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CLASSICAL MODELS OF CONFINEMENT

Harry LEHMANN

II. Institut für Theoretische Physik der Universität Hamburg, Fed. Rep. Germany

Tai Tsun WU[1]

Deutsches Elektronen-Synchrotron DESY, Hamburg, Fed. Rep. Germany

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We investigate the structure of classical models of confinement by analytic methods. The models considered involve just one abelian gauge field and can be viewed as non-linear versions of Maxwell's theory. We study in detail the case of two opposite static point charges. A scaling law is found for small separations of the charges, while a perturbation treatment is developed for large separations. The results obtained for large separations include an exact evaluation of the first correction term to the linear potential and a classification of the asymptotic field configurations.

1. Introduction

For some years there have been attempts to construct classical models as possible approximations to Yang–Mills theory. Examples of such attempts can be found in ref. [1]. For a recent review see Adler and Piran [2].

Motivated by these authors, we investigate with analytic methods a class of models involving just one abelian gauge field. Since we are primarily interested in the case of large separation of static charges, the approach consists mainly of studying the asymptotic behaviour of the classical field equations. Because of the non-linearity of these equations, the first step is to derive a suitable zeroth-order approximation. Starting with this zeroth-order approximation, perturbation calculation is then carried out. In particular, by this method, we shall show that there are qualitatively different asymptotic field configurations, depending on the parameters of the model.

The models can be considered as ordinary electrostatics with a field-dependent dielectric constant ε. In other words, the fields $\boldsymbol{D}$ and $\boldsymbol{E}$ satisfy

$$\nabla \cdot \boldsymbol{D} = \rho \,, \tag{1.1}$$

$$\nabla \times \boldsymbol{E} = 0 \,, \tag{1.2}$$

where $\boldsymbol{D}$ and $\boldsymbol{E}$ are related in a non-linear manner by

$$\boldsymbol{D} = \varepsilon(E)\boldsymbol{E} \,, \tag{1.3}$$

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with $E = |\boldsymbol{E}|$. The special feature of this class of models is

$$\varepsilon(E) = 0\,, \qquad \text{for} \qquad E \leqslant E_0\,, \tag{1.4}$$

which implies that the electrostatic energy of two point charges $\pm Q$ increases linearly for large separations (see footnote 8 of ref. [2]). The model that was studied extensively by Adler and Piran [2, 3] corresponds to the choice

$$\varepsilon(E) = \text{const} \ln(E/E_0)\,, \qquad \text{for } E > E_0\,. \tag{1.5}$$

Another model, which is distinguished by its simplicity, is obtained by the choice

$$\varepsilon(E) = \text{const}\,, \qquad \text{for } E > E_0\,. \tag{1.6}$$

This case has been studied by Giles [4], who gave its exact solution for two space dimensions.

For the general case, a more useful characterization of the models is to express E as a function of D:

$$E = f(D)\,. \tag{1.7}$$

The formulation of the problem and some general properties of the models are given in sect. 2, the limiting case of small separation between two static charges is treated in sect. 3, and the opposite case of large separation in the remaining sections of this paper.

2. Formulation of the problem

Consider the variational principle

$$\delta L = 0\,, \tag{2.1}$$

with

$$L = \int \mathrm{d}^3x [\mathscr{L}(E(\boldsymbol{x})) - \varphi(\boldsymbol{x})\rho(\boldsymbol{x})]\,, \tag{2.2}$$

where

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla\varphi(\boldsymbol{x})\,. \tag{2.3}$$

Eq. (1.2) of course follows from (2.3). In order to get (1.1) and (1.3) from (2.1) and (2.3), we must choose

$$\mathscr{L}(E) = \int \mathrm{d}E\, E\varepsilon(E)\,. \tag{2.4}$$

Throughout this paper, we are interested in the case of two static charges of opposite sign, so that

$$\rho(\boldsymbol{x}) = Q[\delta(\boldsymbol{x} - \boldsymbol{x}_1) - \delta(\boldsymbol{x} - \boldsymbol{x}_2)]\,. \tag{2.5}$$

Near $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, D is necessarily large. Because of the feature (1.4) of the models under consideration, variation with respect to φ shows that there are two domains: in domain (I) (1.1) and (1.3) hold with $E > E_0$, and in domain (II) $E \leqslant E_0$ and $\varepsilon(E) = 0$. Furthermore, both charges are located in domain (I). Let B denote the boundary between these two domains, then variation with respect to B shows that

$$\mathscr{L}(E(\boldsymbol{x})) \text{ is continuous across B} . \tag{2.6}$$

Since $\varepsilon(E)$ is taken to be positive for $E > E_0$, it follows from (2.6) with (2.4) that

$$E(\boldsymbol{x}) = E_0 , \qquad D(\boldsymbol{x}) = 0 , \tag{2.7}$$

for $\boldsymbol{x}$ on B.

Let the two charges of (2.5) be located on the z-axis at $z = \pm R$. Because of rotational symmetry we introduce cylindrical coordinates z and ρ. Eq. (1.1) implies, for $\rho > 0$,

$$\partial_z D_z + \frac{1}{\rho} \partial_\rho (\rho D_\rho) = 0 . \tag{2.8}$$

We can express D in terms of a flux function $\Psi(\rho, z)$ as [3]

$$\rho D_z = \frac{1}{2\pi} \partial_\rho \Psi , \qquad \rho D_\rho = -\frac{1}{2\pi} \partial_z \Psi . \tag{2.9}$$

The boundary conditions on Ψ are

$$\begin{gathered} \Psi = 0 \quad \text{and} \quad \nabla \Psi = 0 \qquad \text{on B} , \\ \Psi(0, z) = \begin{cases} Q, & |z| < R \\ 0, & |z| > R , \end{cases} \\ \Psi \to 0 , \qquad \text{as} \qquad z^2 + \rho^2 \to \infty . \end{gathered} \tag{2.10}$$

Substitution of (2.9) into (1.2) leads to a partial differential equation for Ψ:

$$\partial_z \left[\frac{\partial_z \Psi f(D)}{[(\partial_z \Psi)^2 + (\partial_\rho \Psi)^2]^{1/2}} \right] + \partial_\rho \left[\frac{\partial_\rho \Psi f(D)}{[(\partial_z \Psi)^2 + (\partial_\rho \Psi)^2]^{1/2}} \right] = 0 , \tag{2.11}$$

where f is defined by (1.7). Carrying out the differentiations gives

$$\begin{aligned} &\left[(\partial_z \Psi)^2 + \left(1 + \frac{g}{D}\right)(\partial_\rho \Psi)^2 \right] \partial_{zz} \Psi + \left[\left(1 + \frac{g}{D}\right)(\partial_z \Psi)^2 + (\partial_\rho \Psi)^2 \right] \partial_{\rho\rho} \Psi \\ &\quad - 2 \frac{g}{D} (\partial_z \Psi)(\partial_\rho \Psi)(\partial_{z\rho} \Psi) - \frac{1}{\rho} [(\partial_z \Psi)^2 + (\partial_\rho \Psi)^2] \partial_\rho \Psi = 0 , \end{aligned} \tag{2.12}$$

where

$$g(D) = \frac{f(D)}{f'(D)} - D . \tag{2.13}$$

Eq. (2.12) is elliptic if

$$D(D+g)=\frac{Df(D)}{f'(D)}>0\,. \tag{2.14}$$

Eq. (2.11) is the Euler–Lagrange equation for the variational principle

$$\frac{\delta}{\delta\Psi}\int_{-\infty}^{\infty}\mathrm{d}z\int_{0}^{\infty}\rho\,\mathrm{d}\rho\,\tilde{L}(D)=0\,, \tag{2.15}$$

where

$$\tilde{L}(D)=\int_{0}^{D}\mathrm{d}D'\,f(D')\,. \tag{2.16}$$

As already mentioned in the introduction and to be discussed in more detail in sect. 4, for all models of this class the electrostatic energy between the two charges increases linearly for large separations. Therefore the charges are said to be confined. Moreover, since $D=0$ in domain (II), the electrostatic energy comes entirely from domain (I). For this reason we shall refer to domain (I) as the confinement domain [3].

3. Case of small *R*

In this section, we investigate the behaviour of the field and the confinement domain for small R. In particular, we shall show that as $R\to 0$ the linear dimensions of the confinement domain shrink as $R^{1/3}$ under a fairly general condition.

The condition that we shall use is that

$$\lim_{D\to\infty}\frac{\ln f(D)}{\ln D}=1\,, \tag{3.1}$$

or equivalently by (2.13)

$$\lim_{D\to\infty} g(D)/D\to 0\,. \tag{3.2}$$

Examples of $f(D)$ that satisfy this condition are given by, as $D\to\infty$,

$$f(D)\sim \mathrm{const}\cdot D(\ln D)^{\Delta}\,, \tag{3.3}$$

with any real Δ. Eq. (3.3) is satisfied in the cases studied in [3] and [4].

Under the condition (3.2), (2.12) simplifies greatly for large D to

$$\partial_{zz}\Psi+\partial_{\rho\rho}\Psi-\frac{1}{\rho}\partial_{\rho}\Psi=0\,, \tag{3.4}$$

whose solution is, with the boundary condition (2.10),

$$\Psi=-\tfrac{1}{2}Q\left\{\frac{z-R}{[(z-R)^2+\rho^2]^{1/2}}-\frac{z+R}{[(z+R)^2+\rho^2]^{1/2}}\right\}. \tag{3.5}$$

For small R, z and ρ such that z and ρ are much larger than R, this becomes

$$\Psi \sim \frac{QR\rho^2}{(z^2+\rho^2)^{3/2}}, \tag{3.6}$$

which implies

$$D_z^2 + D_\rho^2 \sim \frac{Q^2R^2}{4\pi^2}\frac{4z^2+\rho^2}{(z^2+\rho^2)^4}. \tag{3.7}$$

Thus D^2 is of order 1 when z and ρ are of order $R^{1/3}$. Therefore we introduce

$$z = R^{1/3}\hat{z}, \qquad \rho = R^{1/3}\hat{\rho}. \tag{3.8}$$

If $\hat{\Psi}$ is defined by

$$\Psi = R^{2/3}\hat{\Psi}, \tag{3.9}$$

then we have from (2.9), (2.12) and (3.7) that

$$\hat{\rho}D_z = \frac{1}{2\pi}\partial_{\hat{\rho}}\hat{\Psi}, \qquad \hat{\rho}D_\rho = -\frac{1}{2\pi}\partial_{\hat{z}}\hat{\Psi}, \tag{3.10}$$

$$\left[(\partial_{\hat{z}}\hat{\Psi})^2 + \left(1+\frac{g}{D}\right)(\partial_{\hat{\rho}}\hat{\Psi})^2\right]\partial_{\hat{z}\hat{z}}\hat{\Psi} + \left[\left(1+\frac{g}{D}\right)(\partial_{\hat{z}}\hat{\Psi})^2 + (\partial_{\hat{\rho}}\hat{\Psi})^2\right]\partial_{\hat{\rho}\hat{\rho}}\hat{\Psi}$$

$$-2\frac{g}{D}(\partial_{\hat{z}}\hat{\Psi})(\partial_{\hat{\rho}}\hat{\Psi})(\partial_{\hat{z}\hat{\rho}}\hat{\Psi}) - \frac{1}{\hat{\rho}}[(\partial_{\hat{z}}\hat{\Psi})^2 + (\partial_{\hat{\rho}}\hat{\Psi})^2]\partial_{\hat{\rho}}\hat{\Psi} = 0, \tag{3.11}$$

and, for small $\hat{z}$ and $\hat{\rho}$

$$\Psi \sim \frac{Q\hat{\rho}^2}{(\hat{z}^2+\hat{\rho}^2)^{3/2}}. \tag{3.12}$$

Since eqs. (3.10)–(3.12) no longer depend on R, they give the leading behaviour of the field and the confinement domain.

For the case of small R in the logarithmic model, see Adler [5].

4. Case of large R

4.1. ORIENTATION

Consider two charges $\pm Q$ separated by a large distance. For any surface that encloses one of the charges but not the other, integration of the normal component of $\boldsymbol{D}$ must give $4\pi Q$. How do we expect the distribution of $\boldsymbol{D}$ to be when the distances to the two charges are of order R?

The qualitative argument follows closely the one for ordinary linear electrostatics. The distribution of $\boldsymbol{D}$ must be such that the total electrostatic energy as given by (2.16) is a minimum. Since $f(D)$ is an increasing function, $\tilde{L}(D)$ is a convex function of D. Therefore, under the constraint of flux conservation, it is energetically unfavorable to have large values of D. We therefore reach the conclusion that D is small far away from the two charges.

The relevant quantity is therefore the behaviour of $f(D)$ for small D. A fairly general case is that

$$f(D) = E_0 + b_\sigma D^\sigma + \cdots , \tag{4.1}$$

for $\sigma > 0$, where b_σ is a positive coefficient. In this case, (2.13) implies that

$$g(D) = \mathrm{O}(D^{1-\sigma}) . \tag{4.2}$$

Let the scale for the coordinate ρ be R^α, that of z being R. Since $\alpha = 1$ for ordinary linear electrostatics, we expect $\alpha < 1$ for the present models. It follows from (2.9) and these scales that

$$D_z = \mathrm{O}(R^{-2\alpha}) , \qquad D_\rho = \mathrm{O}(R^{-1-\alpha}) , \qquad D = \mathrm{O}(R^{-2\alpha}) . \tag{4.3}$$

Therefore, the orders of magnitude for the various terms in (2.12) are R^{-4}, $R^{-2-2\alpha}$, $R^{-2-2\alpha(1-\sigma)}$, and $R^{-4\alpha}$ because of (4.2) and (4.3). The last two orders are the largest; equating these orders of magnitude gives a determination of α

$$\alpha = \frac{1}{1+\sigma} . \tag{4.4}$$

This implies in particular that the transverse dimension of the confinement domain increases as $R^{1/(1+\sigma)}$ for large R.

For the special case of $\sigma = 1$ to be studied in more detail in this paper,

$$\alpha = \tfrac{1}{2} . \tag{4.5}$$

This scaling law (4.5) has been verified numerically [2] for the logarithmic model.

The model studied by Giles [4] corresponds to $\sigma \to \infty$ and $\alpha = 0$. Therefore, the confinement domain in this case has a finite cross section, which may be the case in QCD apart from a logarithmic quantum effect [6].

4.2. ZEROTH-ORDER SOLUTION

In this paper we shall concentrate mostly on the case $\sigma = 1$. In this section, we give the zeroth-order solution for large R. It is the simplicity of this zeroth-order solution which makes the later developments possible. Since this simplicity is largely lost for the general case of arbitrary σ, generalization to arbitrary σ is technically not straightforward beyond the zeroth order.

Let $f(D)$ be expanded into a Taylor series

$$f(D) = \sum_{j=0}^{\infty} b_j D^j . \tag{4.6}$$

Here $b_0 = E_0$. Since $\sigma = 1$, b_1 is positive. By suitable scaling of the three vectors $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{x}$, we choose units such that

$$E_0 = b_1 = Q = 1\,. \tag{4.7}$$

Throughout the rest of this paper, (4.7) is used except for the purpose of a comparison of the total electrostatic energy (see (4.18)–(4.20)). With (4.7), the first few terms of the Taylor series for $f(D)$ are

$$f(D) = 1 + D + b_2 D^2 + b_3 D^3 + \cdots\,. \tag{4.8}$$

It turns out that the value of b_2 is of great importance. There are two different cases depending on whether b_2 is equal to $-\frac{1}{2}$ or not.

For the purpose of obtaining the zeroth-order solution, it is sufficient to use $f(D) = 1 + D$ and hence by (2.13)

$$g(D) \sim 1\,. \tag{4.9}$$

Let $\Psi^{(0)}$ be the zeroth-order approximation to Ψ. Because of (4.5), introduction of the variables z' and ρ' through

$$z' = z/R\,, \qquad \rho' = \rho/\sqrt{R}\,, \tag{4.10}$$

shows that the equation for $\Psi^{(0)}$ is

$$\partial_{z'}\left[\frac{\partial_{z'}\Psi^{(0)}}{\partial_{\rho'}\Psi^{(0)}}\right] - \partial_{\rho'}\left[\frac{1}{2}\left(\frac{\partial_{z'}\Psi^{(0)}}{\partial_{\rho'}\Psi^{(0)}}\right)^2 + \frac{\partial_{\rho'}\Psi^{(0)}}{2\pi\rho'}\right] = 0\,, \tag{4.11}$$

as a consequence of (2.11) and (2.9). If we express ρ' as a function of z' and $\Psi^{(0)}$, the resulting partial differential equation for ρ' can be solved by separation of variables. Using the boundary condition (2.10), the result is

$$\Psi^{(0)}(\rho, z) = \left(1 - \tfrac{1}{2}\sqrt{\pi}\,\frac{R\rho^2}{R^2 - z^2}\right)^2, \tag{4.12}$$

and the boundary B is an ellipsoid of revolution

$$\frac{z^2}{R^2} + \tfrac{1}{2}\sqrt{\pi}\,\frac{\rho^2}{R} = 1\,, \tag{4.13}$$

with semi-major axis R and semi-minor axis $(2R/\sqrt{\pi})^{1/2}$.

4.3. ELECTROSTATIC ENERGY

We proceed to calculate the total electrostatic energy $V(R)$

$$V = \int \mathrm{d}^3x \int_0^D \mathrm{d}D'\, E(D') \sim \int \mathrm{d}^3x\,(D + \tfrac{1}{2}D^2)\,, \tag{4.14}$$

using the zeroth-order solution (4.12). To avoid possible problems in the vicinity of the charges, we integrate over z from $-(R-\varepsilon)$ to $R-\varepsilon$:

$$V(R) \sim 4\pi \int_0^{R-\varepsilon} \mathrm{d}z \int_0^{\rho_b^{(0)}(z)} \rho\, \mathrm{d}\rho \left[-\frac{1}{2\pi\rho}\partial_\rho \Psi^{(0)} - \frac{1}{4\pi\rho}\frac{(\partial_z \Psi^{(0)})^2}{\partial_\rho \Psi^{(0)}} + \frac{1}{8\pi^2\rho^2}(\partial_\rho \Psi^{(0)})^2 \right], \tag{4.15}$$

where

$$\rho_b^{(0)}(z) = (2R/\sqrt{\pi})^{1/2}(1 - z^2/R^2)^{1/2}. \tag{4.16}$$

The substitution of (4.12) into (4.15) gives

$$V(R) = 2R + \frac{2}{3\sqrt{\pi}} \ln R + \mathrm{O}(1), \tag{4.17}$$

for large R.

This result (4.17) holds provided that the units (4.7) are used. In order to compare with the numerical result of Adler and Piran [3] who used the units

$$E_0 = \kappa, \qquad b_1 = \tfrac{16}{9}\pi^2, \qquad Q = 2\sqrt{\tfrac{1}{3}}, \tag{4.18}$$

a rescaling is necessary. They obtained

$$V(R) = 2Q\kappa R + C\kappa^{1/2} \ln(R\sqrt{\kappa}) + \mathrm{O}(1). \tag{4.19}$$

Using (4.17), rescaling to the units (4.18) gives

$$C = \pi^{1/2} 2^{9/2} 3^{-11/4} = 1.954908641, \tag{4.20}$$

to be compared with the numerical result of 1.95 [3].

More generally, when the σ of (4.1) is less than one, then the correction to the $2R$ term in $V(R)$ is of the order $R^{(1-\sigma)/(1+\sigma)}$. On the other hand, for $\sigma > 1$ there is no correction term which is unbounded as $R \to \infty$.

The conclusion (4.17) is actually not as obvious as it may seem. The question is: how can we get two terms respectively of order R and order $\ln R$ from the zeroth-order approximation?

(a) The first term in the integral of (4.15) is

$$-2 \int_0^{R-\varepsilon} \mathrm{d}z \int_0^{\rho_b^{(0)}(z)} \mathrm{d}\rho\, \partial_\rho \Psi^{(0)} = -2 \int_0^{R-\varepsilon} \mathrm{d}z\, [\Psi^{(0)}(\rho_b^{(0)}(z), z) - \Psi^{(0)}(0, z)]. \tag{4.21}$$

If the exact Ψ is used instead of $\Psi^{(0)}$, $\rho_b^{(0)}$ changes correspondingly such that, by (2.10),

$$\Psi(\rho_b(z), z) = \Psi^{(0)}(\rho_b^{(0)}(z), z) = 0,$$

$$\Psi(0, z) = \Psi^{(0)}(0, z) = 1.$$

Therefore this integral contributes $2(R-\varepsilon)$ exactly.

(b) We have implicitly assumed here that the distance between the charge and the z-axis intersection of the boundary B of the confinement domain is of order 1. This is justified as follows.

Since only the derivatives of Ψ appear in (2.11), we can shift z to $z-R=z_1$ without changing the equation. In the limit $R\to\infty$ the boundary condition (2.10) becomes

$$\Psi(0, z_1)=\begin{cases}1\,, & \text{for } z_1<0\\ 0\,, & \text{for } z_1>0\,.\end{cases}$$

Thus R has disappeared from both the partial differential equation and the boundary condition, and hence $\Psi(\rho, z_1)$ exists in this limit. This implies in particular that ρ_b is finite for finite z_1.

4.4. FIRST-ORDER SOLUTION

To obtain the solution to the order $1/R$, we use the following variables. Let $\rho_b(z)$ denote the boundary B,

$$r=\rho_b(z)/\rho_b(0)\,,\qquad \tau=\frac{\rho}{\rho_b(0)r}\,. \tag{4.22}$$

In terms of the variables z' of (4.10) and this τ, (2.12) becomes

$$\begin{aligned}&\frac{g(D)}{r^2D}\frac{\rho_b(0)^2}{R^2}\bigg\{(\partial_{z'}\Psi)^2\partial_{\tau\tau}\Psi+(\partial_\tau\Psi)^2\left[2\frac{\partial_{z'}r}{r}\partial_{z'}\Psi-\frac{\tau\partial_{z'z'}r}{r}\partial_\tau\Psi\right]+(\partial_\tau\Psi)^2\partial_{z'z'}\Psi\\&\quad-2(\partial_\tau\Psi)(\partial_{z'}\Psi)\,\partial_{z'\tau}\Psi\bigg\}\\&\quad+(r^{-1}\partial_\tau\Psi)^2\left[1+\frac{\rho_b(0)^2}{R^2}\left(\tau\partial_{z'}r-r\frac{\partial_{z'}\Psi}{\partial_\tau\Psi}\right)^2\right]\left\{\frac{1}{r^2}\tau\partial_\tau\left(\frac{\partial_\tau\Psi}{\tau}\right)\right.\\&\quad\left.+\frac{\rho_b(0)^2}{R^2}\left[\left(\frac{\partial_{z'}r}{r}\right)^2\tau\partial_\tau(\tau\partial_\tau\Psi)-2\tau\frac{\partial_{z'}r}{r}\partial_{z'\tau}\Psi-\tau(\partial_\tau\Psi)\partial_{z'}\left(\frac{\partial_{z'}r}{r}\right)+\partial_{z'z'}\Psi\right]\right\}=0\,,\end{aligned} \tag{4.23}$$

with

$$D=-\frac{1}{2\pi\rho_b(0)^2}\frac{\partial_\tau\Psi}{\tau r^2}\left[1+\frac{\rho_b(0)^2}{R^2}\left(\tau\partial_{z'}r-r\frac{\partial_{z'}\Psi}{\partial_\tau\Psi}\right)^2\right]^{1/2}. \tag{4.24}$$

These equations are exact. Expansion to first order is of the form

$$\rho_b(0)^2=\frac{2}{\sqrt{\pi}}R\left(1+\frac{c_1}{4\sqrt{\pi}R}\right), \tag{4.25}$$

$$\Psi(\rho, z) = (1-\tau^2)^2 + \frac{1}{4\sqrt{\pi}R}\Psi^{(1)} , \tag{4.26}$$

$$r = (1-z'^2)^{1/2} + \frac{1}{4\sqrt{\pi}R} r^{(1)} , \tag{4.27}$$

where the zeroth-order solution of subsect. 4.2 has been used. We then substitute (4.25)–(4.27) into (4.23) and collect the $1/R$ terms:

$$\begin{aligned} &-8b_2\frac{1-\tau^2}{1-z'^2} - 12\frac{\tau^2 z'^2}{1-z'^2} - (1-z'^2)^{-1/2}[(1-z'^2)^2\partial_{z'z'}r^{(1)} + r^{(1)}] \\ &-\frac{(1-z'^2)^2}{4\tau^2(1-\tau^2)}\left[\partial_{z'z'}\Psi^{(1)} - \frac{2z'}{1-z'^2}\partial_{z'}\Psi^{(1)}\right] \\ &+2c_1 + 4(1-z'^2)^{-1/2}r^{(1)} - \frac{1}{8\tau}\partial_\tau\left(\frac{1}{\tau}\partial_\tau\Psi^{(1)}\right) \\ &+\frac{8z'^2(1-2\tau^2)}{1-z'^2} + \frac{4(1+z'^2)(1-\tau^2)}{1-z'^2} = 0 . \end{aligned} \tag{4.28}$$

Since all terms in (4.28) that do not involve $\Psi^{(1)}$ have τ dependences of the forms 1 or τ^2, and $\Psi^{(1)}$ satisfies the boundary conditions

$$\Psi^{(1)} = 0 , \qquad \text{at } \tau = 0 ,$$

$$\Psi^{(1)} = \partial_\tau\Psi^{(1)} = 0 , \qquad \text{at } \tau = 1 ,$$

$\Psi^{(1)}$ must be of the form

$$\Psi^{(1)} = \tau^2(1-\tau^2)^2 k(z') . \tag{4.29}$$

The substitution of (4.29) into (4.28) yields the two ordinary differential equations

$$\frac{4(1-2b_2)}{1-z'^2} + \frac{32z'^2}{1-z'^2} - \tfrac{1}{4}(1-z'^2)^2\left[\partial_{z'z'}k - \frac{2z'}{1-z'^2}\partial_{z'}k\right] + 3k = 0 , \tag{4.30}$$

$$\partial_{z'z'}r^{(1)} - 3(1-z'^2)^{-2}r^{(1)} + (1-z'^2)^{-3/2}(-2c_1 + k) + 20(1-z'^2)^{-5/2}z'^2 = 0 , \tag{4.31}$$

for the determination of the two functions k and $r^{(1)}$.

With the variable

$$u' = \tanh^{-1} z' , \tag{4.32}$$

the solution of (4.30) is

$$k = -(9-2b_2)\cosh 2u' + \tfrac{2}{3}(7+2b_2) . \tag{4.33}$$

Let

$$r^{(1)} = (1-z'^2)^{1/2} w^{(1)} , \tag{4.34}$$

then (4.31) is

$$\partial_{u'u'} w^{(1)} - 4w^{(1)} = 2c_1 + \tfrac{4}{3}(4-b_2) - (1+2b_2)\cosh 2u' . \tag{4.35}$$

This equation has the interesting feature that the two cases of

$$1+2b_2=0\,, \qquad 1+2b_2 \neq 0\,,$$

are qualitatively different. In the latter case $w^{(1)}(u)$ has a term proportional to $u'\sinh 2u'$. As $z' \to 1$, such a term is of order $\ln(1-z')/(1-z')$. On the other hand, in the case $b_2 = -\frac{1}{2}$, no such logarithmic factor appears to first order. These two cases are treated in more detail in the next two sections.

The first-order solution is

$$\Psi = (1-\tau^2)^2 \left\{ 1 + \frac{\tau^2}{4\sqrt{\pi}R}\left[(2b_2-9)\cosh 2u' + \tfrac{2}{3}(2b_2+7)\right] \right\} , \tag{4.36}$$

with the boundary B given by

$$r = \frac{1}{\cosh u'}\left\{ 1 - \frac{1}{16\sqrt{\pi}R}\left[(2b_2+1)u'\sinh 2u' - (2c_1 - \tfrac{4}{3}(b_2-4)(\cosh 2u'-1))\right] \right\} . \tag{4.37}$$

5. Case of large R with $b_2 = -\frac{1}{2}$

5.1. PROPERTIES OF THE FIRST-ORDER SOLUTION

When $b_2 = -\frac{1}{2}$, (4.37) for the boundary B simplifies to

$$r = \frac{1}{\cosh u'}\left[1 + \frac{1}{8\sqrt{\pi}R}(c_1+3)(\cosh 2u' - 1) \right] . \tag{5.1}$$

In terms of the original cylindrical coordinates, this is

$$\rho_b(z) = \left(\frac{2}{\sqrt{\pi}}R\right)^{1/2}\left(1+\frac{c_1}{8\sqrt{\pi}R}\right)\left(1-\frac{z^2}{R^2}\right)^{1/2}\left[1+\frac{1}{4\sqrt{\pi}R}(c_1+3)\frac{z^2}{R^2-z^2}\right] . \tag{5.2}$$

To the first order in $1/R$, (5.2) can be rewritten more elegantly as

$$\rho_b(z) = \left[\frac{2}{\sqrt{\pi}}\left(R+\frac{c_1}{4\sqrt{\pi}}\right)\right]^{1/2}\left[1-\left(R+\frac{c_1+3}{4\sqrt{\pi}}\right)^{-2} z^2\right]^{1/2} . \tag{5.3}$$

Therefore to the first order, the boundary B remains an ellipsoid of revolution with semi-major axis

$$R + \frac{c_1+3}{4\sqrt{\pi}} , \tag{5.4}$$

and semi-minor axis

$$\left[\frac{2}{\sqrt{\pi}}\left(R+\frac{c_1}{4\sqrt{\pi}}\right)\right]^{1/2}. \tag{5.5}$$

The coefficient c_1 cannot be obtained by asymptotic methods. The reason is that the determination of c_1 requires information about the solution of the partial differential equation near the charges $\pm Q$, while a necessary condition for the applicability of the asymptotic methods is that D is small. Note that the same combination $R+c_1/4\sqrt{\pi}$ appears in both (5.4) and (5.5).

5.2. CHOICE OF VARIABLES

Since the first-order solution presented in subsect. 4.4 is already fairly complicated, the study of higher-order solutions entails a careful choice of variables. It is seen from the first-order expansions (4.25)–(4.27) and the results (5.4) and (5.5) that a convenient parameter for expansions is $4\sqrt{\pi}R+\mathrm{O}(1)$. With this in mind, we introduce the parameters Λ, Λ', and Λ'' which differ from $4\sqrt{\pi}R$ by order 1 and let

$$\rho_{\mathrm{b}}(0)=\left(\frac{\Lambda''}{2\pi}\right)^{1/2}, \tag{5.6}$$

$$\bar{z}=4\sqrt{\pi}z/\Lambda', \tag{5.7}$$

$$\psi=(\Lambda'/\Lambda'')^2\Psi, \tag{5.8}$$

$$\Lambda=\Lambda'^2/\Lambda''. \tag{5.9}$$

With this notation, (2.12) becomes

$$\frac{8g(D)}{r^2D\Lambda}\left\{(\partial_{\bar{z}}\psi)^2\partial_{\tau\tau}\psi+(\partial_\tau\psi)^2\left[2\frac{\partial_{\bar{z}}r}{r}\partial_{\bar{z}}\psi-\frac{\tau\partial_{\bar{z}\bar{z}}r}{r}\partial_\tau\psi\right]+(\partial_\tau\psi)^2\partial_{\bar{z}\bar{z}}\psi-2(\partial_\tau\psi)(\partial_{\bar{z}}\psi)\partial_{\bar{z}\tau}\psi\right\}$$
$$+(r^{-1}\partial_\tau\psi)^2\left[1+\frac{8}{\Lambda}\left(\tau\partial_{\bar{z}}r-r\frac{\partial_{\bar{z}}\psi}{\partial_\tau\psi}\right)^2\right]\left\{\frac{1}{r^2}\tau\partial_\tau\left(\frac{\partial_\tau\psi}{\tau}\right)\right.$$
$$\left.+\frac{8}{\Lambda}\left[\left(\frac{\partial_{\bar{z}}r}{r}\right)^2\tau\partial_\tau(\tau\partial_\tau\psi)-2\tau\frac{\partial_{\bar{z}}r}{r}\partial_{\bar{z}\tau}\psi-\tau(\partial_\tau\psi)\partial_{\bar{z}}\left(\frac{\partial_{\bar{z}}r}{r}\right)+\partial_{\bar{z}\bar{z}}\psi\right]\right\}=0, \tag{5.10}$$

with

$$D=-\frac{\partial_\tau\psi}{\tau r^2\Lambda}\left[1+\frac{8}{\Lambda}\left(\tau\partial_{\bar{z}}r-r\frac{\partial_{\bar{z}}\psi}{\partial_\tau\psi}\right)^2\right]^{1/2}. \tag{5.11}$$

Note that we are still at liberty to vary Λ', and hence Λ, by an amount of order 1. It is this freedom that enables us to absorb the unknown coefficient c_1 and thus render the perturbation calculation more manageable.

Similar to (4.32), let

$$u=\tanh^{-1}\bar{z}\,. \tag{5.12}$$

The perturbation expansions to be considered in this sect. 5 are

$$\psi=(1-\tau^2)^2+\Lambda^{-1}\psi^{(1)}+\Lambda^{-2}\psi^{(2)}+\Lambda^{-3}\psi^{(3)}+\cdots, \tag{5.13}$$

$$r=\frac{1}{\cosh u}+\Lambda^{-1}r^{(1)}+\Lambda^{-2}r^{(2)}+\Lambda^{-3}r^{(3)}+\cdots. \tag{5.14}$$

Note that the $r^{(1)}$ here is not quite the same as that of subsect. 4.4, but this should cause no confusion.

The present $r^{(1)}$ and also $\psi^{(1)}$ are easily obtained by rescaling the first-order results of subsect. 4.4:

$$\psi^{(1)}=(1-\tau^2)^2[6+\tau^2(-10\cosh 2u+4)]\,, \tag{5.15}$$

$$r^{(1)}=0\,. \tag{5.16}$$

In writing down (5.15) and (5.16), we have chosen

$$\Lambda=\Lambda''+6\,. \tag{5.17}$$

This choice, which we are at liberty to make as pointed out above, leads to a particularly simple forms for $\psi^{(1)}$ and $r^{(1)}$. That $r^{(1)}=0$ shows that the boundary B remains an ellipsoid of revolution in first order.

5.3. SECOND-ORDER SOLUTION

We have seen that to first order in $1/\Lambda$ the boundary B remains essentially unchanged for one particular value of the Taylor coefficient $b_2=-\frac{1}{2}$. As we discuss in this section, the same phenomenon occurs to second order if, in addition to $b_2=-\frac{1}{2}$, we also choose

$$b_3=\tfrac{7}{12}\,. \tag{5.18}$$

To second order

$$\psi=(1-\tau^2)^2\left\{1+\frac{2}{\Lambda}[3-\tau^2(5\cosh 2u-2)]\right\}+\frac{1}{\Lambda^2}\psi^{(2)}, \tag{5.19}$$

$$r=\frac{1}{\cosh u}+\frac{1}{\Lambda^2}r^{(2)}, \tag{5.20}$$

$$g(D)=1+D+\tfrac{3}{2}(1-2b_3)D^2\,. \tag{5.21}$$

Expanding (5.10) to order $1/\Lambda^2$ and following the procedure of subsect. 4.4, we obtain

$$\psi^{(2)}=\tfrac{1}{12}(1-\tau^2)^2\{3(163-12b_3)$$
$$-2\tau^2[(145-12b_3)\cosh 4u+4(133-12b_3)\cosh 2u-3(51+12b_3)]$$
$$+\tau^4[(703-12b_3)\cosh 4u-4(161+12b_3)\cosh 2u+3(155-12b_3)]\}\,, \tag{5.22}$$

$$r^{(2)}=\frac{7-12b_3}{144\cosh u}(\cosh 4u-\cosh 2u+12u\sinh 2u)\,. \tag{5.23}$$

Note that, by (2.10), (5.8) and (5.9), (5.22) implies that

$$\Lambda''=\Lambda-6-\frac{1}{4\Lambda}(163-12b_3)\,,$$

or

$$\Lambda=\Lambda''+6+\frac{1}{4\Lambda''}(163-12b_3)\,. \tag{5.24}$$

This involves the choice of a constant in the second order just like (5.17). That $r^{(2)}=0$ when (5.18) is satisfied proves our assertion that the boundary B remains an ellipsoid of revolution in second order when $b_2=-\frac{1}{2}$ and $b_3=\frac{7}{12}$.

5.4. THIRD-ORDER SOLUTION

Having found that for a special choice of the Taylor coefficients b_2 and b_3 the boundary B remains an ellipsoid of revolution in the first and second orders, curiosity motivated us to look into the situation in the third order.

Taking $b_2=-\frac{1}{2}$ and $b_3=\frac{7}{12}$, we have to third order

$$\psi=(1-\tau^2)^2\left\{1+\frac{2}{\Lambda}[3-\tau^2(5\cosh 2u-2)]\right.$$
$$+\frac{1}{\Lambda^2}[39-\tau^2(23\cosh 4u+84\cosh 2u-29)$$
$$\left.+\tau^4(58\cosh 4u-56\cosh 2u+37)]\right\}+\frac{1}{\Lambda^3}\psi^{(3)}\,, \tag{5.25}$$

$$r=\frac{1}{\cosh u}+\frac{1}{\Lambda^3}r^{(3)}\,, \tag{5.26}$$

$$g(D)=1+D-\tfrac{1}{4}D^2-(\tfrac{19}{4}+4b_4)D^3\,. \tag{5.27}$$

Expanding (5.10) to order $1/\Lambda^3$ and following the same procedure once again, we obtain for $r^{(3)}$:

$$r^{(3)}=\frac{1}{1200\cosh u}[(27\cosh 6u-28\cosh 4u+\cosh 2u)$$

$$-30(b_4+\tfrac{103}{120})(\cosh 6u+16\cosh 4u-17\cosh 2u+120\,u\sinh 2u)]\,, \qquad (5.28)$$

and for $\psi^{(3)}$ at $\tau=0$:

$$\psi^{(3)}(u,0)=265-8(b_4+\tfrac{103}{120})\,. \qquad (5.29)$$

The complete expression for $\psi^{(3)}$ is quite complicated and not instructive. Therefore in the present case

$$\Lambda''=\Lambda-6-\frac{39}{\Lambda}-\frac{1}{\Lambda^2}[499-8(b_4+\tfrac{103}{120})]\,,$$

or

$$\Lambda=\Lambda''+6+\frac{39}{\Lambda''}+\frac{1}{\Lambda''^2}[265-8(b_4+\tfrac{103}{120})]\,. \qquad (5.30)$$

Once again, this involves the choice of a constant in third order.

The new feature is that, as seen from (5.28), the boundary B is no longer an ellipsoid of revolution no matter what the value of b_4 is. For $b_4=-\frac{103}{120}$, both $r^{(3)}$ and $\psi^{(3)}$ are elementary functions of z without logarithms. In this case,

$$f(D)=1+D-\tfrac{1}{2}D^2+\tfrac{7}{12}D^3-\tfrac{103}{120}D^4+\cdots\,. \qquad (5.31)$$

6. Case of large R with $b_2\neq-\frac{1}{2}$

6.1. FORMULATION OF THE PROBLEM

We have found that, for $b_2\neq-\frac{1}{2}$ and to the first order in $1/R$, the equation (4.37) for the boundary B has a term proportional to $(u'\sinh u')/R$. This implies that, in the notation of subsect. 5.2, r has a term proportional to $(u\sinh u)/\Lambda$. This correction term becomes comparable to the leading order when

$$\frac{u\,e^{2u}}{\Lambda}=O(1)\,. \qquad (6.1)$$

It is the purpose of this section to study, for the case $b_2\neq-\frac{1}{2}$, the behaviour of the boundary B and the field in the region given by (6.1). For this purpose, it is necessary to sum the leading terms in the $1/\Lambda$ power series expansion of r. These terms are of the orders

$$e^{-u}\,,\qquad \frac{u\,e^{u}}{\Lambda}\,,\qquad \frac{u^2\,e^{3u}}{\Lambda^2}\,,\qquad \frac{u^3\,e^{5u}}{\Lambda^3}\,,\qquad \ldots\,. \qquad (6.2)$$

Consequently, for $w = r\cosh u$, we expect the sum of the leading terms to be a function of

$$\xi = \frac{u\,\mathrm{e}^{2u}}{\Lambda}. \tag{6.3}$$

Inclusion of the subleading terms leads, in the region (6.1), to an expansion for w of the form

$$w = w_0(\xi) + \frac{1}{u} w_1(\xi) + \cdots, \tag{6.4}$$

and a corresponding expansion for ψ

$$\psi = \psi_0(\xi, \tau) + \frac{1}{u}\psi_1(\xi, \tau) + \cdots. \tag{6.5}$$

It is seen from (4.36) that

$$\psi_0(\xi, \tau) = (1-\tau^2)^2. \tag{6.6}$$

In order to obtain $w_0(\xi)$, we rewrite (5.10) in the form, after replacing both $\cosh u$ and $\sinh u$ by $\frac{1}{2}\mathrm{e}^u$,

$$\begin{aligned}
&g(D)\, w^4 \left\{1 + \frac{2}{\Lambda}\mathrm{e}^{2u}\left[\tau(w - \partial_u w) + w\frac{\partial_u\psi}{\partial_\tau\psi}\right]^2\right\}^{-3/2} \\
&\quad\times\left\{\tau\left(1 - \frac{\partial_{uu}w}{w}\right) + \partial_u\frac{\partial_u\psi}{\partial_\tau\psi} - \frac{\partial_u\psi}{\partial_\tau\psi}\left[\partial_\tau\frac{\partial_u\psi}{\partial_\tau\psi} + 2\frac{\partial_u w}{w}\right]\right\} \\
&= \tfrac{1}{8}\partial_\tau\frac{\partial_\tau\psi}{\tau} + \frac{w^2\,\mathrm{e}^{2u}}{4\Lambda}\left\{\left[3 - 4\frac{\partial_u w}{w} + 2\left(\frac{\partial_u w}{w}\right)^2 - \frac{\partial_{uu}w}{w}\right]\partial_\tau\psi + \frac{2}{\tau}\partial_u\psi\right. \\
&\quad\left. + \left(1 - \frac{\partial_u w}{w}\right)^2 \tau\partial_{\tau\tau}\psi + 2\left(1 - \frac{\partial_u w}{w}\right)\partial_{u\tau}\psi + \frac{1}{\tau}\partial_{uu}\psi\right\},
\end{aligned} \tag{6.7}$$

where

$$D = -\frac{\mathrm{e}^{2u}}{\Lambda}\,\frac{\partial_\tau\psi}{4\tau w^2}\left\{1 + \frac{2}{\Lambda}\mathrm{e}^{2u}\left[\tau(w - \partial_u w) + w\frac{\partial_u\psi}{\partial_\tau\psi}\right]^2\right\}^{1/2}. \tag{6.8}$$

Taking the limit $\Lambda \to \infty$ with fixed ξ, we obtain from (6.7)

$$w_0^4\left[1 - \frac{4}{w_0}\xi\partial_\xi\xi\partial_\xi w_0\right] = 1. \tag{6.9}$$

With the condition $w_0 = 1$ at $\xi = 0$, the solution of (6.9) is given by

$$w_0 = (1 + A\xi)^{1/2}, \tag{6.10}$$

where the constant A will be determined in subsect. 6.3.

6.2. DETERMINATION OF ψ_1

In order to find the ψ_1 of (6.5), we need to expand (6.7) to next order. It is sufficient to use $g(D) = 1 - 2b_2 D$ with

$$D = \frac{1}{u}\,\frac{1-\tau^2}{w_0^2}\,\xi\,, \tag{6.11}$$

from (6.8). Inspection of (6.7) to next order with reference to its τ dependence shows that ψ_1 is of the form

$$\psi_1(\xi,\tau) = \tau^2(1-\tau^2)^2 h(\xi)\,, \tag{6.12}$$

where the boundary conditions

$$\psi_1(\xi,0) = \psi_1(\xi,1) = \partial_\tau\psi_1(\xi,1) = 0\,, \tag{6.13}$$

have been used. With these approximations, (6.7) reduces to

$$\begin{aligned}
&w^4\left[1 - \frac{2b_2}{u}\,\frac{1-\tau^2}{w^2}\,\xi\right]\left[1 + \frac{2}{u}\,\xi(w-\partial_u w)^2\tau^2\right]^{-3/2}\\
&\quad\times\left[\tau\left(1 - \frac{\partial_{uu}w}{w}\right) - \frac{1}{u}\,\tau(1-\tau^2)\xi\partial_\xi\xi\partial_\xi h + \frac{1}{u}\,\frac{\partial_u w}{w}\,\tau(1-\tau^2)\xi\partial_\xi h\right]\\
&= \tfrac{1}{8}\partial_\tau\frac{\partial_\tau\psi}{\tau} + \frac{1}{4}\,\frac{1}{u}\,w^2\xi\left\{\left[3 - 4\frac{\partial_u w}{w} + 2\left(\frac{\partial_u w}{w}\right)^2 - \frac{\partial_{uu}w}{w}\right](-4\tau)(1-\tau^2)\right.\\
&\quad\left.+\left(1 - \frac{\partial_u w}{w}\right)^2(-4\tau)(1-3\tau^2)\right\}.
\end{aligned} \tag{6.14}$$

In all the correction terms, i.e. terms with the factor $1/u$, w can be replaced by the w_0 of (6.10). Therefore, to the present order, the differential equation is

$$\begin{aligned}
&w^4\left[1 - \frac{2b_2}{u}\,\frac{1-\tau^2}{1+A\xi}\,\xi - \frac{3}{4}\,\xi\frac{\tau^2}{1+A\xi}\right]\\
&\quad\times\left[\tau\left(1 - \frac{\partial_{uu}w}{w}\right) - \frac{1}{u}\,\tau(1-\tau^2)\xi\partial_\xi\xi\partial_\xi h + \frac{1}{u}\,\frac{A\xi}{1+A\xi}\,\tau(1-\tau^2)\xi\partial_\xi h\right]\\
&= \tau + \frac{1}{u}\,\tau(-2+3\tau^2)h - \frac{2\tau}{u}\,\frac{1}{1+A\xi}\,\xi(2-3\tau^2)\,.
\end{aligned} \tag{6.15}$$

For the purpose of determining h, we study the coefficient of τ^3/u:

$$\xi\partial_\xi\xi\partial_\xi h - \frac{A\xi}{1+A\xi}\,\xi\partial_\xi h - \frac{3}{(1+A\xi)^2}\,h = \frac{(9-2b_2)\xi}{(1+A\xi)^3}\,. \tag{6.16}$$

The only solution without fractional powers of ξ for small ξ is

$$h(\xi)=\tfrac{1}{2}\sqrt{\tfrac{1}{3}}(9-2b_2)\left\{h_2(\xi)\int_0^\xi d\xi'\,h_1(\xi')(1+A\xi')^{-4}-h_1(\xi)\int_0^\xi d\xi'\,h_2(\xi')(1+A\xi')^{-4}\right\},\tag{6.17}$$

where

$$\begin{aligned}h_1(\xi)&=\xi^{\sqrt{3}}(1+A\xi)^{-1}F(\sqrt{3}-2,\sqrt{3}-1;2\sqrt{3}+1;-A\xi)\,,\\ h_2(\xi)&=\xi^{-\sqrt{3}}(1+A\xi)^{-1}F(-\sqrt{3}-2,-\sqrt{3}-1;-2\sqrt{3}+1;-A\xi)\,,\end{aligned}\tag{6.18}$$

are solutions of the homogeneous equation. Here F is the hypergeometric function. In particular, it follows from (6.16) that

$$h(\xi)=\frac{9-2b_2}{4A}\frac{1}{1+A\xi}[\ln|1+A\xi|+O(1)]\tag{6.19}$$

for $1+A\xi$ near zero.

6.3. DETERMINATION OF w_1

In order to find the w_1 of (6.4), we set $\tau=1$ in (6.15):

$$w^3(w-\partial_{uu}w)=1+\frac{1}{u}\left[h(\xi)+\frac{5\xi}{1+A\xi}\right].\tag{6.20}$$

Let

$$w_1(\xi)=\frac{\xi}{(1+A\xi)^{1/2}}v(\xi)\,,\tag{6.21}$$

then

$$\xi(1+A\xi)v''(\xi)+(3+2A\xi)v'(\xi)=-\frac{1}{4\xi}\left[\frac{h(\xi)}{\xi}+\frac{5}{1+A\xi}+A(2+A\xi)\right].\tag{6.22}$$

Since $v(\xi)$ has a power-series expansion at $\xi=0$, there cannot be a pole on the right-hand side of (6.22) at $\xi=0$. Since

$$\lim_{\xi\to 0}f(\xi)/\xi=\tfrac{1}{2}(2b_2-9)\,,\tag{6.23}$$

as seen from (6.16), this absence of a pole implies the condition that

$$A=-\tfrac{1}{4}(2b_2+1)\,.\tag{6.24}$$

This could have been obtained by comparing (6.10) with (4.37). However, the present derivation does not depend on the $1/R$ perturbation calculation.

With this value of A, the solution of (6.22) is

$$v(\xi)=\frac{9-2b_2}{8\sqrt{3}}[I(-A\xi)-I(-A\xi)|_{\sqrt{3}\to-\sqrt{3}}]-\tfrac{5}{8}\ln\frac{-A\xi}{1+A\xi}-\tfrac{1}{4}A\ln(-A\xi)\,, \qquad (6.25)$$

where

$$\begin{aligned}I(x)=\frac{1}{2(1-\sqrt{3})(2-\sqrt{3})}\Big\{&x^{-1-\sqrt{3}}(1-x)^{-1}[(5-2x)F(-\sqrt{3}),-1-\sqrt{3};1-2\sqrt{3};x)\\&-\sqrt{3}(3-4x)F(1-\sqrt{3},-1-\sqrt{3};1-2\sqrt{3};x)]\\&\times\int_0^x dx'\,x'^{\sqrt{3}}(1-x')^{-5}F(-2+\sqrt{3},-1+\sqrt{3};1+2\sqrt{3};x')\\&+\int_0^x dx'\,x'^{-1}(1-x')^{-7}F(-2+\sqrt{3},-1+\sqrt{3};1+2\sqrt{3};x')\\&\times\Big[\frac{(1-2x)x'^2}{x^2}((2+x')F(-\sqrt{3},-1-\sqrt{3};1-2\sqrt{3};x')\\&-\sqrt{3}(1-x')F(1-\sqrt{3},-1-\sqrt{3};1-2\sqrt{3};x'))\\&-((7-10x')F(-\sqrt{3},-1-\sqrt{3};1-2\sqrt{3};x')-2\sqrt{3}(1-x')\\&\times(2-3x')F(1-\sqrt{3},-1-\sqrt{3};1-2\sqrt{3};x'))\Big]\Big\}\,.\end{aligned} \qquad (6.26)$$

In (6.26), it is understood that

$$\int_0^x dx'\,x'^{-1}=\ln x\,,\qquad \int_0^x dx'\,x'^{-\sqrt{3}}=\frac{x^{1-\sqrt{3}}}{1-\sqrt{3}}\,. \qquad (6.27)$$

With this convention, all terms of the form $\ln(-A\xi)$ cancel each other. The derivation of this solution of (6.22) is straightforward but tedious.

For the discussion of sect. 7, we need the behaviour of $v(\xi)$ when $1+A\xi$ is near zero. This can be obtained directly from the differential equation (6.22) with the help of (6.19).

$$v(\xi)=-\tfrac{1}{64}(9-2b_2)[\ln(1+A\xi)]^2+\mathrm{O}[\ln(1+A\xi)]\,. \qquad (6.28)$$

7. Discussion

7.1. QUALITATIVE BEHAVIOUR FOR $b_2\neq-\frac{1}{2}$

It is seen from sect. 5 and sect. 6 that the cases $b_2=-\frac{1}{2}$ and $b_2\neq-\frac{1}{2}$ are qualitatively different. In both cases there is a central domain where $\rho_b=\mathrm{O}(\sqrt{R})$. In the case $b_2\neq-\frac{1}{2}$, we have learned from (6.10) that the boundary ρ_b is given approximately by

$$\rho_b = \left[\frac{2}{\sqrt{\pi}} R(1+A\xi)\right]^{1/2} \Big/ \cosh u$$
$$= \left(\frac{2}{\sqrt{\pi}}\right)^{1/2} \left[R(1-\bar{z}^2) + \frac{A}{2\sqrt{\pi}} \ln \frac{1+\bar{z}}{1-\bar{z}}\right]^{1/2}, \qquad (7.1)$$

where A is given by (6.24). Therefore the correction term becomes comparable to the leading term when

$$\rho_b = O(\sqrt{\ln R}) . \qquad (7.2)$$

It follows from (7.2) that the central domain for $b_2 \neq -\frac{1}{2}$ is characterized by $\rho_b \gg (\ln R)^{1/2}$.

Next we repeat this argument for the second domain where (7.2) holds. For this purpose, we consider the correction term to (7.1) as given by (6.28)

$$\rho_b = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2} \frac{1}{\cosh u} [R(1+A\xi)]^{1/2} \left\{1 - \tfrac{1}{64}(9-2b_2) \frac{\xi}{u(1+A\xi)} [\ln(1+A\xi)]^2\right\} . \qquad (7.3)$$

This formula holds for

$$\bar{z} = 1 + \frac{A}{4\sqrt{\pi} R} \ln R + O\left(\frac{\ln R}{R}\right) . \qquad (7.4)$$

Therefore, the correction term

$$-\tfrac{1}{64}(9-2b_2) \frac{\xi}{u(1+A\xi)} [\ln(1+A\xi)]^2 ,$$

is comparable to 1 when

$$1 + A\xi = O\left(\frac{[\ln(1+A\xi)]^2}{\ln R}\right) = O\left(\frac{[\ln \ln R]^2}{\ln \mathrm{R}}\right), \qquad (7.5)$$

and thus

$$\rho_b = O(\ln \ln R) . \qquad (7.6)$$

In summary, we have found that in the case $b_2 \neq -\frac{1}{2}$ there are at least three overlapping subdomains of the confinement domain: subdomain 1 (central domain) characterized by

$$\rho_b \gg (\ln R)^{1/2} ; \qquad (7.7)$$

subdomain 2 characterized by

$$\ln \ln R \ll \rho_b \ll R^{1/2} ; \qquad (7.8)$$

and subdomain 3 characterized partially by

$$\rho_b \ll (\ln R)^{1/2} . \qquad (7.9)$$

7.2. QUALITATIVE BEHAVIOUR FOR $b_2 = -\frac{1}{2}$

We now discuss the results obtained in sect. 5 for the case $b_2 = -\frac{1}{2}$. Then to first order on $1/R$ the boundary B remains an ellipsoid of revolution. Moreover, we have found in subsect. 5.3 that this remains true to order $1/R^2$ provided that the next Taylor coefficient b_3 of $f(D)$ is $\frac{7}{12}$. (For completeness, we mention that, in the case $b_2 = -\frac{1}{2}$ but $b_3 \neq \frac{7}{12}$, the central domain is characterized by $\rho_b \gg (\ln R)^{1/4}$.) When $b_2 = -\frac{1}{2}$ and $b_3 = \frac{7}{12}$, to order $1/R^3$ the situation is as follows:

(a) the boundary B cannot remain an ellipsoid of revolution no matter what b_4 is;

(b) only for the special case $b_4 = -\frac{103}{120}$: both the boundary B and the field are given by elementary functions of $\bar{z}$ without logarithms.

We do not know whether the elementary structure described in (b) remains valid to all orders in $1/R$ with an appropriate choice of the higher Taylor coefficients of $f(D)$. Inspection of the differential equations for ψ and r in higher orders leads us to suspect that logarithmic functions of $\bar{z}$ may appear in certain orders in $1/R$ when a diophantic equation is satisfied. These dangerous orders are 8th, 49th, 288th etc. If such logarithmic functions actually do not appear, then perhaps a unique function $f(D)$ exists such that there are only two subdomains where ρ is respectively of order $\sqrt{R}$ and 1. This problem remains to be investigated in more detail.

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