph $\Gamma(s_{L_1L_2}, c_1, \dots, c_m)$.

Suppose a certain vortex $s_{L_1L_2}$ of total length $2(L_1+L_2)$ is given. Because of its rectangular shape there are $\binom{L_1L_2}{m}$ possible arrangements of m loops c_j winding through $s_{L_1L_2}$ and Λ_β as illustrated in fig. 8. This gives a factor $\binom{L_1L_2}{m}$ from Σ_{Q_1}. Furthermore $a(Q)=1$ because Q_1 is a one-polymer cluster. Since

$$\sum_{m=0}^{L_1L_2}\binom{L_1L_2}{m}(\tanh^\beta\kappa)^m=(1+\tanh^\beta\kappa)^{L_1L_2}, \tag{4.22}$$

we find

$$\sum_{Q}\left|a(Q)\prod_{\mathrm{Pol}\in Q}\phi(\mathrm{Pol})\right|\geqslant\sum_{\substack{L_1\leqslant d_1\\ L_2\leqslant d_2}}(1+\tanh^\beta\kappa)^{L_1L_2}e^{-2g_0^{-2}\cdot 2(L_1+L_2)}, \tag{4.23a}$$

or

$$\sum_{Q}\left|a(Q)\prod_{\mathrm{Pol}\in Q}\phi(\mathrm{Pol})\right|\geqslant\sum_{L_1\leqslant d_1,\,L_2\leqslant d_2}\exp\{L_1L_2\ln(1+\tanh^\beta\kappa)-4g_0^{-2}(L_1+L_2)\}$$

$$\to\infty \qquad \text{as } d_1, d_2\to\infty, \tag{4.23b}$$

$\exp\{\dots\}>1$ for all L_1, L_2 that satisfy (4.24)

$$\frac{L_1L_2}{L_1+L_2}\geqslant\frac{4g_0^{-2}}{\ln(1+\tanh^\beta\kappa)}. \tag{4.24}$$

Therefore the partial sum of clusters Q_1 destroys the absolute convergence of $\ln Z_\pm$ in the limit $d_1, d_2\to\infty$.

Region (II) of the phase diagram characterized by the analyticity property of the free energy could shrink to a line $\kappa=0$, $g_{0c}^{-2}\leqslant g_0^{-2}\leqslant\infty$ when the temperature β^{-1} is strictly positive. The line corresponds to pure gauge models.

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