ANOMALY FREE CONDITION AND $SU(3)_c \times U(1)_{em}$ REALITY IN GRAND UNIFIED THEORIES

K.S. SOH [1]

II. Institut für Theoretische Physik der Universität Hamburg, Hamburg, Fed. Rep. Germany

Received 3 September 1983

If fermions are assigned to totally antisymmetric representation of SU(N), $SU(3)_c \times U(1)_{em}$ reality is a sufficient condition for vanishing anomaly. The anomaly free condition and the $SU(3)_c \times U(1)_{em}$ reality become equivalent if and only if $Q = \text{diag}(-1/3, -1/3, -1/3, 1, 0, ..., 0)$.

Grand unification of strong, weak, and electromagnetic interactions has been a subject of intensive studies since the simplest model based upon SU(5) was introduced by Georgi and Glashow [1]. One of the main efforts to extend the simple theory is to enlarge the unifying group in order to explain multiple families of fermions. For this purpose various authors have proposed sets of plausible assumptions to find a family unifying group out of many possibilities [2–4]. The following four assumptions are frequently used:

(A1) Unifying group G is a simple compact group and representation of left-handed (LH) fermions is complex with respect to G.

(A2) The LH fermion representation contains only singlet, triplet, and anti-triplet representations of the color SU(3) subgroup.

(A3) The representation of the LH fermions must be real with respect to the $SU(3)_c \times U(1)_{em}$ subgroup.

(A4) The representation of the LH fermions should be anomaly free.

The purpose of this paper is to show that the above conditions are not all independent, i.e., the condition (A4) follows from the other three. Before we present this as a theorem, we explain each assumption briefly. As a consequence of (A1) and (A2) G is limited to SU(N), SO($4N$ + 2) and E_6, whose admissible representations are totally antisymmetric tensors of SU(N), spinors of SO($4N$ + 2), and the 27-dimensional representation of E_6 [5]. Since E_6 is too small to accomodate multiple families of fermions, we will consider SU(N) and SO($4N$ + 2) only.

A slight modification of Georgi's first law [3] was introduced by Kim et al. [6] in the form of the assumption (A3), which was used to determine a possible electromagnetic charge operator in SU(N) and SO($4N$ + 2). The assumption (A3) is motivated by the non-observation of leptons and quarks whose representations are complex with respect to $SU(3)_c \times U(1)_{em}$. Even though this law is not without theoretical supports, we emphasize (A3) as a phenomenological statement without concerning its theoretical justification. Some thoughts on this point can be found in Barr and Zee [7], where reality with respect to $U(1)_{em}$ is considered.

While the motivation behind (A2) and (A3) is to extrapolate the ideas extracted from low energy observation, the assumption (A4) is called for by theoretical needs to make grand unifying gauge theory renormalizable. Therefore it is quite surprising that a purely technical requirement (A4) is guaranteed by the apparently unrelated phenomenological conditions (A2) and (A3).

Now, we state the theorem.

Theorem. Let the LH fermions be assigned to the totally antisymmetric representation of SU(N), and the color subgroup be a regular subgroup of SU(N).

[1] Alexander von Humboldt Fellow.

0.031-9163/83/$ 03.00 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

Then the reality of the LH fermion representation with respect to $SU(3)_c \times U(1)_{em}$ implies that the representation is anomaly free.

Proof. Let the electromagnetic charge operator be

$$Q = \mathrm{diag}(a, a, a, b_1, ..., b_{N-3}) , \quad a \neq 0 , \qquad (1)$$

where $b \equiv \Sigma_{i=1}^{N-3} b_i = -3a$, and the first three elements belong to the color group. Let $[m]$ be the mth order totally antisymmetric representation of SU(N), and all the LH fermions are assigned to some superposition of them $\Sigma\, C_m\, [m]$.

The reality with respect to $SU(3)_c \times U(1)_{em}$ means that each fermion in the representation $(\mathbf{3}, q)$ must be paired with another fermion $(\mathbf{3}^*, -q)$ and each fermion $(\mathbf{1}, q)$ must be paired with another $(\mathbf{1}, -q)$. A necessary condition for this pairing is that

$$\sum_{\text{(sum over all color singlet fermions)}} q = 0 , \qquad (2)$$

$$\sum_{\text{(sum over all triplet and antitriplet fermions)}} q = 0, \qquad (3)$$

where q is the electromagnetic charge of each LH fermion. This necessary condition which we call linear condition will be shown to be equivalent to the anomaly free condition.

First we proceed with the color singlet sector by computing total charge in the representation $[m]$. Let $q_{[m]}$ be the total charge obtained by adding charges of the color singlet fermions in $[m]$. Then we have

$$q_{[m]} = \sum_{i_1<...<i_m} (b_{i_1} + ... + b_{i_m})$$

$$+ \sum_{i_1<...<i_{m-3}} (3a + b_{i_1} + ... + b_{i_{m-3}})$$

$$= b\,\frac{m}{N-3}\binom{N-3}{m} + b\,\frac{m-N}{N-3}\binom{N-3}{m-3}$$

$$= b\,\frac{(N-3)!(N-2m)}{(m-1)!(N-m-1)!} \simeq bA_{N,m} \, . \qquad (4)$$

Note that the final form is proportional to $A_{N,m}$ which is the anomaly of the representation $[m]$. In the cases of $m = 1$, and 2 the second term of the first line does not exist, but the final form is still correct. Now the charge summed over all the color singlet fermions must satisfy

$$\sum C_m q_{[m]} = b\Big(\sum C_m A_{N,m}\Big) = 0 \qquad (5)$$

which is exactly the anomaly free condition. Similar computations on the color triplet–antitriplet sector lead to the same formula except b is replaced by $(-b)$.

To sum up we have proved that a vanishing anomaly is a necessary condition for $SU(3)_c \times U(1)_{em}$ reality in SU(N) when the assumptions (A1) and (A2) are given as premises.

A few remarks are in order:

(R1) The above theorem holds when $SU(3)_c$ is replaced by $SU(n)_c$ for $n = 2$ or 3 only. This is somewhat interesting in connection with the symmetry breaking pattern because SU(3) has almost unique properties in relation with the anomaly, which is not shared by other SU(n). [See also (R3).] So, at least partially the symmetry breaking patterns may be constrained by group theory itself without detail dynamics.

(R2) In the case of SO($4N + 2$), the spin representation is anomaly free and broken down to SU($2N + 1$) representations as [8]

$$\chi_+ = [0] + [2] + [4] + ... + [2N] \, . \qquad (6)$$

The charge operator Q can be of two components, in general,

$$Q = Q_{SU(2N+1)} + Q_{U(1)} \, . \qquad (7)$$

The $SU(3)_c \times U(1)_{em}$ linear reality is satisfied for the $Q_{SU(2N+1)}$ part. The rest part $Q_{U(1)}$, has charge for each $[2m]$,

$$b_{[2m]} = (2N + 1 - 4m)z , \qquad (8)$$

where z is a real number, and $m = 0, 1, ..., N$. It is a straightforward calculation to show that the linear reality condition is also satisfied for any z in the case of $SU(3)_c \times U(1)_{em}$, or $SU(2)_c \times U(1)_{em}$, which can be checked in SO(10) or SO(14).

(R3) Now, we consider when the vanishing anomaly becomes a sufficient condition for $SU(3)_c \times U(1)_{em}$ reality. For this we investigate quadratic reality on a sum over all charge squared of each fermion, which provides a new constraint for the color triplet–antitriplet sector. There are no such constraints in the color singlet sector or in the $SU(2)_c \times U(1)_{em}$ case. The quadratic reality is defined by

$$\sum_{\text{(sum over all triplet fermions)}} q^2 = \sum_{\text{(sum over all antitriplet fermions)}} q^2 \qquad (9)$$

By explicit evaluations we obtain

$$a^2\left(\sum C_m A_{N,m}\right) + x^2\left(\sum C_m B_{N,m}\right) = 0\,, \qquad (10)$$

where $A_{N,m}$ is the anomaly of $[m]$, and

$$B_{N,m} = \frac{(N-5)!(N-2m)}{(m-2)!(N-m-2)!}\,.$$

Here $x^2 = (\Sigma_{i=1}^{N-3} b_i^2) - 9a^2$. Eq. (10) means that the anomaly free condition is equivalent to the $SU(3)_c \times U(1)_{em}$ reality (up to quadratic) if and only if $x^2 = 0$. Since $\Sigma\, b_i = -3a$, $\Sigma b_i^2 = 9a^2$ has a unique solution $b_4 = -3a$, $b_i = 0$, $i > 4$ (modulo permutation), we obtain a unique (up to scale) charge operator

$$Q_0 = \mathrm{diag}(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 1, 0, 0, ..., 0)\,. \qquad (11)$$

Usually in grand unified theories, Q_0 is an input to fit the known charges of quarks and leptons. It is remarkable that a pure group theoretical consideration leads to the unique charge operator. This means that charges of quarks, leptons and W bosons, and the bare Weinberg angle are in a sense derivable rather than being phenomenological inputs. Of course, Q does not have to be Q_0. Then the extra condition ($C_m B_{N,m} = 0$) besides the anomaly free condition provides useful constraints in model building. An application in SU(9) will be presented elsewhere. It is easy to see that if eq. (11) is satisfied the vanishing anomaly is equivalent to $SU(3)_c \times U(1)_{em}$ reality (not only up to quadratic).

In this paper everything is proved by explicit computations. There may be a simpler and more manifest way of proving the theorem, which could reveal some deep reasons in these apparently coincidental relations.

I am grateful to Professor G. Kramer, and Dr. B. Schrempp and Dr. F. Schrempp for their kind hospitality at Hamburg University. I also thank the Alexander von Humboldt Foundation for financial support.

References

[1] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438.
[2] M. Gell-Mann, P. Ramond and R. Slansky, Rev. Mod. Phys. 50 (1978) 721.
[3] H. Georgi, Nucl. Phys. B156 (1979) 126.
[4] P. Frampton, Phys. Lett. 89B (1980) 352; C.W. Kim and C. Roiesnel, Phys. Lett. 93B (1980) 343.
[5] P. Langacker, Phys. Rep. 72C (1981) 185.
[6] J. Kim et al., Nucl. Phys. B181 (1981) 531.
[7] S. Barr and A. Zee, J. Math. Phys. 22 (1981) 2263.
[8] F. Wilczek and A. Zee, Phys. Rev. D25 (1982) 553.