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CONTINUUM LIMIT IMPROVED LATTICE ACTION FOR PURE YANG–MILLS THEORY (II)

P WEISZ[1] and R WOHLERT

II Institut fur Theoretische Physik der Universitat Hamburg, Hamburg, Federal Republic of Germany

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Wilson loops are calculated to one-loop order using lattice actions including 6-link loops The static potential is extracted and its small-a expansion performed using coefficients determined at tree level The Λ-parameter for Symanzik's improved action is thereby obtained and constraints placed on the coefficients appearing in the action at order g^2

1. Introduction

Monte Carlo techniques applied to lattice regularised quantum field theories present possibilities to explore non-perturbative phenomena. To extract quantitative physical information, in asymptotically free theories, the continuum limit must be approached, in which case the bare coupling must *a priori* be chosen sufficiently small and the lattice size sufficiently large such that the constraints

$$\text{lattice spacing ``}a\text{''} \ll \text{correlation length } \xi \ll \text{lattice size } L\,, \qquad (1.1)$$

are fulfilled. Unfortunately, practical limitations, often force us, at present to violate these restrictions, and this leads to expected systematic errors. It has, however, been discovered by Symanzik [1, 2] that the corrections to the continuum theory stemming from finite lattice spacing can be systematically diminished by the use of a judicially chosen lattice action.

Symanzik has described the construction of the improved action for both ϕ_4^4 [3] theory and the non-linear σ-model in 2 dimensions [4]. The improved action for the non-linear σ-model was completely specified to 1-loop order [4] and used in Monte-Carlo calculations [5–7], and dramatic improvement in the scaling behaviour and other properties was claimed. There have been also some studies [8] involving an improved Gross–Neveu model lattice action which also indicates the merit of using such actions.

The initial success of Symanzik's programme for the above models encourages the pursual of the programme for Yang–Mills theory and to the full QCD. For the Yang–Mills theory the following motivated ansatz for the improved action was

[1] Heisenberg Foundation Fellow

made [9]

$$S_{\rm Imp} = -\frac{2}{g^2} \sum_{\text{Repts } R} \text{Re}\Big\{ c_{R0}(g^2) \sum_{\Box} \text{tr}\,(1 - u_R(\Box))$$
$$+ c_{R1}(g^2) \sum_{\text{rectangle}} \text{tr}\,(1 - u_R((\text{rectangle}))$$
$$+ c_{R2}(g^2) \sum_{\text{chair}} \text{tr}\,(1 - u_R(\text{chair})) + c_{R3}(g^2) \sum_{\text{cube}} \text{tr}\,(1 - u_R(\text{cube}))\Big\}. \qquad (1.2)$$

Monte-Carlo calculations have already been carried out [10] using tree-level improved coefficients. The topic of this paper is to place some constraints on the coefficients to 1-loop order.

We want first to take this opportunity to clarify a misconception introduced by one of us in ref. [9] (referred to as (I), in the following). In (I) it was incorrectly implied that to tree level $c_2(0)$, $c_3(0)$ ($c_i = \sum_R 2t_R c_{Ri}$) satisfied $c_2(0) + c_3(0) = 0$ but were separately undetermined. The correct statement is that the constraint $c_2(0) + c_3(0) = 0$ is the maximal information that one can extract from consideration of the Wilson loops at lowest order, but

$$c_2(0) = c_3(0) = 0\,, \qquad c_0(0) = \tfrac{5}{3}\,, \qquad c_1(0) = -\tfrac{1}{12}\,, \qquad (1.3)$$

is obtained by consideration of the improvement of the full classical action. A clear form of these considerations due to Lüscher is presented in sect. 2. Curci, Menotti and Paffuti [11] were the first to correctly determine the tree-level action, although their derivation followed calculation of corrections to the static potential at 1-loop order.

In sect. 3 we calculate Wilson loops with the improved action (1.2) on an infinite lattice to 1-loop order. In sect. 4 we extract the static potential and perform its small-a expansion. We thereby obtain a value for the ratio of Λ-parameters of improved and unimproved actions, and also place constraints on the 1-loop coefficients, $c_i'(0)$. Again, not all the coefficients are determined since the Wilson loop itself contains insufficient information at this order. However our result shows that bent plaquettes will be required at 1-loop order. The results are discussed briefly in the conclusion. Sect 5 deals with a definition of an improvement of the χ-variable, introduced by Creutz [14] to extract the string tension.

2. Improvement of the classical action

This section, which deals with the improvement of the classical action and hence of the quantum effective action to tree level, is due to Lüscher, who kindly permitted us to include his clear exposition in our paper.

Consider SU(n) gauge fields $U(n, \mu)$, $n \in \mathbb{Z}^d$, $\mu = 1, \ldots, d$, $d \geq 2$. An admissible classical lattice action S is required to fulfil the following.

(a) S is local,

$$S=\sum_{n} L(n)\,, \tag{2.1}$$

where $L(n)$ depends on a finite number of link variables in the neighbourhood of n.

(b) $L(n)$ is real and a scalar under translations of the fields. S is invariant under cubic rotations and reflections.

(c) $L(n)$ is a gauge invariant polynomial in the link variables.

(d) S has the correct classical continuum limit i.e. let $A_\mu(x)$ be an arbitrary C^∞ gauge field with compact support. For every $a>0$, define

$$U(n,\mu)=\mathrm{P}\exp\left[a\int_0^1 \mathrm{d}t\,A_\mu(an+a\hat{\mu}-ta\hat{\mu})\right]. \tag{2.2}$$

Then we require

$$\lim_{a\to 0} a^{d-4}S=\tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu}\,, \tag{2.3}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu, A_\nu]\,. \tag{2.4}$$

Remarks.

(i) Because of the classical nature of the consideration the coupling constant plays no rôle.

(ii) The idea associated with (2.2) is that a lattice with spacing "a" is embedded in space-time. The $U(n,\mu)$'s are then just the parallel transporters from $an+a\hat{\mu}$ to an determined by the A_μ's. The construction is thus gauge covariant: when A_μ is transformed by a gauge transformation then $U(n,\mu)$ is transformed by the corresponding lattice gauge transformation. (a)–(d) imply that

$$a^{d-4}S=\tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu})+\mathrm{O}(a^{2p})\,, \tag{2.5}$$

where, in the usual case, $p=1$. In the special case that $p\geq 1$, we call the action an improved classical action. In the following it is the aim to determine such an action. Consider first the case $d=2$. Define for each closed path ℓ on the lattice

$$W(\ell)=\mathrm{tr}\,U(\ell), \tag{2.6}$$

where $U(\ell)$ is the product of U's along ℓ. Further define

$$L_0(n)=N-\mathrm{Re}\,W(\ell_0), \tag{2.7}$$

$$L_1(n)=2N-\mathrm{Re}\,W(\ell_1)-\mathrm{Re}\,W(\ell_1')\,, \tag{2.8}$$

where ℓ_0, ℓ_1, ℓ_1' are the paths in fig. 1.

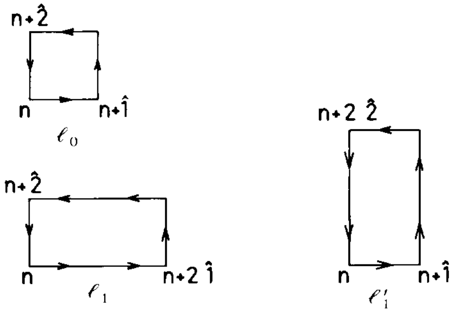

Fig 1 Loops appearing in the action (2 9)

The action

$$S = -\sum_n \{\alpha L_0(n) + \beta L_1(n)\}\,, \tag{2.9}$$

fulfils properties (a)–(c). The constants α, β should now be determined such that also (d) holds in the form of (2.5) with $p=2$ Consider then A_μ, a classical field as in (d), and define $l_0(x, a)$ and $l_1(x, a)$ analogously to L_0, L_1 except that the corresponding paths ℓ start at x instead of at an. In particular then

$$L_i(n) = l_i(an, a)\,. \tag{2.10}$$

It then follows from the theorem proved in appendix B that

$$a^{-2}S = \sum_{\nu=0}^{7} \frac{1}{\nu!} a^{\nu-4} \int \mathrm{d}^2x \frac{\partial^\nu}{\partial a^\nu} \{\alpha l_0(x, a) + \beta l_1(x, a)\}_{a=0} + \mathrm{O}(a^4)\,. \tag{2.11}$$

The integrands $(\partial^\nu/\partial a^\nu)\{\alpha l_0(x, a) + \beta l_1(x, a)\}_{a=0}$ in (2.11) are polynomials in $A_\mu(x)$ and their derivatives. The dimension of each contribution is exactly ν and only gauge invariant combinations occur. The sum extends over only even ν since terms with odd ν would violate parity. One immediately establishes that the integrands vanish for $\nu=0, 2$. It then follows that

$$a^{-2}S = \int \mathrm{d}^2x \left\{ r_0 \operatorname{tr} F^2 + a^2 r_1 \sum_{\mu=1}^{2} \operatorname{tr}(D_\mu F)^2 \right\}, \tag{2.12}$$

with

$$F = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]\,, \tag{2.13}$$

$$D_\mu F = \partial_\mu F + [A_\mu, F]\,. \tag{2.14}$$

To determine the coefficients r_0 and r_1 we consider an abelian field

$$A_\mu(x) = a_\mu(x) T, \qquad a_\mu(x) \in \mathbb{R}\,, \tag{2.15}$$

$$T = \frac{1}{2i}\begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}. \tag{2.16}$$

Then we have

$$\begin{aligned} F(x) &= f(x)T, \qquad f = \partial_1 a_2 - \partial_2 a_1 , \\ D_\mu F(x) &= \partial_\mu f(x) T, \\ \operatorname{tr} F^2 &= -\tfrac{1}{2} f^2 , \qquad \operatorname{tr}(D_\mu F)^2 = -\tfrac{1}{2}(\partial_\mu f)^2 , \end{aligned} \tag{2.17}$$

$$l_0(x, a) = 2\left\{1 - \cos\left(\tfrac{1}{2}a^2 \int_0^1 \int_0^1 \mathrm{d}^2 z\, f(x + az)\right)\right\}, \tag{2.18}$$

$$\begin{aligned} l_1(x, a) &= 2\left\{1 - \cos\left(\tfrac{1}{2}a^2 \int_0^1 \int_0^2 \mathrm{d}^2 z\, f(x + az)\right)\right\} \\ &\quad + 2\left\{1 - \cos\left(\tfrac{1}{2}a^2 \int_0^2 \int_0^1 \mathrm{d}^2 z\, f(x + az)\right)\right\}. \end{aligned} \tag{2.19}$$

Expanding for $a \to 0$

$$l_0(x, a) = \tfrac{1}{4}a^4\left\{ f^2 - \tfrac{1}{12}a^2 \sum_\mu (\partial_\mu f)^2 + \text{divergence} + \mathrm{O}(a^4)\right\}, \tag{2.20}$$

$$l_1(x, a) = \tfrac{1}{4}a^4\{8f^2 - a^2 \tfrac{5}{3} \sum_\mu (\partial_\mu f)^2 + \text{divergence} + \mathrm{O}(a^4)\}, \tag{2.21}$$

and inserting this in (2.11) together with (2.17) yields (2.12) with

$$r_0 = \tfrac{1}{2}\alpha + 4\beta , \tag{2.22}$$

$$r_1 = \tfrac{1}{24}\alpha + \tfrac{5}{6}\beta . \tag{2.23}$$

To get the improved classical action we chose

$$r_0 = 1 , \qquad r_1 = 0 \tag{2.24}$$

and hence require

$$\alpha = \tfrac{10}{3}, \qquad \beta = -\tfrac{1}{6}. \tag{2.25}$$

For dimension $d \geq 2$, for $\mu \neq \nu$, define $L_0^{\mu\nu}(n)$ and $L_1^{\mu\nu}(n)$ as $L_0(n)$, $L_1(n)$ above with the loops in the μ, ν plane. It then follows from the theorem in the appendix and the considerations for the case $d = 2$ that the action

$$S = \sum_n \sum_{\mu < \nu} \{\tfrac{10}{3} L_0^{\mu\nu}(n) - \tfrac{1}{6} L_1^{\mu\nu}(n)\} \tag{2.26}$$

is an improved classical action for arbitrary d. Now if one takes the classical continuum limit of the classical action

$$S=-a^{d-4}\frac{1}{g^2}\sum_{i=0}^{3}\sum_{\ell\in T_i}\sum_{R} c_{R_i}\,\mathrm{tr}\,(1-U_R(\ell))\,, \tag{2.27}$$

with the sum going over 4 and 6 link loops (see (I)) one obtains

$$S=\int d^dx\left\{k_0S_0+a^2\sum_{i=1}^{3}k_iS_i\right\}, \tag{2.28}$$

with

$$S_0=\mathrm{tr}\sum_{\mu,\nu}F_{\mu\nu}^2\,, \tag{2.29}$$

and the S_i operators are of dimension 6 (see (I)). The coefficients k_j are linear functions of the c's. Since every operator S_j can be projected out, the system of equations

$$k_0=\tfrac{1}{2}g^2\,,\qquad k_i=0\,, \tag{2.30}$$

has exactly one solution. As shown above then

$$c_0=\tfrac{5}{3}\,,\qquad c_1=-\tfrac{1}{12}\,,\qquad c_2=c_3=0\,, \tag{2.31}$$

is a solution and hence unique.

In the quantum case, the classical action equals the effective action to order g^{-2}. To improve the quantum action at tree level one then needs in particular

$$c_2(0)=c_3(0)=0\,. \tag{2.32}$$

In the following sections we will set $a=1$, "a" can always be reintroduced by dimensional analysis. For any undefined notation the reader is requested to consult paper (I).

3. Calculation of the Wilson Loop to second-order perturbation theory

Define the coefficients $w_{nR}(\ell)$ by

$$\ln\frac{1}{d_R}\langle\mathrm{tr}\,U_R(\ell)\rangle=-\sum_{n=1}^{\infty}\frac{g^{2n}}{(2n)!}w_{nR}(\ell) \tag{3.1}$$

These are expressible in terms of correlations $U^{(n)}$ involving products of n A fields defined through

$$\frac{1}{d_R}\langle\mathrm{tr}\,(U_R(\ell)-1)\rangle=\sum_{n=2}^{\infty}\frac{g^n}{n!}U_R^{(n)}(\ell,g)\,, \tag{3.2}$$

with

$$U_R^{(2)}(\ell,g)=-\frac{1}{d_R}\mathrm{tr}\,R^aR^b\left\langle\sum_{l_1,l_2}A_{l_1}^aA_{l_2}^b\right\rangle, \tag{3.3}$$

$$U_{\rm R}^{(3)}(\ell, g) = -i\frac{1}{d_{\rm R}}\,{\rm tr}\,R^aR^bR^c\Big\langle \sum_{l_1,l_2,l_3} A^a_{l_1}A^b_{l_2}A^c_{l_3} + 3\sum_{l_1<l_2<l_3}(A^a_{l_1}A^b_{l_2}A^c_{l_3} - A^a_{l_3}A^b_{l_2}A^c_{l_1})\Big\rangle\,, \tag{3.4}$$

$$\begin{aligned} U_{\rm R}^{(4)}(\ell, g) = \frac{1}{d_{\rm R}}\,{\rm tr}\,R^aR^bR^cR^d\Big\langle 24 &\sum_{l_1<l_2<l_3<l_4} A^a_{l_1}A^b_{l_2}A^c_{l_3}A^d_{l_4} \\ +12 &\sum_{l_1<l_2<l_3}(A^a_{l_1}A^b_{l_1}A^c_{l_2}A^d_{l_3} + A^a_{l_1}A^b_{l_2}A^c_{l_2}A^d_{l_3} + A^a_{l_1}A^b_{l_2}A^c_{l_3}A^d_{l_3}) \\ + &\sum_{l_1<l_2}(6A^a_{l_1}A^b_{l_1}A^c_{l_2}A^d_{l_2} + 4A^a_{l_1}A^b_{l_1}A^c_{l_1}A^d_{l_2} + 4A^a_{l_1}A^b_{l_2}A^c_{l_2}A^d_{l_2}) \\ + &\sum_{l} A^a_lA^b_lA^c_lA^d_l\Big\rangle\,. \end{aligned} \tag{3.5}$$

The sums are over links $l_i \in \ell$ and $l_1 < l_2 < \cdots$ denotes path ordering starting at some arbitrary point on ℓ.

Expanding

$$U_{\rm R}^{(n)}(\ell, g) = \sum_{\substack{p=0 \\ p+n=0 \bmod 2}}^{\infty} g^p U_{p{\rm R}}^{(n)}(\ell)\,, \tag{3.6}$$

we have

$$-w_1 = U_0^{(2)}\,, \tag{3.7}$$

$$-w_2 = (U_0^{(4)} - 3[U_0^{(2)}]^2) + 4U_1^{(3)} + 12U_2^{(2)}\,. \tag{3.8}$$

w_1 for an L, T loop is given by

$$w_{1{\rm R}}(L, T) = C_{\rm R}\mathcal{I}(L, T)\,, \tag{3.9}$$

with

$$\mathcal{I}(L, T) = \int_k \left(\frac{\sin(\frac{1}{2}k_1L)}{\sin(\frac{1}{2}k_1)}\right)^2 \left(\frac{\sin(\frac{1}{2}k_dT)}{\sin(\frac{1}{2}k_d)}\right)^2 D_{1d,1d}(k)\,. \tag{3.10}$$

The numerical values for the lowest $\mathcal{I}$'s are

$$\mathcal{I}(1,1) = \begin{cases} \frac{1}{2}\,, & \text{for } c_1 = 0 \\ 0.366262\,, & \text{for } c_1 = -\frac{1}{12}\,, \end{cases} \tag{3.11}$$

$$\mathcal{I}(1,2) = \begin{cases} 0.862251\,, & \text{for } c_1 = 0 \\ 4\mathcal{I}(1,1) + \frac{1}{2c_1}(\frac{1}{2} - \mathcal{I}(1,1))\,, & c_1 \neq 0 \\ 0.662624\,, & \text{for } c_1 = -\frac{1}{12}\,. \end{cases} \tag{3.12}$$

The expression for w_2 is rather more involved. We use covariant gauge fixing with Feynman rules given in detail in the appendix and find,

$$
\begin{aligned}
&U_{0\mathrm{R}}^{(4)}(\ell_{LT})-3(U_{0\mathrm{R}}^{(2)}(\ell_{LT}))^2 \\
&\quad=-6NC_{\mathrm{R}}\Bigg\{\tfrac{1}{3}\theta_1(L,T,4)\int_K D_{11}(k) \\
&\qquad+\int_{k,k'}\Bigg[\!\!\Bigg[-D_{11}(k)D_{11}(k')\frac{1}{s_1^2 s_1'^2}\Bigg(\bar{s}_d^{+2}\Bigg[\bar{s}_1^2\bar{s}_1'^2-\tfrac{1}{8}(s_1^{+2}+s_1^{-2})\Bigg(\frac{\bar{s}_1^+}{s_1^+}-\frac{\bar{s}_1^-}{s_1^-}\Bigg)^2\Bigg] \\
&\qquad+\tfrac{1}{2}\bar{s}_d^2(s_1^+-s_1^-)(\bar{s}_1^++\bar{s}_1^-)\Bigg(\frac{\bar{s}_1^+}{s_1^+}-\frac{\bar{s}_1^-}{s_1^-}\Bigg)\Bigg) \\
&\qquad+D_{11}(k)D_{dd}(k')\cos(k_1'L)\cos(k_dT)\Bigg(\frac{\bar{s}_1}{s_1}\Bigg)^2\Bigg(\frac{\bar{s}_d'}{s_d'}\Bigg)^2 \\
&\qquad+2D_{11}(k)D_{1d}(k')\frac{1}{s_1^2 s_1' s_d'}\bar{s}_d'^2\Bigg(\tfrac{1}{4}(s_1^++s_1^-)(\bar{s}_1^+-\bar{s}_1^-)\Bigg(\frac{\bar{s}_1^+}{s_1^+}-\frac{\bar{s}_1^-}{s_1^-}\Bigg) \\
&\qquad+\cos(k_dT)\Bigg[-\bar{s}_1^2(1-3\bar{s}_1'^2)+\tfrac{1}{4}(\bar{s}_1^++\bar{s}_1^-)\Bigg(\frac{\bar{s}_1^+}{s_1^+}s_1^-+\frac{\bar{s}_1^-}{s_1^-}s_1^+\Bigg)\Bigg]\Bigg) \\
&\qquad+D_{1d}(k)D_{1d}(k')\frac{1}{s_1 s_1' s_d s_d'}\Bigg(5\bar{s}_1^2\bar{s}_1'^2\bar{s}_d^2\bar{s}_d'^2 \\
&\qquad-\frac{1}{4}\frac{\bar{s}_1^+}{s_1^+}\frac{\bar{s}_d^+}{s_d^+}(s_1^-\bar{s}_1^+-s_1^+\bar{s}_1^-)(s_d^-\bar{s}_d^+-s_d^+\bar{s}_d^-) \\
&\qquad+\bar{s}_d^2\bar{s}_d'^2\Bigg[-2\bar{s}_1^2+\frac{1}{2}\frac{\bar{s}_1^+}{s_1^+}\frac{\bar{s}_1^-}{s_1^-}(s_1^{+2}+s_1^{-2})\Bigg]\Bigg)+(L\leftrightarrow T)\Bigg]\!\!\Bigg]\Bigg\},
\end{aligned}
\tag{3.13}
$$

where $s_1\equiv\sin(\frac{1}{2}k_1)$, $\bar{s}_1\equiv\sin(\frac{1}{2}k_1L)$, $s_d\equiv\sin(\frac{1}{2}k_d)$, $\bar{s}_d\equiv\sin(\frac{1}{2}k_dT)$ and similarly for k', $k^\pm=k\pm k'$.

Further

$$
\begin{aligned}
U_{1\mathrm{R}}^{(3)}(\ell_{LT})&=6iNC_{\mathrm{R}}\int_{k,k',k''}(2\pi)^4\delta(k+k'+k'') \\
&\quad\times\sum_i c_i V_{\lambda\rho\tau}^{(i)}(k,k',k'')D_{\lambda1}(k)D_{\rho1}(k')D_{\tau\sigma}(k'')\Bigg[\sin^2(\tfrac{1}{2}k_d''T)\frac{\sin(\frac{1}{2}k_1''L)}{s_1''} \\
&\quad\times\Bigg(\frac{\sin(\frac{1}{2}k_1L)}{s_1}\cos(\tfrac{1}{2}k_1'L)c_1-\frac{\sin(\frac{1}{2}k_1'L)}{s_1'}\cos(\tfrac{1}{2}k_1L)c_1'\Bigg) \\
&\quad\times\Bigg(\frac{\delta_{\sigma1}}{s_1''}-\frac{\delta_{\sigma d}}{s_d''}\Bigg)+\frac{\delta_{\sigma d}}{s_d''}\cdot\frac{\sin(\frac{1}{2}k_1L)}{s_1}\cdot\frac{\sin(\frac{1}{2}k_1'L)}{s_1'} \\
&\quad\times\cos(\tfrac{1}{2}k_1''L)\Bigg(\sin^2(\tfrac{1}{2}k_dT)-\sin^2(\tfrac{1}{2}k_d'T)\Bigg)\Bigg]+(L\leftrightarrow T),
\end{aligned}
\tag{3.14}
$$

$$U_{2R}^{(2)}(\ell_{LT}) = -C_R \int_k \left(\frac{\sin(\frac{1}{2}k_1 L)}{s_1}\right)^2 \left(\frac{\sin(\frac{1}{2}k_d T)}{s_d}\right)^2 (\hat{k}_1 \delta_{d\tau} - \hat{k}_d \delta_{1\tau})(\hat{k}_1 \delta_{d\sigma} - \hat{k}_d \delta_{1\sigma})$$
$$\times D_{\tau\mu}(k) \pi_{\mu\nu}(k) D_{\nu\sigma}(k) , \qquad (3.15)$$

with

$$\pi = \pi_{\text{meas}} + \pi^{\text{gh}} + \pi' + \pi^{\text{W}} + \pi^{\text{V}_4} + \pi^{\text{V}_3} , \qquad (3.16)$$

where

$$\pi_{\mu\nu}^{\text{meas}}(k) = -\tfrac{1}{12} N \delta_{\mu\nu} , \qquad (3.17)$$

$$\pi_{\mu\nu}^{\text{gh}}(k) = -\tfrac{1}{24} N \delta_{\mu\nu} + \tfrac{1}{4} N \int_{k',k''} (2\pi)^4 \delta(k + k' + k'') \frac{1}{\hat{k}'^2 \hat{k}''^2}$$
$$\times [\hat{k}_\mu \hat{k}_\nu - (\widehat{k' - k''})_\mu (\widehat{k' - k''})_\nu] , \qquad (3.18)$$

$$\pi'_{\mu\nu}(k) = (\hat{k}_\lambda \delta_{\mu\nu} - \hat{k}_\mu \delta_{\lambda\nu}) \hat{k}_\lambda [(c'_1 - c'_2 - c'_3)(\hat{k}_\mu^2 + \hat{k}_\lambda^2) + (c'_2 + c'_3)\hat{k}^2] , \qquad (3.19)$$

$$\pi_{\mu\nu}^{\text{W}}(k) = \tfrac{1}{12} \sum_s 2 t_s (6 C_s - N)(\hat{k}_\lambda \delta_{\mu\nu} - \hat{k}_\mu \delta_{\lambda\nu}) \hat{k}_\lambda$$
$$\times \int_{k'} D_{14,14}(k') \Big\{ (c_{s0} + 8 c_{s2}) + 16 c_{s1} (c_\lambda^2 + c_\mu^2) c_1'^2$$
$$+ c_{s2} \sum_{\rho \neq \lambda,\mu} (c_\rho^2 \hat{k}_2'^2 + c_2'^2 \hat{k}_\rho^2) + 12 (c_{s2} + c_{s3}) \sum_{\rho \neq \lambda,\mu} c_2'^2 c_\rho^2 \Big\} , \qquad (320)$$

$$\pi_{\mu\nu}^{\text{V}_4}(k) = \tfrac{1}{3} N \int_{k'} D_{\lambda\rho}(k') \sum_i c_i (V_{\lambda\rho\mu\nu}^{(i)}(k', -k', k, -k) - V_{\lambda\mu\rho\nu}^{(i)}(k', k, -k', -k) \qquad (3.21)$$

$$= \tfrac{1}{12} N \int_{k'} D_{\lambda\rho}(k') \Big[\!\Big[\delta_{\mu\nu} \Big[\delta_{\lambda\rho} \delta_{\lambda\mu} \Big(12 c_\mu^2 c_\mu'^2 + \sum_\tau (\hat{k}_\tau^2 c_\tau'^2 + \hat{k}_\tau'^2 c_\tau^2) \Big)$$
$$- 4 \delta_{\lambda\mu} \hat{k}'_\mu \hat{k}'_\rho c_\rho^2 - 2 \delta_{\lambda\rho} (6 - 3 \hat{k}_\lambda^2 - 3 \hat{k}_\mu'^2 + \hat{k}_\lambda^2 \hat{k}_\mu'^2) \Big]$$
$$- \hat{k}_\mu \hat{k}_\nu [\tfrac{1}{4} \delta_{\lambda\mu} \delta_{\rho\nu} \hat{k}'_\mu \hat{k}'_\nu + 2 \delta_{\lambda\rho} (\delta_{\lambda\mu} c_\nu'^2 + \delta_{\lambda\nu} c_\mu'^2)]$$
$$+ c_1 \Big\{ \delta_{\mu\nu} \Big[\delta_{\lambda\rho} \delta_{\lambda\mu} \Big(12 c_\mu^2 c_\mu'^2 (-2 \hat{k}_\mu'^2 - 2 \hat{k}_\mu^2 + \hat{k}_\mu^2 \hat{k}_\mu'^2)$$
$$+ \sum_\tau (12 (\hat{k}_\tau^2 + \hat{k}_\tau'^2) - (\hat{k}_\tau^4 + \hat{k}_\tau'^4) - 7 (\hat{k}_\mu^2 \hat{k}_\tau^2 + \hat{k}_\mu'^2 \hat{k}_\tau'^2) - 12 \hat{k}_\tau^2 \hat{k}_\tau'^2$$
$$- 3 (\hat{k}_\mu^2 \hat{k}_\tau'^2 + \hat{k}_\mu'^2 \hat{k}_\tau^2) + 2 \hat{k}_\tau^2 \hat{k}_\tau'^2 (\hat{k}_\tau^2 + \hat{k}_\tau'^2) + \tfrac{7}{2} \hat{k}_\tau^2 \hat{k}_\tau'^2 (\hat{k}_\mu^2 + \hat{k}_\mu'^2)$$
$$+ \tfrac{3}{2} \hat{k}_\mu^2 \hat{k}_\mu'^2 (\hat{k}_\tau^2 + \hat{k}_\tau'^2) - \tfrac{1}{2} \hat{k}_\tau^4 \hat{k}_\tau'^4 - \tfrac{5}{4} \hat{k}_\mu^2 \hat{k}_\mu'^2 \hat{k}_\tau^2 \hat{k}_\tau'^2) \Big)$$

$$+8\delta_{\lambda\mu}\hat{k}'_{\mu}\hat{k}'_{\rho}(-9c_{\mu}^{2}+\tfrac{7}{2}\hat{k}'^{2}_{\mu}+\tfrac{9}{2}\hat{k}^{2}_{\rho}+\tfrac{1}{2}\hat{k}'^{2}_{\rho}-\tfrac{3}{4}\hat{k}^{2}_{\mu}\hat{k}'^{2}_{\mu}-\tfrac{7}{8}\hat{k}^{2}_{\mu}\hat{k}^{2}_{\rho}$$
$$-\tfrac{5}{4}\hat{k}'^{2}_{\mu}\hat{k}^{2}_{\rho}c^{2}_{\mu}-\tfrac{1}{2}\hat{k}^{4}_{\rho}c'^{2}_{\rho}-\tfrac{1}{2}\hat{k}^{2}_{\rho}\hat{k}'^{2}_{\rho})$$
$$-8\delta_{\lambda\rho}(-9(\hat{k}^{2}_{\lambda}+\hat{k}'^{2}_{\mu})-\tfrac{3}{2}(\hat{k}'^{2}_{\lambda}+\hat{k}^{2}_{\mu})+\tfrac{3}{4}(\hat{k}^{2}_{\mu}\hat{k}^{2}_{\lambda}+\hat{k}'^{2}_{\mu}\hat{k}'^{2}_{\lambda})$$
$$+3(\hat{k}^{4}_{\lambda}+\hat{k}'^{4}_{\mu})+3(\hat{k}^{2}_{\lambda}\hat{k}'^{2}_{\lambda}c^{2}_{\lambda}+\hat{k}^{2}_{\mu}\hat{k}'^{2}_{\mu}c'^{2}_{\mu})+6\hat{k}^{2}_{\lambda}\hat{k}'^{2}_{\mu}$$
$$\left.-\hat{k}^{2}_{\lambda}\hat{k}'^{2}_{\mu}(\hat{k}^{2}_{\lambda}+\hat{k}'^{2}_{\mu})-\hat{k}^{2}_{\lambda}\hat{k}'^{2}_{\mu}(\hat{k}'^{2}_{\lambda}c^{2}_{\lambda}+\hat{k}^{2}_{\mu}c'^{2}_{\mu}))\right]$$
$$+\hat{k}_{\mu}\hat{k}_{\nu}([\delta_{\lambda\mu}\delta_{\rho\nu}\hat{k}'_{\mu}\hat{k}'_{\nu}(-9+\tfrac{13}{4}(\hat{k}^{2}_{\mu}+\hat{k}'^{2}_{\mu})-\hat{k}^{2}_{\mu}\hat{k}'^{2}_{\mu})$$
$$+\delta_{\lambda\rho}\delta_{\lambda\mu}(-36c'^{2}_{\mu}+14\hat{k}^{2}_{\mu}+18\hat{k}'^{2}_{\nu}+2\hat{k}^{2}_{\nu}-3\hat{k}^{2}_{\mu}\hat{k}'^{2}_{\mu}$$
$$\left.-\tfrac{7}{2}\hat{k}'^{2}_{\mu}\hat{k}'^{2}_{\nu}-5\hat{k}^{2}_{\mu}\hat{k}'^{2}_{\nu}c'^{2}_{\mu}-2\hat{k}'^{4}_{\nu}c^{2}_{\nu}-2\hat{k}^{2}_{\nu}\hat{k}'^{2}_{\nu})]+[\mu\leftrightarrow\nu])\right\}$$
$$+\text{terms}\propto c_{2},\,c_{3}\,, \tag{3.22}$$

$$\pi^{V_3}_{\mu\nu}=-\tfrac{1}{2}N\int_{k',k''}(2\pi)^{4}\delta(k+k'+k'')D_{\rho\sigma}(k')D_{\tau\kappa}(k'')$$
$$\times\sum_{i,j}c_{i}c_{j}V^{(i)}_{\mu\rho\tau}(k,k',k'')\,V^{(j)}_{\nu\sigma\kappa}(k,k',k'')\,. \tag{3.23}$$

In Formulae (3.19)–(3.23) the coefficients c_i appearing are understood to be evaluated at $g^2=0$. One can check that the condition $\pi_{\mu\nu}(0)=0$ holds as it should.

Numerical evaluation of w_2 for finite L, T for the improved action remain to be done. For the Wilson case we agree with Di Giacomo and Rossi [13] for the value of $w_2(1,1)$ and find only small corrections [14] to the estimates for the χ's quoted by Hattori and Kawai [15].

4. The limit $T\to\infty$ and the small-a expansion

To extract the static potential we perform the limit $T\to\infty$. To first order

$$\lim_{T\to\infty}\frac{1}{T}\,w_{1\mathrm{R}}(L,T)=C_{\mathrm{R}}\int_{k}\sin^{2}(\tfrac{1}{2}k_{1}L)D(k)|_{k_d=0}\,, \tag{4.1}$$

where we have introduced the notation $D(k)\equiv D_{dd}(k)$. Note

$$D^{-1}(k)|_{k_d=0}=\hat{\boldsymbol{k}}^{2}-c_{1}\sum_{i}\hat{k}^{4}_{i}$$
$$=\boldsymbol{k}^{2}+\mathrm{O}(\boldsymbol{k}^{6})\,,\qquad \text{for } c_{1}=-\tfrac{1}{12}\,. \tag{4.2}$$

The behaviour of $D^{-1}(k)$ for small k noted above ensures the improvement of the cutoff dependence of the static potential in lowest-order perturbation theory. In second order we find

$$\lim_{T\to\infty}\frac{1}{T}(U_{0R}^{(4)}-3[U_{0R}^{(2)}]^2)=-6NC_R\Big[\tfrac{4}{3}\int_{k'}D(k')\int_{\boldsymbol{k}}\sin^2(\tfrac{1}{2}k_1L)D(k)|_{k_d=0}$$
$$-\int_{\boldsymbol{k},k'}\sin^2(\tfrac{1}{2}k_1^+L)D(k')D(k)|_{k_d=k_d'}$$
$$-4\int_{\boldsymbol{k},k'}\frac{\cos(\tfrac{1}{2}k_d')}{s_d'}\{\sin^2(\tfrac{1}{2}k_1L)D(k)|_{k_d=0}$$
$$-\sin^2(\tfrac{1}{2}k_1^+L)D(k)|_{k_d=k_d'}\}\frac{\partial}{\partial k_d'}D(k')\Big], \tag{4.3}$$

$$\lim_{T\to\infty}\frac{1}{T}U_{1R}^{(3)}=-6iNC_R\int_{k,k'}2\pi\delta(k_d)\sin^2(\tfrac{1}{2}k_1L)D(k)$$
$$\times\frac{\cos(\tfrac{1}{2}k_d')}{s_d'}D_{\lambda d}(k')D_{\rho d}(k^+)\sum_i c_iV_{d\lambda\rho}^{(i)}(k,k',-k^+) \tag{4.4}$$

and

$$\lim_{T\to\infty}\frac{1}{T}U_{2R}^{(2)}=-4C_R\int_k 2\pi\delta(k_d)\sin^2(\tfrac{1}{2}k_1L)D^2(k)\pi_{dd}(k)\,. \tag{4.5}$$

Hence summing (4.2)–(4.4) we have

$$\lim_{T\to\infty}\frac{1}{T}w_{2R}(L,T)=48C_R\int_{\boldsymbol{k}}\sin^2(\tfrac{1}{2}k_1L)\tilde{w}_2(\boldsymbol{k})D^2(k)|_{k_d=0} \tag{4.6}$$

with

$$\tilde{w}_2(\boldsymbol{k})=\Big[\pi_{dd}(k)+\tfrac{1}{6}ND^{-1}(k)\int_{k'}D(k')-\tfrac{1}{8}ND^{-2}(k)\int_{k'}D(k^+)D(k')$$
$$-\tfrac{1}{2}ND^{-1}(k)\int_{k'}(1-D^{-1}(k)D(k^+))\frac{c_d'}{s_d'}\frac{\partial}{\partial k_d'}D(k')$$
$$+\tfrac{1}{2}iND^{-1}(k)\int_{k'}\frac{c_d'}{s_d'}D_{\rho d}(k')D_{\tau d}(k^+)\sum_i c_iV_{d\rho\tau}^{(i)}(k,k',-k^+)\Big]_{k_d=0}. \tag{4.7}$$

It can be checked that $\tilde{w}_2(\boldsymbol{k})$ is indeed independent of the gauge parameter α as required. Rewriting (4.7) we have

$$\tilde{w}_2(\boldsymbol{k})=x(\boldsymbol{k})+Ny(\boldsymbol{k})+z(\boldsymbol{k})\,, \tag{4.8}$$

with

$$x(\boldsymbol{k})=d_0\hat{\boldsymbol{k}}^2\mathcal{I}(1,1)+4d_1(2\hat{\boldsymbol{k}}^2-\tfrac{1}{4}\sum_i\hat{k}_i^4)\mathcal{I}(1,2)\,, \tag{4.9}$$

where

$$d_i \equiv \tfrac{1}{12}\sum_s 2t_s(6C_s - N)c_{si}\,. \tag{4.10}$$

For the case where just the fundamental representation is included one has $d_i = (2N^2-3)c_i/12N$. Further

$$z(\boldsymbol{k}) = (c_1' - c_2' - c_3')\sum_i \hat{k}_i^4 + (c_2' + c_3')(\hat{k}^2)^2\,, \tag{4.11}$$

$$\begin{aligned}
y(\boldsymbol{k}) = {} & \tfrac{1}{12}\hat{\boldsymbol{k}}^2 \int_l [(7 - \tfrac{5}{2}\hat{l}_1^2)D(l) + \hat{l}_1\hat{l}_d D_{1d}(l) \\
& + c_1\{(84 - 60\hat{l}_1^2 - 27\hat{l}_d^2 + 10\hat{l}_1^4 + \tfrac{23}{2}\hat{l}_1^2\hat{l}_d^2)D(l) + (36 - 14\hat{l}_1^2)D_{1d}(l)\hat{l}_1\hat{l}_d\}] \\
& - \tfrac{1}{12}c_1 \sum_i \hat{k}_i^4 \int_l [(25 - 10\hat{l}_1^2 - 6\hat{l}_d^2 + 2\hat{l}_1^2\hat{l}_d^2 + \tfrac{1}{2}\hat{l}_1^4)D(l) + (4 - \hat{l}_1^2)D_{1d}(l)\hat{l}_1\hat{l}_d] \\
& + [D^{-1}(k)\tfrac{1}{6}\int_l D(l) - \tfrac{1}{2}D^{-1}(k)\int_{k'} (1 - D^{-1}(k)D(k^+))\frac{c_d'}{s_d'}\frac{\partial}{\partial k_d'}D(k') \\
& - \tfrac{1}{8}D^{-2}(k)\int_{k'} D(k^+)D(k') - \int_{k'} \frac{c_d'^2\hat{k}_d'^2}{\hat{k}'^2}\left(\frac{1}{\hat{k}^{+2}} - \frac{1}{\hat{k}'^2}\right) \\
& + \tfrac{1}{2}iD^{-1}(k)\int \frac{c_d'}{s_d'} D_{\rho d}(k')D_{\tau d}(k^+)\sum_i c_i V_{d\rho\tau}^{(i)}(k, k', -k^+) \\
& - \tfrac{1}{2}\int\left(\left[D_{\rho\sigma}(k')D_{\tau\kappa}(k^+)\sum_{i,j} c_i c_j V_{d\rho\tau}^{(i)}(k, k', -k^+)V_{d\sigma\kappa}^{(j)}(k, k', -k^+)\right]\right. \\
& \left.\left. - [\boldsymbol{k} = 0]\right)\right]_{k_d = 0}\,,
\end{aligned} \tag{4.12}$$

with

$$\begin{aligned}
& \sum_{i=0,1} c_i V_{d\rho\tau}^{(i)}(k, k', -k^+)|_{k_d=0} \\
& = ic_d'\{\delta_{d\rho}\widehat{(k-k')}_\tau + 2c_\rho\hat{k}_d'\delta_{\rho\tau} - \delta_{d\tau}\widehat{(k^+ + k)}_\rho \\
& \quad + c_1[\delta_{d\rho}(-\hat{k}_d'(\delta_{\tau d}\hat{k}^{+2} - \hat{k}_d^+\hat{k}_\tau^+) - \widehat{(k-k')}_\tau(\hat{k}_\tau^2 + \hat{k}_\tau'^2 + 2\hat{k}_d'^2)) \\
& \quad + 2\delta_{\rho\tau}\hat{k}_d'(\hat{k}^2\delta_{\rho d} - 2c_\rho\hat{k}_d'^2 - \hat{k}_\rho'\hat{k}_\rho^+(1 - \tfrac{1}{2}\hat{k}_\rho^2) - 2c_\rho\hat{k}_\rho^2) \\
& \quad + \delta_{d\tau}(-\hat{k}_d'(\delta_{\rho d}\hat{k}'^2 - \hat{k}_d'\hat{k}_\rho') + \widehat{(k^+ + k')}_\rho(\hat{k}_\rho^{+2} + \hat{k}_\rho^2 + 2\hat{k}_d'^2))]\}\,.
\end{aligned} \tag{4.13}$$

For the Wilson case $c_1 = 0$ our expression reduces to that first obtained by Müller and Rühl [16] in the axial gauge,

$$x(\boldsymbol{k})|_{c_1=0} = \frac{2N^2-3}{24N}\hat{\boldsymbol{k}}^2\,, \tag{4.14}$$

$$y(\boldsymbol{k})|_{c_1=0}=\hat{k}^2\left(\tfrac{3}{4}\int_{k'}\frac{1}{\hat{k}'^2}-\frac{5}{96}\right)+\left[2\hat{\boldsymbol{k}}^2\int_{k'}\left(1-\frac{\hat{k}^2}{\hat{k}^{+2}}\right)\frac{c_d'^2}{(\hat{k}'^2)^2}-\tfrac{1}{8}(\hat{\boldsymbol{k}}^2)^2\int_{k'}\frac{1}{\hat{k}'^2\hat{k}^{+2}}\right.$$
$$\left.+\int_{k'}\frac{c_d'^2}{\hat{k}'^2\hat{k}^{+2}}\left(2\hat{k}^2c_d'^2-\sum_i\hat{k}_i^2\hat{k}_i'^2\right)+4\int_{k'}\frac{c_d'^2\hat{k}_d'^2}{\hat{k}'^2}\left(\frac{1}{\hat{k}^{+2}}-\frac{1}{\hat{k}'^2}\right)\right]_{k_d=0}. \tag{4.15}$$

To carry out the improvement programme we require the expansion of $\tilde{w}_2$ up to terms $\sim k^4$. $x(\boldsymbol{k})$ is of no problem

$$x(\boldsymbol{k})=\boldsymbol{k}^2(d_0\mathscr{I}(1,1)+8d_1\mathscr{I}(1,2))-\tfrac{1}{12}\sum_i k_i^4(d_0\mathscr{I}(1,1)+20d_1\mathscr{I}(1,2))+\mathrm{O}(k^6)\,. \tag{4.16}$$

The expansion of $y(k)$ is complicated due to the horrible algebraic form of the propagator and vertices. The procedure we followed was as follows. We first broke $y(k)$ into three pieces

$$y(\boldsymbol{k})=y(\boldsymbol{k})|_{c_1=0}+c_1\frac{\partial}{\partial c_1}y(\boldsymbol{k})|_{c_1=0}+c_1^2R(\boldsymbol{k},c_1)\,. \tag{4.17}$$

The Wilson part $y(k)|_{c_1=0}$ contains all the terms $\boldsymbol{k}^2\ln\boldsymbol{k}^2$. Each additional factor of c_1 carries with it additional powers of momenta in the integrand. Hence the second and third have logs starting at order k^4 and k^6 respectively. The expansion of the first two pieces was performed by hand and the resulting coefficient evaluated numerically. The third piece was evaluated by performing a Taylor expansion of the integrand using "Reduce" and then integrating numerically. Our final result is

$$y(\boldsymbol{k})=-\frac{\beta_0}{N}\boldsymbol{k}^2\ln\boldsymbol{k}^2+a_1(c_1)\boldsymbol{k}^2+\frac{1}{(4\pi)^2}(1+12c_1)\ln\boldsymbol{k}^2$$
$$\times\left[\tfrac{1}{10}(\boldsymbol{k}^2)^2+\tfrac{59}{90}\sum_i k_i^4\right]+a_2(c_1)\sum_i k_i^4+a_3(c_1)(\boldsymbol{k}^2)^2+\mathrm{O}(k^6\ln\boldsymbol{k}^2)\,, \tag{4.18}$$

with

$$\beta_0=\frac{11N}{3(4\pi)^2}\,, \tag{4.19}$$

$$a_1(0)=\tfrac{11}{3}P_2+\tfrac{5}{36}P_1+\frac{28}{9(4\pi)^2}-\tfrac{1}{48}=0.108435\,, \tag{4.20}$$

$$a_1(-\tfrac{1}{12})=a_1(0)-0.031381\,, \tag{4.21}$$

$$a_2(0)=-\tfrac{59}{90}P_2+\tfrac{59}{2160}P_1+\tfrac{1}{576}+\frac{1}{(4\pi)^2}\{-\tfrac{124}{105}\ln 2-\tfrac{1019}{12\,600}\}$$
$$=-0.015470\,, \tag{4.22}$$

$$a_2(-\tfrac{1}{12})=-0.002002\,, \tag{4.23}$$

$$a_3(0) = -\tfrac{13}{5}P_3 + \tfrac{9}{40}P_2 - \tfrac{23}{1440}P_1 + \frac{1}{(4\pi)^2}\{\tfrac{32}{35}\ln 2 - \tfrac{363}{700}\}$$

$$= -0.002188\,, \tag{4.24}$$

$$a_3(-\tfrac{1}{12}) = 0.003695\,, \tag{4.25}$$

where

$$P_1 = \int_l \frac{1}{\hat{l}^2} = 0.1549334\,, \tag{4.26}$$

$$P_2 = \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4}\left[\frac{1}{(\hat{l}^2)^2}\prod_\mu \theta(\pi - |l_\mu|) - \frac{1}{(l^2)^2} + \frac{1}{(l^2+1)^2}\right]$$

$$= 0.0240132\,, \tag{4.27}$$

$$P_3 = \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4}\left[\frac{1}{(\hat{l}^2)^3}\prod_\mu \theta(\pi - |l_\mu|) - \frac{1}{(l^2)^3} - \left(\frac{1}{(l^2)^2} - \frac{1}{(l^2+1)^2}\right)\frac{1}{4}\frac{\sum_\mu l_\mu^4}{(l^2)^2}\right]$$

$$= 0.0022482\,. \tag{4.28}$$

The coefficients of the $k^4 \ln k^2$ terms for $c_1 = 0$ agree with those calculated by Stehr [17]. The important result is that the $k^4 \ln k^2$ terms vanish for $c_1 = -\frac{1}{12}$ as first pointed out by Curci et al. [11]. Setting $c_2(0) = -c_3(0) \neq 0$ would introduce an unwanted $(\boldsymbol{k}^2)^2 \ln \boldsymbol{k}^2$ term and hence we must have $c_2(0) = c_3(0) = 0$, thereby reproducing the result in sect. 2. In fact it has been proven by Symanzik [18] that for scalar field theories, the inclusion of next-nearest neighbours to achieve improvement to the tree level, also ensures the absence of all $a^2 \ln a$ terms in the 1-loop graphs. The above result indicates that the proof also extends to lattice gauge theories.

Comparing the result of Fischler [19] for the static potential in the continuum minimal subtraction scheme with the $O(k^2)$ terms in $\tilde{w}(k)$ obtained above, one obtains the ratio of the lattice to continuum Λ's. For the Wilson case one recovers the Hasenfratz and Hasenfratz [20] result [21]

$$\frac{\Lambda_{\mathrm{L,Wilson}}}{\Lambda_{\mathrm{min}}} = \exp\left\{J + \beta_0^{-1}\left[\frac{1}{16N} - NP\right]\right\}, \tag{4.29}$$

where

$$J = \tfrac{1}{2}(\ln 4\pi - \gamma) = 0.9769042\,, \tag{4.30}$$

$$P = \tfrac{5}{72}P_1 + \tfrac{11}{6}P_2 + \tfrac{1}{32} - \frac{1}{6(4\pi)^2} = 0.0849780\,. \tag{4.31}$$

For the ratio of the Λ's for the improved Wilson action we obtain

$$\frac{\Lambda_{\mathrm{L,imp\ Wilson}}}{\Lambda_{\mathrm{L,Wilson}}} = \exp\left\{\beta_0^{-1}\left[N(0.043300) - \frac{1}{N}(0.041414)\right]\right\}$$

$$= \begin{cases} 4.133 \pm 0.004\,, & \text{for } N = 2 \\ 5.294 \pm 0.004\,, & \text{for } N = 3\,. \end{cases} \tag{4.32}$$

These values agree with those obtained independently by Bernreuther and Wetzel [22] but lie outside the range initially estimated by Curci et al. [11]. Λ-ratios for other improved actions (e.g. mixed action) are easily deduced from the above formulae.

Finally the improvement of the action requires the absence of k^4 terms in $\tilde{w}_2(\boldsymbol{k})$. Hence, combining the results above we demand

$$c_2' + c_3' = -N(0.003695)\,, \tag{4.33}$$

$$c_1' = -N(0.001693) + d_0(0.030522) + d_1(1.104373)\,,$$

$$\underset{\text{Wilson case}}{=} -N(0.008553) + \frac{1}{N}(0.010290)\,, \tag{4.34}$$

$$c_0' + 8c_1' + 16c_2' + 8c_3' = 0\,. \tag{4.35}$$

We note that our corrections are numerically very small for all reasonable g^2N, similar in order of magnitude to those found by Symanzik [4] for the non-linear σ-model.

5. An improved χ

To implement the improvement programme consistently it is necessary to check that the observables measured have sufficiently weak intrinsic cutoff dependence. For example, looking forward to the theory with fermions, local currents defined via point-splitting would in general have to be improved. An example of a non-local expression in the pure gauge theory which requires improvement, and which we will briefly discuss in the following, is Creutz's χ [12], often used in Monte Carlo calculations to extract the string tension. χ is defined by

$$a^{-2}\chi(R, a) = a^{-2}(V(R, R, \Lambda, a) - 2V(R-a, R, \Lambda, a) + V(R-a, R-a, \Lambda, a))\,, \tag{5.1}$$

where V is the logarithm of the $R \times T$ Wilson loop

$$V(R, T, \Lambda, a) = -\ln\left(\frac{1}{N}\langle \mathrm{tr}\; U(\ell_{RT})\rangle\right). \tag{5.2}$$

In the continuum limit ($a \to 0$, R, T, Λ fixed), V approaches a finite value v modulo linear and kink divergences. A possible definition of the string tension is

$$\chi = \lim_{R,T\to\infty} \frac{\partial}{\partial R}\frac{\partial}{\partial T} v(R, T, \Lambda)\,. \tag{5.3}$$

$a^{-2}\chi$ in (5.1) defined by Creutz, has the merits that firstly it is absent of the perimeter and kink divergences mentioned above; and secondly, in the continuum limit, it approaches $(\partial/\partial R)(\partial/\partial T)v(R, T, \Lambda)|_{T=R}$, which for large R gives the string tension.

Suppose now, we consider the situation from the point of view of the improvement programme, and suppose that Symanzik's improved action has been constructed and observables such as V have diminished cutoff dependence. Then for small a/R it follows that

$$a^{-2}\chi(R,a)=\frac{\partial}{\partial R}\frac{\partial}{\partial T}v(R,T,\Lambda)|_{T=R}-\tfrac{1}{2}a\left(\frac{\partial}{\partial R}+\frac{\partial}{\partial T}\right)\frac{\partial}{\partial R}\frac{\partial}{\partial T}v(R,T,\Lambda)|_{T=R}$$
$$+\tfrac{1}{6}a^2\left(\frac{\partial^2}{\partial R^2}+\frac{3}{2}\frac{\partial}{\partial R}\frac{\partial}{\partial T}+\frac{\partial^2}{\partial T^2}\right)\frac{\partial}{\partial R}\frac{\partial}{\partial T}v(R,T,\Lambda)|_{T=R}+\cdots. \qquad (5.4)$$

In other words, χ as defined in (5 1), has corrections $O(a^2)$ to the continuum expression and its use would theoretically not be in the spirit of the improvement efforts made so far. The corrections arise because of the explicit dependence on the cutoff in the definition of χ. We can, however, easily find a modified χ (which we call $\hat{\chi}$) so that the corrections are $O(a^4)$,

$$a^{-2}\hat{\chi}(R,a)=\frac{\partial}{\partial R}\frac{\partial}{\partial T}v(R,T,\Lambda)|_{T=R}+O(a^4). \qquad (5.5)$$

It just imvolves a slightly more involved sum

$$a^{-2}\hat{\chi}(R,a)=\sum_{m,n}e_{m,n}V(R+m,R+n,\Lambda,a), \qquad (5.6)$$

with coefficients $e_{m,n}$ satisfying

$$\sum_{m,n}e_{m,n}(m^k+n^k)=0, \qquad \text{for } k=0,1,2,3,4, \qquad (5.7)$$

$$\sum_{m,n}e_{mn}mn(m^k+n^k)=\begin{cases}1, & \text{for } k=0\\ 0, & \text{for } k=1,2,\end{cases} \qquad (5.8)$$

$$\sum_{m,n}e_{m,n}m^2n^2=0. \qquad (5.9)$$

Define

$$e^+_{m,n}=\tfrac{1}{2}(e_{m,n}+e_{n,m}). \qquad (5.10)$$

A reasonably practical solution can be found involving just $e^+_{m,n}$ with $-2\leq m,n\leq 1$ and $|m-n|\leq 2$. There are nine such e^+'s and given any one of them (apart from $e^+_{0,-1}$) the others are determined thus:

$$e^+_{1,1}=y+\tfrac{1}{3}, \qquad e^+_{0,0}=3y+\tfrac{5}{4}, \qquad e^+_{-1,-1}=1-3y,$$
$$e^+_{-2,-2}=-y-\tfrac{1}{12}, \qquad e^+_{1,0}=-2y-\tfrac{1}{3}, \qquad e^+_{-1,-2}=2y,$$
$$e^+_{1,-1}=y, \qquad e^+_{0,-2}=-y+\tfrac{1}{12}, \qquad e^+_{0,-1}=-1. \qquad (5.11)$$

$\hat{\chi}$ is, in theory, an improved observable. But we are faced with practical problems, and whether or not it is better to work with such a $\hat{\chi}$ is a delicate question. Since we want to exclude loops of "too small" a size we would require good data on larger loops (e.g. 5×5). Here we will discuss these difficulties no further; we have merely touched on the problem concerning which more should be understood.

6. Conclusion

By calculating the small-a expansion of the static potential in 1-loop order we have, (a) verified the result of Curci et al. [11] on the form of the tree-level improved action and (b) placed some constraints on the coefficients of the improved action at 1-loop order. We still require a further calculation to specify c_0', c_2', c_3' separately, either by calculating the effective action completely to 1-loop or calculating the Wilson loop to 2-loops (such a calculation is under way). It would also be satisfying to have consistency checks of the ansatz by ensuring that other physical quantities are also improved. We see no reason at present why c_2' (or c_3') could be chosen zero despite the fact that the currents $J_\nu^a = D_\mu F_{\mu\nu}^a$ are classically zero and that the composite operator $\int J_\mu^a J_\mu^a$ formally acts as a trivial perimeter-measuring operator on Wilson loops. However if we set $c_2'=0$ by hand in our constraints (4.33)–(4.35) we obtain corrections which have a slope consistent with the trial values found by Wilson [23] in his renormalisation group studies. It has been suggested to us by Symanzik that $c_2(g^2)$ or $c_3(g^2)$ could possibly be set identically to zero at the cost of systematically altering the definition of the observables.

There are many difficulties still to be overcome. One type of problem is associated with the definition of improved observables, touched upon in sect. 5. Further, simple methods to obtain bounds on the glueball spectrum used for the Wilson action, e.g. use of the positivity of the transfer matrix, will have to be reviewed in the improved case. Also some studies of the phase structure, e.g. mean field studies, would be appreciated.

With the appearance of parallel processors the extra complication of the action does not present a major problem, as long as enough effort is spent in optimising the programmes. We would like to stress that Symanzik's programme is not an alternative to the use of larger lattices but rather complements these advances. As for the results we would also like to repeat that although observation of scaling-like behaviour for physical masses is a necessary condition for the relevance of the continuum limit, the thereby individual mass/Λ ratios extracted in practice have large uncertainties due to perturbative corrections. A better test of improvement is the smoother behaviour of the ratios of physical masses.

Many situations may occur in practice – perhaps tree-level or 1-loop improved actions show scaling windows where the standard action did not really show any (as in the σ-model). But there is no a priori reason that the improved 1-loop actions should have dramatic success over the standard actions in regions where the coupling is not too small. In the latter case the improvement coefficients as functions of g must be found by trial and error [2]; the 1-loop forms just aid the search.

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Appendix A

FEYNMAN RULES FOR THE IMPROVED ACTION

We employ covariant gauge fixing and thereby end up with the following total action to perform perturbative calculations

$$S_{\rm total}(A,c)=S_{\rm measure}(A)+S_{\rm ghost}(A,c)+S(A)+S_{\rm gf}(A)\,, \tag{A.1}$$

with

$$S_{\rm measure}(A)=-\tfrac{1}{24}Ng^2\int_k \tilde{A}^a_\mu(k)\tilde{A}^a_\mu(-k)+{\rm O}(g^4)\,, \tag{A.2}$$

$$\begin{aligned} S_{\rm ghost}(A,c)=-\int_{k,k'}\bar{\tilde{c}}^a(k)\tilde{c}^b(k')\Big[&(2\pi)^4\delta(k+k')\delta^{ab}\hat{k}^2+igf_{cab}\\ &\times\int_p(2\pi)^4\delta(k+k'+p)\tilde{A}^c_\mu(p)\hat{k}_\mu\cos(\tfrac12 k'_\mu)+\tfrac{1}{12}g^2f_{cae}f_{dbe}\\ &\times\int_{p,p'}(2\pi)^4\delta(k+k'+p+p')\tilde{A}^c_\mu(p)\tilde{A}^d_\mu(p')\hat{k}_\mu\hat{k}'_\mu+{\rm O}(g^3)\Big]\,, \end{aligned} \tag{A.3}$$

$$S(A)+S_{\rm gf}(A)=-\sum_{n=2}^{\infty}\frac{g^{n-2}}{n!}S_n(A,g)\,, \tag{A.4}$$

where

$$\begin{aligned} S_2(A)=\int_k\big[&\tilde{A}^a_\mu(k)\tilde{A}^a_\nu(-k)D^{-1}_{\mu\nu}(k)\\ &-\tfrac12 g^2\tilde{f}^a_{\mu\nu}(k)\tilde{f}^a_{\mu\nu}(-k)((c_1'-c_2'-c_3')(\hat{k}^2_\mu+\hat{k}^2_\nu)+(c_2'+c_3')\hat{k}^2)\big]\,, \end{aligned} \tag{A.5}$$

$$\begin{aligned} S_3(A,g)=\int_{k_1,k_2,k_3}&(2\pi)^4\delta(k_1+k_2+k_3)\\ &\times\tilde{A}^a_\lambda(k_1)\tilde{A}^b_\rho(k_2)\tilde{A}^c_\tau(k_3)f_{abc}\sum_i c_iV^{(i)}_{\lambda\rho\tau}(k_1,k_2,k_3)\,, \end{aligned} \tag{A.6}$$

$$S_4(A,g)=\int_{k_1,k_2,k_3,k_4}(2\pi)^4\delta(k_1+k_2+k_3+k_4)\tilde{A}^a_\lambda(k_1)\tilde{A}^b_\rho(k_2)\tilde{A}^c_\tau(k_3)\tilde{A}^d_\sigma(k_4)$$
$$\times\sum_i\left[c_i f_{abe}f_{cde}(V^{(i)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)-V^{(i)}_{\rho\lambda\tau\sigma}(k_2,k_1,k_3,k_4))\right.$$
$$\left.-\sum_{\mathrm{R}}c_{\mathrm{R}i}S^{abcd}_{\mathrm{R}}W^{(i)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)\right]. \qquad (A.7)$$

The propagator $D_{\mu\nu}(k)$ in (A.5) was discussed in detail in (I) and is given by

$$D_{\mu\nu}(k)=(\hat{k}^2)^{-2}\left[\alpha\hat{k}_\mu\hat{k}_\nu+\sum_\sigma(\hat{k}_\sigma\delta_{\mu\nu}-\hat{k}_\nu\delta_{\mu\sigma})\hat{k}_\sigma A_{\sigma\nu}(k)\right], \qquad (A.8)$$

with ($c_1=-\frac{1}{12}$)

$$A_{\mu\nu}(k)=A_{\nu\mu}(k)=(1-\delta_{\mu\nu})\Delta(k)^{-1}\left[(\hat{k}^2)^2-c_1\hat{k}^2\left(2\sum_\rho\hat{k}^4_\rho+\hat{k}^2\sum_{\rho\neq\mu,\nu}\hat{k}^2_\rho\right)\right.$$
$$\left.+c_1^2\left(\left(\sum_\rho\hat{k}^4_\rho\right)^2+\hat{k}^2\sum_\rho\hat{k}^4_\rho\sum_{\tau\neq\mu,\nu}\hat{k}^2_\tau+(\hat{k}^2)^2\prod_{\rho\neq\mu,\nu}\hat{k}^2_\rho\right)\right], \qquad (A.9)$$

where

$$\Delta(k)=\left(\hat{k}^2-c_1\sum_\rho\hat{k}^4_\rho\right)\left[\hat{k}^2-c_1\left((\hat{k}^2)^2+\sum_\tau\hat{k}^4_\tau\right)\right.$$
$$\left.+\tfrac{1}{2}c_1^2\left((\hat{k}^2)^3+2\sum_\tau\hat{k}^6_\tau-\hat{k}^2\sum_\tau\hat{k}^4_\tau\right)\right]-4c_1^3\sum_\rho\hat{k}^4_\rho\prod_{\tau\neq\rho}\hat{k}^2_\tau. \qquad (A.10)$$

(A corrected version of (A.16) in paper (I)).

The coefficients c_i' appearing in (A.5) are defined by

$$c_i'(g^2)=g^{-2}(c_i(g^2)-c_i(0)). \qquad (A.11)$$

The three-point vertices in (A.6) have the property

$$V^{(i)}_{\lambda\rho\tau}(k_1,k_2,k_3)=-V^{(i)}_{\rho\lambda\tau}(k_2,k_1,k_3)=-V^{(i)}_{\tau\rho\lambda}(k_3,k_2,k_1), \qquad (A.12)$$

and are given by

$$V^{(0)}_{\lambda\rho\tau}(k_1,k_2,k_3)=i[\delta_{\lambda\rho}(\widehat{k_1-k_2})_\tau c_{3\lambda}+2\text{ cyclic perms}], \qquad (A.13)$$

$$V^{(1)}_{\lambda\rho\tau}(k_1,k_2,k_3)=8V^{(0)}_{\lambda\rho\tau}(k_1,k_2,k_3)$$
$$+i[\delta_{\lambda\rho}\{c_{3\lambda}((\widehat{k_1-k_2})_\lambda(\delta_{\lambda\tau}\hat{k}_3^2-\hat{k}_{3\lambda}\hat{k}_{3\tau})-(\widehat{k_1-k_2})_\tau(\hat{k}^2_{1\tau}+\hat{k}^2_{2\tau}))$$
$$+(\widehat{k_1-k_2})_\tau(\hat{k}_{1\lambda}\hat{k}_{2\lambda}-2c_{1\lambda}c_{2\lambda}\hat{k}^2_{3\lambda})\}+2\text{ cyclic perms}], \qquad (A.14)$$

$$V^{(2)}_{\lambda\rho\tau}(k_1,k_2,k_3)=16V^{(0)}_{\lambda\rho\tau}(k_1,k_2,k_3)$$
$$-i\Big[\delta_{\lambda\rho}(1-\delta_{\lambda\tau})c_{3\lambda}\sum_{\sigma\neq\lambda,\tau}((\widehat{k_1-k_2})_\tau(\hat{k}^2_{1\sigma}+\hat{k}^2_{2\sigma}+\hat{k}^2_{3\sigma})$$
$$+\hat{k}_{3\tau}(\hat{k}^2_{1\sigma}-\hat{k}^2_{2\sigma}))+(1-\delta_{\lambda\rho})(1-\delta_{\lambda\tau})$$
$$\times(1-\delta_{\rho\tau})\hat{k}_{1\lambda}\hat{k}_{2\rho}(\widehat{k_1-k_2})_\tau+2\text{ cyclic perms}\Big], \tag{A.15}$$

$$V^{(3)}_{\lambda\rho\tau}(k_1,k_2,k_3)=8V^{(0)}_{\lambda\rho\tau}(k_1,k_2,k_3)-i\Big[\delta_{\lambda\rho}(1-\delta_{\lambda\tau})c_{3\lambda}(\widehat{k_1-k_2})_\tau\sum_{\sigma\neq\lambda,\tau}(\hat{k}^2_{1\sigma}+\hat{k}^2_{2\sigma})$$
$$+\tfrac{1}{2}(1-\delta_{\lambda\rho})(1-\delta_{\lambda\tau})(1-\delta_{\rho\tau})(\widehat{k_1-k_2})_\tau$$
$$\times\{\hat{k}_{1\lambda}\hat{k}_{2\rho}-\tfrac{1}{3}(\widehat{k_3-k_1})_\rho(\widehat{k_2-k_3})_\lambda\}+2\text{ cyclic perms}\Big], \tag{A.16}$$

where we have introduced the notation

$$c_{n\lambda}=\cos(\tfrac{1}{2}k_{n\lambda}). \tag{A.17}$$

Finally the four-point vertices in (A.7). The $V^{(i)}_{\lambda\rho\tau\sigma}$ have the properties

$$V^{(i)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)=V^{(i)}_{\rho\tau\sigma\lambda}(k_2,k_3,k_4,k_1)=V^{(i)}_{\sigma\tau\rho\lambda}(k_4,k_3,k_2,k_1). \tag{A.18}$$

They are given by the following unwieldy expressions (which we have not attempted to reduce to an optimal form),

$$V^{(0)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)$$
$$=2\delta_{\rho\sigma}\delta_{\lambda\tau}\cos\tfrac{1}{2}(k_1-k_3)_\rho\cos\tfrac{1}{2}(k_2-k_4)_\lambda$$
$$-[\delta_{\lambda\rho}\delta_{\sigma\tau}(\cos\tfrac{1}{2}(k_1-k_2)_\tau\cos\tfrac{1}{2}(k_3-k_4)_\lambda-\tfrac{1}{4}\hat{k}_{1\tau}\hat{k}_{2\tau}\hat{k}_{3\lambda}\hat{k}_{4\lambda})+1\text{ cyclic perm}]$$
$$-\tfrac{1}{6}[\delta_{\lambda\rho}\delta_{\lambda\tau}\hat{k}_{4\lambda}(\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma}+c_{1\sigma}(\widehat{k_2-k_3})_\sigma+c_{3\sigma}(\widehat{k_2-k_1})_\sigma)+3\text{ cyclic perms}]$$
$$+\tfrac{1}{12}\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\Big[2\sum_\nu\hat{k}_{1\nu}\hat{k}_{2\nu}\hat{k}_{3\nu}\hat{k}_{4\nu}+2(\widehat{k_1+k_3})^2-(\widehat{k_1+k_2})^2-(\widehat{k_1+k_4})^2\Big], \tag{A.19}$$

$$V^{(1)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)$$
$$=8\delta_{\lambda\tau}\delta_{\rho\sigma}[c_{1\lambda}c_{3\lambda}\cos(k_2-k_4)_\lambda\cos\tfrac{1}{2}(k_1-k_3)_\rho+(\lambda\leftrightarrow\rho,\ k_1\leftrightarrow k_2,\ k_3\leftrightarrow k_4)]$$
$$-2[\delta_{\lambda\rho}\delta_{\sigma\tau}\{(\cos\tfrac{1}{2}(k_1+k_2)_\lambda\cos(k_1+k_2)_\lambda\cos\tfrac{1}{2}(k_1+k_2)_\tau$$
$$+\cos(k_1+k_2)_\lambda c_{1\lambda}c_{2\lambda}\hat{k}_{1\tau}\hat{k}_{2\tau}+2\cos\tfrac{1}{2}(k_1+k_2)_\lambda\cos\tfrac{1}{2}(k_1+k_2)_\tau c_{1\tau}c_{2\tau}\hat{k}_{1\tau}\hat{k}_{2\tau}$$
$$+\cos\tfrac{1}{2}(k_1+k_2)_\lambda\cos\tfrac{1}{2}(2k_1-2k_2-k_3+k_4)_\tau)$$
$$+(\lambda\leftrightarrow\tau,\ k_1\leftrightarrow k_3,\ k_2\leftrightarrow k_4)\}+1\text{ cyclic perm}]$$
$$+[\delta_{\lambda\rho}\delta_{\lambda\tau}\{(\tfrac{2}{3}\hat{k}_{4\lambda}\hat{k}_{4\sigma}(c^2_{4\sigma}+c^2_{4\lambda})+2\hat{k}_{4\lambda}\hat{k}_{1\sigma}c_{1\sigma}c_{4\sigma}\cos(k_1+k_4)_\sigma$$

$$+4\hat{k}_{4\lambda}\hat{k}_{1\sigma}c_{1\lambda}c_{4\lambda}\cos\tfrac{1}{2}(k_1+k_4)_\lambda\cos\tfrac{1}{2}(k_1+k_4)_\sigma$$

$$+c_{1\lambda}(\widehat{2k_4+k_3-k_2})_\lambda(\widehat{k_1-k_2-k_3})_\sigma)+(k_1\leftrightarrow k_3)\}+3\text{ cyclic perms}]$$

$$+\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\Bigg[\sum_\nu(-\tfrac{4}{3}\hat{k}_{1\nu}^2\{c_{1\nu}^2+c_{1\lambda}^2\}+\tfrac{1}{2}\sin^2(k_1+k_2)_\nu$$

$$-\tfrac{1}{8}(\widehat{k_1+k_2})_\nu^2(\widehat{k_1+k_2})_\lambda^2-2\hat{k}_{1\nu}\hat{k}_{2\nu}c_{1\nu}c_{2\nu}\cos(k_1+k_2)_\nu$$

$$-4\hat{k}_{1\nu}\hat{k}_{2\nu}\cos\tfrac{1}{2}(k_1+k_2)_\nu c_{1\lambda}c_{2\lambda}\cos\tfrac{1}{2}(k_1+k_2)_\lambda+\tfrac{1}{4}(\widehat{k_1+k_2})_\nu^2(\widehat{k_1+k_4})_\lambda^2)$$

$$+3-3(\widehat{k_1+k_2})_\lambda^2+\tfrac{1}{2}(\widehat{k_1+k_2})_\lambda^4-\cos(k_1+k_2)_\lambda\cos(k_1+k_4)_\lambda$$

$$-c_{1\lambda}(\widehat{k_1-k_2-k_3})_\lambda(\widehat{2k_4+k_3-k_2})_\lambda-c_{1\lambda}(\widehat{k_1-k_4-k_3})_\lambda(\widehat{2k_2+k_3-k_4})_\lambda$$

$$+2\cos\tfrac{1}{2}(k_1+k_2)_\lambda\cos\tfrac{1}{2}(2k_3-2k_4+k_2-k_1)_\lambda$$

$$-4c_{1\lambda}c_{3\lambda}\cos\tfrac{1}{2}(k_1-k_3)_\lambda\cos(k_2-k_4)_\lambda$$

$$+(3\text{ cyclic permutations of the momenta})\Bigg],\qquad\text{(A.20)}$$

$$V^{(2)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)$$

$$=4\delta_{\lambda\tau}\delta_{\sigma\rho}(1-\delta_{\lambda\rho})\cos\tfrac{1}{2}(k_1-k_3)_\rho\cos\tfrac{1}{2}(k_2-k_4)_\lambda\sum_{\mu\neq\lambda,\rho}\left(4-\tfrac{1}{2}\sum_{a=1}^4\hat{k}_{a\mu}^2\right)$$

$$+4\Bigg[\delta_{\lambda\rho}\delta_{\sigma\tau}(1-\delta_{\lambda\tau})\Bigg\{\cos\tfrac{1}{2}(k_1+k_2)_\lambda\sum_{\mu\neq\lambda,\tau}\Bigg(\Bigg(-1+\tfrac{1}{4}\sum_{a=1}^4\hat{k}_{a\mu}^2$$

$$-\tfrac{1}{8}(\widehat{k_1+k_2})_\mu^2-\tfrac{1}{4}(\widehat{k_1+k_4})_\mu^2\Bigg)\cos\tfrac{1}{2}(k_1+k_2)_\tau-\hat{k}_{1\tau}\hat{k}_{2\tau}c_{1\mu}c_{2\mu}\cos\tfrac{1}{2}(k_1+k_2)_\mu\Bigg)$$

$$+(\lambda\leftrightarrow\tau,\,k_1\leftrightarrow k_3,\,k_2\leftrightarrow k_4)\Bigg\}+1\text{ cyclic perm}\Bigg]$$

$$+\Bigg[\delta_{\lambda\rho}\delta_{\lambda\tau}(1-\delta_{\lambda\sigma})\hat{k}_{4\lambda}\sum_{\mu\neq\lambda,\sigma}((\tfrac{1}{3}+\tfrac{1}{12}\hat{k}_{4\mu}^2+\tfrac{1}{2}\hat{k}_{1\mu}\hat{k}_{4\mu}\cos\tfrac{1}{2}(k_1+k_4)_\mu)\hat{k}_{4\sigma}$$

$$+2c_{4\mu}^2\hat{k}_{1\sigma}\cos\tfrac{1}{2}(k_1+k_4)_\sigma+c_{4\mu}\cos\tfrac{1}{2}(2k_1+k_4)_\mu(\widehat{2k_1+k_4})_\sigma$$

$$+(k_1\leftrightarrow k_3))+3\text{ cyclic perms}\Bigg]$$

$$+\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\sum_{\lambda\neq\mu\neq\nu\neq\lambda}[\tfrac{2}{3}\hat{k}_{1\mu}^2(2-\tfrac{1}{4}\hat{k}_{1\nu}^2)-(\widehat{k_1+k_2})_\mu^2(1-\tfrac{1}{8}(\widehat{k_1+k_2})_\nu^2)$$

$$+3\text{ permutations of momenta}]$$

$$+[\delta_{\lambda\tau}(1-\delta_{\lambda\sigma})(1-\delta_{\lambda\rho})(1-\delta_{\sigma\rho})\cos\tfrac{1}{2}(k_2-k_4)_\lambda(2\hat{k}_{4\sigma}\hat{k}_{4\rho}\cos\tfrac{1}{2}(k_1-k_3)_\rho$$

$$+(\widehat{2k_1+k_4})_\sigma(\widehat{2k_3+k_2})_\rho+(\rho\leftrightarrow\sigma,\,k_2\leftrightarrow k_4))+1\text{ cyclic perm}]$$

$$+[\delta_{\lambda\rho}(1-\delta_{\lambda\sigma})(1-\delta_{\lambda\tau})(1-\delta_{\sigma\tau})(\cos\tfrac{1}{2}(k_1+k_2)_\sigma \hat{k}_{3\sigma}\{-\cos\tfrac{1}{2}(k_3-k_4)_\lambda \hat{k}_{3\tau}$$
$$+\cos\tfrac{1}{2}(k_1+k_2)_\lambda (\widehat{2k_4+k_3})_\tau\} + \tfrac{1}{4}\hat{k}_{3\lambda}\hat{k}_{4\lambda}\hat{k}_{4\sigma}(\widehat{2k_4+k_3})_\tau$$
$$+\cos\tfrac{1}{2}(k_1+k_2)_\lambda (\widehat{2k_1+k_4})_\sigma \{(\widehat{2k_4+k_3})_\tau - \tfrac{1}{2}(\widehat{2k_2+k_3})_\tau\}$$
$$+(\tau\leftrightarrow\sigma,\, k_1\leftrightarrow k_2,\, k_3\leftrightarrow k_4)) + 3 \text{ cyclic perms}]\,, \qquad (A.21)$$

$$V^{(3)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4)$$
$$= 8\delta_{\lambda\tau}\delta_{\sigma\rho}(1-\delta_{\lambda\rho})\cos\tfrac{1}{2}(k_1-k_3)_\rho \cos\tfrac{1}{2}(k_2-k_4)_\lambda$$
$$\times \sum_{\mu\neq\lambda,\rho} \cos\tfrac{1}{2}(k_1-k_3)_\mu \cos\tfrac{1}{2}(k_2-k_4)_\mu$$
$$+2\Bigg[\delta_{\lambda\rho}\delta_{\sigma\tau}(1-\delta_{\lambda\tau})\Bigg\{\cos\tfrac{1}{2}(k_1+k_2)_\lambda \sum_{\mu\neq\lambda,\tau}\cos\tfrac{1}{2}(k_1+k_2)_\mu$$
$$\times(\cos\tfrac{1}{2}(k_1+k_2)_\mu \cos\tfrac{1}{2}(k_1+k_2)_\tau - 2\cos\tfrac{1}{2}(k_1-k_2)_\mu \cos\tfrac{1}{2}(k_1-k_2)_\tau)$$
$$+(\lambda\leftrightarrow\tau,\, k_1\leftrightarrow k_3,\, k_2\leftrightarrow k_4)\Bigg\} + 1 \text{ cyclic perm}\Bigg]$$
$$+\Bigg[\delta_{\lambda\rho}\delta_{\lambda\tau}(1-\delta_{\lambda\sigma})\hat{k}_{4\lambda}\sum_{\mu\neq\lambda,\sigma}((-\tfrac{1}{3}+\tfrac{1}{12}\hat{k}^2_{4\mu})\hat{k}_{4\sigma}$$
$$+c_{4\mu}\cos\tfrac{1}{2}(2k_1+k_4)_\mu(\widehat{2k_1+k_4})_\sigma + (k_1\leftrightarrow k_3)) + 3 \text{ cyclic perms}\Bigg]$$
$$+\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\sum_{\lambda\neq\mu\neq\nu\neq\lambda}[\tfrac{2}{3}\hat{k}^2_{1\mu}(1-\tfrac{1}{4}\hat{k}^2_{1\nu}) - (\widehat{k_1+k_2})^2_\mu(\tfrac{1}{2}-\tfrac{1}{8}(\widehat{k_1+k_2})^2_\nu)$$
$$+3 \text{ permutations of momenta}]$$
$$-[\delta_{\lambda\tau}(1-\delta_{\lambda\sigma})(1-\delta_{\lambda\rho})(1-\delta_{\sigma\rho})\cos\tfrac{1}{2}(k_2-k_4)_\lambda$$
$$\times\{(\widehat{2k_1+k_4})_\sigma(\widehat{2k_1+k_2})_\rho + (k_1\leftrightarrow k_3)\} + 1 \text{ cyclic perm}]$$
$$+[\delta_{\lambda\rho}(1-\delta_{\lambda\sigma})(1-\delta_{\lambda\tau})(1-\delta_{\sigma\tau})(\tfrac{1}{2}\cos\tfrac{1}{2}(k_1+k_2)_\lambda\{-\hat{k}_{3\tau}(\widehat{2k_3+k_4})_\sigma$$
$$-(\widehat{2k_2+k_3})_\tau(\widehat{2k_1+k_4})_\sigma\} + \tfrac{1}{4}\cos\tfrac{1}{2}(k_3-k_4)_\lambda\{\hat{k}_{3\tau}\hat{k}_{4\sigma}$$
$$+(\widehat{2k_4+k_3})_\tau(\widehat{2k_3+k_4})_\sigma\} + (\tau\leftrightarrow\sigma,\, k_1\leftrightarrow k_2,\, k_3\leftrightarrow k_4)) + 3 \text{ cyclic perms}]\,. \qquad (A.22)$$

The tensor appearing in (A.7) is totally symmetric and defined by

$$S^{abcd}_{\mathrm{R}} = \tfrac{1}{12}\operatorname{tr}(\{R^a, R^b\}\{R^c, R^d\} + \{R^a, R^c\}\{R^b, R^d\} + \{R^a, R^d\}\{R^b, R^c\})\,. \qquad (A.23)$$

Upon contraction one obtains

$$\sum_a S^{aabc}_{\mathrm{R}} = \delta^{bc} t_{\mathrm{R}}(C_{\mathrm{R}} - \tfrac{1}{6}N)\,. \qquad (A.24)$$

The vertices $W^{(i)}_{\lambda\rho\tau\sigma}$ are totally symmetric (have no continuum analogue),

$$W^{(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4}(k_1, k_2, k_3, k_4) = W^{(1)}_{\lambda_{\rho(1)}\lambda_{\rho(2)}\lambda_{\rho(3)}\lambda_{\rho(4)}}(k_{\rho(1)}, k_{\rho(2)}, k_{\rho(3)}, k_{\rho(4)})\,. \qquad (A.25)$$

They are given in terms of the $V^{(i)}_{\lambda\rho\tau\sigma}$ by

$$W^{(i)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) = 4(V^{(i)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) + V^{(i)}_{\lambda\rho\sigma\tau}(k_1, k_2, k_4, k_3) + V^{(i)}_{\lambda\tau\rho\sigma}(k_1, k_3, k_2, k_4)) . \tag{A.26}$$

For completeness we list the resulting expressions

$$W^{(0)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) = 2\Big\{\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\sum_{\nu}\hat{k}_{1\nu}\hat{k}_{2\nu}\hat{k}_{3\nu}\hat{k}_{4\nu} + (\delta_{\lambda\rho}\delta_{\sigma\tau}\hat{k}_{3\lambda}\hat{k}_{4\lambda}\hat{k}_{1\tau}\hat{k}_{2\tau} + 2\text{ perms}) - (\delta_{\lambda\rho}\delta_{\lambda\tau}\hat{k}_{4\lambda}\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma} + 3\text{ perms})\Big\}, \tag{A.27}$$

$$W^{(1)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) = 32\Big\{\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\sum_{\nu}\hat{k}_{1\nu}\hat{k}_{2\nu}\hat{k}_{3\nu}\hat{k}_{4\nu}(K_\nu + K_\lambda) + (\delta_{\lambda\rho}\delta_{\sigma\tau}\hat{k}_{3\lambda}\hat{k}_{4\lambda}\hat{k}_{1\tau}\hat{k}_{2\tau}(K_\lambda + K_\tau) + 2\text{ perms}) - (\delta_{\lambda\rho}\delta_{\lambda\tau}\hat{k}_{4\lambda}\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma}(K_\lambda + K_\sigma) + 3\text{ perms})\Big\}, \tag{A.28}$$

where we have introduced the notation

$$K_\nu = c_{1\nu}c_{2\nu}c_{3\nu}c_{4\nu}, \tag{A.29}$$

$$\begin{aligned} &W^{(2)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) \\ &= W^{(3)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) + 4(d-2)W^{(0)}_{\lambda\rho\tau\sigma}(k_1, k_2, k_3, k_4) \\ &+ 2\Big\{\Big[\delta_{\lambda\rho}\delta_{\lambda\tau}(1-\delta_{\lambda\sigma})\sum_{\mu\neq\lambda,\sigma}\hat{k}_{4\lambda}\hat{k}_{4\mu}(\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma}\hat{k}_{4\mu} - \hat{k}_{1\mu}\hat{k}_{2\mu}\hat{k}_{3\mu}\hat{k}_{4\sigma}) \\ &+ 3\text{ perms}\Big] - \Big[\delta_{\lambda\rho}\delta_{\sigma\tau}(1-\delta_{\lambda\sigma})\sum_{\mu\neq\lambda,\sigma}(\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\lambda}\hat{k}_{4\lambda}(\widehat{k_1+k_2})^2_\mu \\ &+ 2\hat{k}_{1\mu}\hat{k}_{2\mu}\hat{k}_{3\mu}\hat{k}_{4\mu}(\hat{k}_{1\sigma}\hat{k}_{2\sigma}c_{3\lambda}c_{4\lambda} + c_{1\sigma}c_{2\sigma}\hat{k}_{3\lambda}\hat{k}_{4\lambda})) + 2\text{ perms}\Big] \\ &+ [\delta_{\lambda\rho}(1-\delta_{\lambda\sigma})(1-\delta_{\lambda\tau})(1-\delta_{\tau\sigma})(2c_{3\lambda}c_{4\lambda}\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma}\hat{k}_{1\tau}\hat{k}_{2\tau}\hat{k}_{4\tau} \\ &- \hat{k}_{3\lambda}\hat{k}_{4\lambda}\{\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma}c_{4\tau}(\widehat{k_1+k_2})_\tau + \hat{k}_{1\tau}\hat{k}_{2\tau}\hat{k}_{4\tau}c_{3\sigma}(\widehat{k_1+k_2})_\sigma \\ &- \hat{k}_{3\sigma}\hat{k}_{4\tau}(c_{1\sigma}c_{2\sigma}\hat{k}_{1\tau}\hat{k}_{2\tau} + c_{1\tau}c_{2\tau}\hat{k}_{1\sigma}\hat{k}_{2\sigma})\} + 5\text{ perms}]\Big\}, \end{aligned} \tag{A.30}$$

$$
\begin{aligned}
&W^{(3)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)\\
&=8\Big\{\delta_{\lambda\rho}\delta_{\lambda\tau}\delta_{\lambda\sigma}\sum_{\lambda\neq\mu\neq\nu\neq\lambda}(\hat{k}_{1\mu}\hat{k}_{2\mu}\hat{k}_{3\mu}\hat{k}_{4\mu}K_\nu\\
&\quad+(\hat{k}_{1\mu}\hat{k}_{2\mu}c_{3\mu}c_{4\mu}\hat{k}_{3\nu}\hat{k}_{4\nu}c_{1\nu}c_{2\nu}+(2\to3\to4\to2)+(2\to4\to3\to2)))\\
&\quad-\Big[\delta_{\lambda\rho}\delta_{\lambda\tau}(1-\delta_{\lambda\sigma})\hat{k}_{4\lambda}\sum_{\mu\neq\lambda,\sigma}(\hat{k}_{1\sigma}\hat{k}_{2\sigma}\hat{k}_{3\sigma}K_\mu\\
&\quad+(\hat{k}_{1\sigma}c_{2\sigma}c_{3\sigma}\hat{k}_{2\mu}\hat{k}_{3\mu}c_{1\mu}c_{4\mu}+(1\to2\to3\to1)+(1\to3\to2\to1))+3\text{ perms}\Big]\\
&\quad+\Big[\delta_{\lambda\rho}\delta_{\sigma\tau}(1-\delta_{\lambda\sigma})\sum_{\mu\neq\lambda,\sigma}(\hat{k}_{3\lambda}\hat{k}_{4\lambda}c_{\mu3}c_{\mu4}+\hat{k}_{3\mu}\hat{k}_{4\mu}c_{\lambda3}c_{\lambda4})\\
&\quad\times(\hat{k}_{1\sigma}\hat{k}_{2\sigma}c_{\mu1}c_{\mu2}+\hat{k}_{1\mu}\hat{k}_{2\mu}c_{\sigma1}c_{\sigma2})+2\text{ perms}\Big]\\
&\quad+[\delta_{\lambda\rho}(1-\delta_{\lambda\sigma})(1-\delta_{\lambda\tau})(1-\delta_{\tau\sigma})(\hat{k}_{3\lambda}\hat{k}_{4\lambda}c_{4\tau}c_{3\sigma}\{\hat{k}_{1\tau}\hat{k}_{2\sigma}c_{2\tau}c_{1\sigma}+\hat{k}_{2\tau}\hat{k}_{1\sigma}c_{1\tau}c_{2\sigma}\}\\
&\quad-c_{3\lambda}c_{4\lambda}\hat{k}_{4\tau}\hat{k}_{3\sigma}\{c_{1\tau}c_{2\tau}\hat{k}_{1\sigma}\hat{k}_{2\sigma}+\hat{k}_{1\tau}\hat{k}_{2\tau}c_{1\sigma}c_{2\sigma}\})+5\text{ perms}]\Big\}. \qquad (A.31)
\end{aligned}
$$

Note that the $W^{(i)}$'s are transverse

$$\hat{k}_{1\lambda}W^{(i)}_{\lambda\rho\tau\sigma}(k_1,k_2,k_3,k_4)=0. \qquad (A.32)$$

Appendix B

AUXILIARY THEOREM

Let $f(x,a)$, $x\in\mathbb{R}^d$, $0\leqslant a\leqslant 1$ be a C^∞ function with compact support, i.e. there exists $k>0$ such that

$$f(x,a)=0\,,\qquad \text{for } |x|>k\,, \qquad (B.1)$$

Then for all $q=0,1,2$

$$\sum_n f(an,a)=\sum_{\nu=0}^{q}\frac{1}{\nu!}a^{-d+\nu}\int \mathrm{d}^dx\frac{\partial^\nu}{\partial a^\nu}f(x,a)|_{a=0}+\mathrm{O}(a^{-d+q+1})\,. \qquad (B.2)$$

Proof. From Taylor's theorem

$$f(x,a)=\sum_{\nu=0}^{a}\frac{1}{\nu!}a^\nu\frac{\partial^\nu}{\partial a^\nu}f(x,a)|_{a=0}+a^{a+1}R\,, \qquad (B.3)$$

$$|R|\leqslant C\,,\qquad R(x,a)=0\,,\qquad \text{for } |x|>K\,.$$

It follows that

$$\Big|\sum_n R(an,a)\Big| \leq \sum_{n<K/a} C = O(a^{-d}) , \tag{B.4}$$

and thus

$$\sum f(an,a) = \sum_{\nu=0}^{q} \frac{1}{\nu!} a^{\nu} \sum_n \frac{\partial^{\nu}}{\partial a^{\nu}} f(x,a)\Big|_{\substack{a=0\\ x=an}} + O(a^{-d+q+1}) . \tag{B.5}$$

All the terms on the right-hand side of (B.5) are of the form

$$\sum_n g(an) , \qquad g(x)\ C^{\infty} , \qquad g(x)=0 , \qquad \text{for } |x|>k$$

Define

$$\tilde{g}(p) = \int d^d x\, e^{-ipx} g(x) . \tag{B.6}$$

The Poisson summation formula yields

$$\sum_n g(an) = a^{-d} \sum_n \tilde{g}\left(\frac{2\pi n}{a}\right) . \tag{B.7}$$

It is well-known that for every r

$$|\tilde{g}(p)| \leq c_r (1+|p|)^{-r} , \tag{B.8}$$

from which follows for arbitrarily large r

$$\sum_n g(an) = a^{-d}\tilde{g}(0) + O(a^{-d+r}) , \tag{B.9}$$

and hence the theorem.

References

[1] K Symanzik, in Recent developments in gauge theories, eds G 't Hooft et al (Plenum, New York, 1980) p 313

[2] K Symanzik, Proc of Trieste Workshop on non-perturbative field theory and QCD, Dec. 1982, (World Scientific) p 61

[3] K Symanzik, Nucl Phys B226 (1983) 187

[4] K Symanzik, Nucl Phys B226 (1983) 205

[5] B Berg, S Meyer, I Montvay and K Symanzik, Phys Lett 126B (1983) 467, and in preparation

[6] S Meyer, Talk at Santa Barbara Workshop 20 6–1 7 83

[7] M Falcioni, G Martinelli, M L Paciello, B Taglienti and G Parisi, Nucl Phys B225 [FS9] (1983) 313

[8] T Eguchi and R Nakayama, Phys Lett 126B (1983) 89

[9] P Weisz, Nucl Phys B212 (1983) 1

[10] S Belforte, G Curci, P Menotti and G P Paffuti, Pisa preprint IFUP-TH 83/6,
B Berg, A Billoire, S Meyer and C Panagiotakopoulos, Phys Lett 133B (1983) 359,
Y Iwasaki and T Yoshie, Phys Lett 131B (1983) 159,
F Gutbrod and I Montvay, in preparation

[11] G Curci, P Menotti and G P Paffuti, Phys Lett 130B (1983) 205
[12] M Creutz, Phys Rev Lett 45 (1980) 313
[13] A Di Giacomo and G.C Rossi, Phys Lett. 100B (1981) 481
[14] T Hattori and H Kawai, Phys Lett 105B (1981) 43
[15] P Weisz, W Wetzel and R Wohlert, in preparation
[16] V Muller and W Ruhl, Ann of Phys 133 (1981) 240
[17] J Stehr, private communication
[18] K Symanzik, private communication
[19] W Fischler, Nucl Phys. B129 (1977) 157
[20] A Hasenfratz and P Hasenfratz, Phys Lett 93B (1980) 165
[21] P Weisz, Phys Lett 100B (1981) 331
[22] W Bernreuther and W Wetzel, Phys Lett 132B (1983) 382
[23] K Wilson, Recent developments in gauge theories, eds G 't Hooft et al (Plenum, New York, 1980) p 243