GLUEBALL MASSES ON LARGE LATTICES

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We calculate the scalar and tensor glueball masses on large lattices (ranging from 8^4 to $10^3 \times 12$) for $\beta = 2.2$ to 2.5 in the case of SU(2) and for $\beta = 5.5$ to 5.9 in the case of SU(3). In comparing our results to those previously obtained on much smaller lattices we find only small finite-size effects. We confirm previous results on the continuum renormalization group behaviour of the SU(3) 0^{++} and 2^{++} glueball masses.

The application of lattice [1] Monte Carlo [2] techniques to the calculation of the (glueball) spectrum of SU(2) and SU(3) gauge theories has had remarkable success. The method [3] which has become standard, because it is the most direct and is least affected by systematic biases, can be crudely summarized as follows. Construct a trial glueball wave-functional, $\phi(\boldsymbol{n}, t)$, "centred" on the site $(\boldsymbol{n}, t)$. (ϕ consists of a combination of closed loops of links such as to have the desired J^{PC} content.) Make the $\boldsymbol{p} = 0$ translation invariant sum

$$\phi(\boldsymbol{p} = 0, t) = \sum_{\boldsymbol{n}} \phi(\boldsymbol{n}, t) \qquad (1)$$

and measure (on the Monte Carlo generated gauge field configurations) the correlation function

$$G(\boldsymbol{p} = 0, t) \equiv \langle \phi(\boldsymbol{p} = 0, t)\phi(\boldsymbol{p} = 0, t = 0)\rangle \,. \qquad (2)$$

Vary ϕ over a class of sensibly chosen wave-functionals in order to obtain a large enough projection onto the desired glueball, so that $G(\boldsymbol{p} = 0, t)$ is dominated by the lowest-mass glueball intermediate state for $t \geqslant a$ (where a is the lattice spacing). Then the desired glueball mass is given by

$$ma = \ln[G(\boldsymbol{p} = 0, t = a)/G(\boldsymbol{p} = 0, t = 2a)] \,. \qquad (3)$$

The reason for using $\boldsymbol{p} = 0$ wave-functionals, as in (1), is that this makes the extraction of the mass, as in (3), very direct. The price for this convenience is that on an $L_s^3 \times L_t$ lattice one will get only L_t measurements of G per generated configuration. Thus the computer time required to achieve a given signal/error ratio is proportional to L_s^3. For this reason most calculations have been performed on small lattices, typically with $L_s = 4$. That calculations on such small lattices are not implausible is due to the numerical evidence, in both SU(2) [4] and SU(3) [5], that the typical glueball is only about $2a$ in diameter for the couplings of relevance. Nonetheless a systematic calculation on much larger lattices is obviously very desirable. In this letter we summarize the results of such a calculation.

The secret to calculating glueball masses on large lattices, as originally pointed out in ref. [6], is to observe that the number of low momenta, say $|\boldsymbol{p}| < 2m$, also increases as L_s^3. If we perform our calculation with a wave-functional of non-zero momentum

$$\phi(\boldsymbol{p}, t) = \sum_{\boldsymbol{n}} \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{n})\phi(\boldsymbol{n}, t) \,, \qquad (4)$$

we will obtain the corresponding glueball energy from

$$E(\boldsymbol{p})a = \ln[G(\boldsymbol{p}, t = a)/G(\boldsymbol{p}, t = 2a)] \qquad (5)$$

and hence a measurement of the mass from

$$(ma)^2 = [E(p)a]^2 - (pa)^2 . \qquad (6)$$

The total number of such measurements is proportional to L_s^3, and hence the computer time required to achieve a given percentage error on the mass estimate is roughly *independent* of lattice size (for large enough lattices). Of course at this point eq. (6), the continuum energy–momentum dispersion relation, is an assumption, albeit a plausible assumption for small momenta in a range of couplings where one expects to obtain continuum physics. However one can use the same data we employ herein to *demonstrate* the validity of the continuum dispersion relation. This has been done in an accompanying letter [7], and so we take eq. (6) to be valid for the range of couplings and small momenta of interest in this paper. (As $|p|/m$ becomes large, so does the error on the extracted m so that it contributes insignificantly to our final mass estimates.)

In the present calculation we use the standard Wilson action [1] with periodic boundary conditions. The lattices we employ are 8^4 at $\beta = 5.5, 5.7, 5.9$ in the SU(3) case, $8^3 \times 10$ at $\beta = 2.2, 2.3, 2.4$ and $10^3 \times 12$ at $\beta = 2.5$ in the SU(2) case. The number of configurations is 3500, 4000, 4500, 5400, 24000, 10500, 12000, respectively. For the error analysis the configurations at each β were split into between 8 and 24 groups. Since the final masses and errors were usually obtained by least χ^2 fits, accurate error estimates were crucial. Details will appear in a forthcoming longer paper. We now turn to our results: finite-size effects; 0^+ and 2^+ masses in SU(2); 0^{++} and 2^{++} masses in SU(3).

Finite-size effects. The most interesting finite-size effects are those that appear in the glueball mass estimates themselves. However, since the mass is derived from the longer-distance fall-off of correlation functions, the errors are large enough to conceal small finite-size effects. To perform a high-resolution search for such effects we consider instead the quantity $G(p = 0, a)/G(p = 0, 0)$ for which we have accurate measurements on an extensive range of lattice sizes. Typically this quantity has an O(50%) contribution from the lowest-mass glueball, with the remainder being contributed by higher-mass states. Apart from accidental cancellations any finite-size effects in the lowest glueball mass should be reflected in changes in $G(p = 0, a)/G(p = 0, 0) (= G(a)/G(0))$.

In fig. 1a we plot the SU(2) values of $G(a)/G(0)$ for various lattice sizes, for both 0^+ and 2^+, and for wavefunctionals ϕ based on either the 1×1 or the 2×2 loops. We have data at both $\beta = 2.3$ and 2.5; according to the usual perturbative relation $a(\beta)$ decreases by $\approx 40\%$ between these values of the coupling. The $8^3 \times 10$ and $10^3 \times 12$ lattices are large enough for the physics to be well inside the low-temperature con-

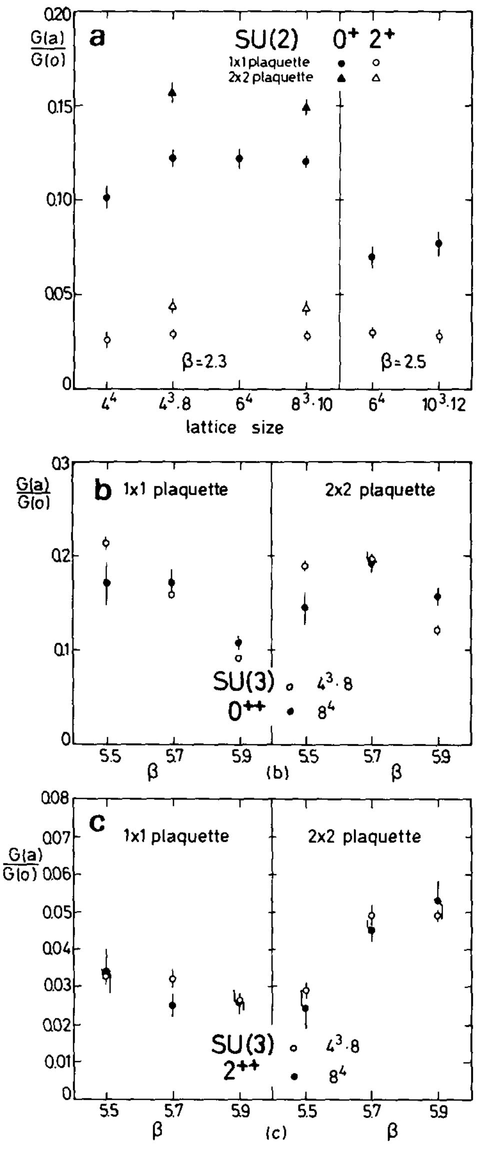

Fig. 1. Comparison of correlation functions $G(a)/G(0)$ measured on lattices of differing sizes: (a) 0^+ and 2^+ at $\beta = 2.3$ and 2.5 in SU(2); (b) 0^{++} in SU(3); (c) 2^{++} in SU(3).

fining phase of QCD. (The data on small lattices come from ref. [6] and the second paper in ref. [3].) The smallness of any finite-size effects is extraordinary.

In figs. 1b, c we plot similar data for the SU(3) case in a somewhat different format. The 2^{++} correlation functions show no finite-size effects just as in the SU(2) case. The 0^{++} case does, however, exhibit significant finite-size effects. The largest effects are at $\beta = 5.5$ which is close to the maximum of the SU(3) specific-heat peak. The drop in the correlation function is a direct reflection of the observed [8] flattening of the SU(3) specific-heat peak with increasing lattice size. It is associated with an increasing mass for the 0^{++} glueball (see below). Presumably one would have seen similar effects at $\beta = 2.2$ in the SU(2) case. This effect is to have been expected. A large lattice has a narrower Boltzman peak and will not sample gauge field configurations characteristic of the nearby critical point (with its associated zero-mass 0^{++} glueball [9]). Significant finite-size effects are also visible at $\beta = 5.9$. At this value of β a 4^3 spatial lattice is indeed very small. The direction of the observed effect suggests a decreasing glueball mass and/or an increasing projection onto the lowest-mass glueball. How much this is reflected in the actual mass estimates will be seen below. Finally we note that there are no significant finite-size effects at $\beta = 5.7$, which is the value of β we have previously [5] used for estimating the 0^{++} and 2^{++} glueball masses.

Glueball masses. In the present calculation we employ the standard method [3] with two trial wave-functionals, the 1×1 loop and the 2×2 loop. We do not perform a more extensive variational calculation, since we know from previous work on smaller lattices [3–6,10] that, where the glueball is smaller than about $2a$ across [$\beta \lesssim 2.3$ in SU(2) and $\beta \lesssim 5.7$ in SU(3)], one or both of these wave-functionals will be good enough for eq. (3) to be valid even for the lightest scalar glueball. On the other hand, once we increase β so that the glueball is more than $\approx 2a$ in diameter, a trial wave-functional needs to be much more complex in terms of loops of links on the lattice if it is to represent the increasingly structured glueball wave-function. No variational calculation of the sophistication necessary has been performed as yet. We shall not attempt such a novel calculation. Instead we accept that our wave-functional gets worse as β increases, that eq. (3) breaks down, and that in order to get a reliable mass estimate one must measure the correlation function out to ever larger distances. In practice we shall measure the scalar glueball correlation function out to four lattice spacings. For the heavier tensor glueball we shall use eq. (3) at all β relying on the more rapid fall-off of this correlation function to sieve out higher-mass contributions beyond one lattice spacing.

We extract the masses from our measured values of $E(p)$ using eq. (6) and a least χ^2 procedure. The details are not always straightforward and will be described in a longer paper. In plotting the data we transform the measured dimensionless products $m(\beta)a(\beta)$ into $m(\beta)a(\beta = \beta_0)$, for some convenient fixed β_0, using the usual perturbative renormalization group formula. If we are indeed in the continuum limit, and if moreover we are deep enough in this limit for the perturbative connection between $a(\beta)$ and β to be accurate, then we should expect $m(\beta)a(\beta = \beta_0)$ to be *independent* of β. This is the standard test for continuum physics. Given finite statistical errors one requires a measure of the *significance* of any apparent β (in-)dependence. In the present context an obvious yardstick to use is the perturbative variation of $a(\beta)$ over the range of β being considered. Of course some variation in $m(\beta)a(\beta_0)$ would not be unexpected, since there is no reason to expect the perturbative expression for $a(\beta)$ to be completely correct in the range of β we investigate. A more model-independent criterion for continuum physics is that dimensionless ratios of physical quantities should become independent of β. In our case that would be the ratio of tensor and scalar glueball masses.

We now turn to our SU(2) results. In table 1 and fig. 2a we display our 0^+ mass estimates. As β increases we present results obtained from further out along the correlation function. We observe that at $\beta = 2.3$ eq. (3) is still accurate. However, at higher β this is no longer the case. We observe (within large errors) a signal for some increase in $m(\beta)a(2.3)$ with β. However this increase is certainly much less than the factor of 2 by which the perturbative $a(\beta)$ varies from $\beta = 2.2$ to 2.5. The 2^+ mass is shown in table 1 and fig. 2b and shows very similar behaviour. In fig. 2c we plot the ratio of tensor and scalar masses versus β. The results are consistent with scaling with any systematic increase or decrease of this ratio being $\lesssim 30\%$.

Turning now to the SU(3) case we display in table 2 and fig. 3a our mass estimates for the 0^{++} glueball on

Table 1

	$m(\beta)a(\beta)$				Obtained from
	$\beta = 2.2$	$\beta = 2.3$	$\beta = 2.4$	$\beta = 2.5$	
0^+	1.57 ± 0.11	1.36 ± 0.06	1.48 ± 0.08	$1.24^{+0.12}_{-0.11}$	$G(p, a)/G(p, 2a)$
		1.20 ± 0.13	$1.14^{+0.29}_{-0.25}$	$1.16^{+0.26}_{-0.19}$	$G(p, 2a)/G(p, 3a)$
				$0.90^{+0.55}_{-0.35}$	$G(p, 3a)/G(p, 4a)$
2^+	$2.45^{+0.40}_{-0.30}$	2.70 ± 0.20	$2.35^{+0.35}_{-0.25}$	$2.05^{+0.20}_{-0.15}$	$G(p, a)/G(p, 2a)$

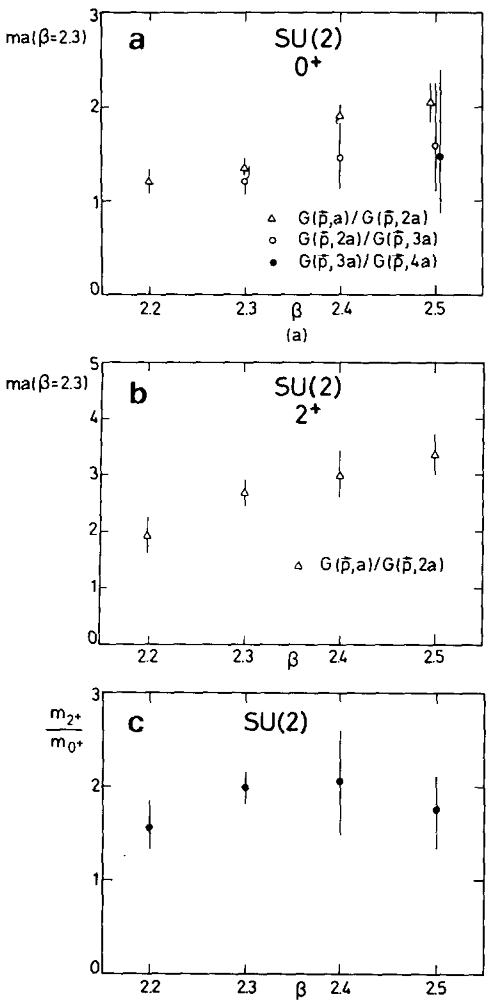

Fig. 2. SU(2) glueball masses $m(\beta)a(\beta = 2.3)$ as a function of β: (a) the scalar; (b) the tensor; (c) the ratio of tensor to scalar.

the 8^4 lattice. In fig. 3a we also show our previous [5] mass estimates obtained on a smaller $4^3 \times 8$ lattice. The main change is an increased mass at $\beta = 5.5$. This is presumably due to a flattening (with increasing lattice size) of the specific-heat peak located at this value [8]. At $\beta = 5.7$ any change is very small. At $\beta = 5.9$ the errors are large and we can only rule out very large finite-size effects. We note that the change at $\beta = 5.5$ further improves what was already a reasonably good continuum renormalization group dependence from $\beta = 5.1$ to 5.9. In table 2 and fig. 3b we plot our 2^{++} mass estimates. Note that for $\beta \leqslant 5.5$ the $4^3 \times 8$ estimates were obtained from $G(a)/G(0)$ assuming a projection of 0.9 ± 0.1 onto the lowest-mass tensor glueball (this estimate being obtained by an extrapolation from $\beta = 5.9$ and 5.7). We had no useful $p = 0$ signal for $G(2a)/G(a)$. Now with far fewer configurations we are able to get a usefully accurate signal. We observe that the $4^3 \times 8$ and 8^4 results are mutually consistent and that any β dependence is much weaker than that of the perturbative $a(\beta)$ (which changes by a factor ≈ 2.5 between $\beta = 5.1$ and $\beta = 5.9$). The ratio of tensor and scalar masses is constant for $\beta \geqslant 5.5$, showing a decrease (in the direction of decreasing β) as one approaches the strong-coupling regime.

In summary, our large lattice glueball calculations support previous results on smaller lattices. No large finite-size effects are found in either SU(2) or SU(3), with the exception of an increased mass for the scalar glueball right on the specific-heat peak – which was to be expected. Previous evidence for continuum scaling of both the 0^{++} [5,10] and 2^{++} [5] glueballs receives extra support from the present calculation. A more detailed presentation will appear in a longer paper.

Table 2

	$m(\beta)a(\beta)$			Obtained from
	$\beta = 5.5$	$\beta = 5.7$	$\beta = 5.9$	
0^{++}	1.37 ± 0.10	1.13 ± 0.07	$1.22^{+0.16}_{-0.14}$	$G(p, a)/G(p, 2a)$
			$0.66^{+0.34}_{-0.30}$	$G(p, 2a)/G(p, 3a)$
2^{++}	$2.90^{+0.60}_{-0.55}$	$2.40^{+0.35}_{-0.27}$	$2.05^{+0.55}_{-0.40}$	$G(p, a)/G(p, 2a)$

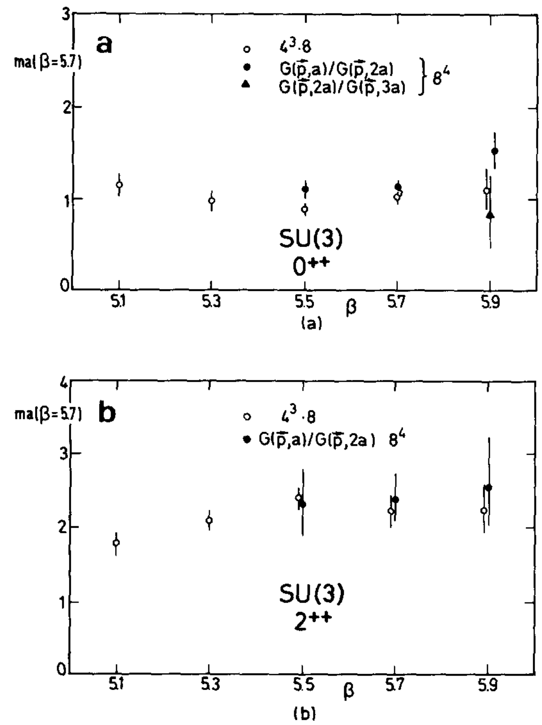

Fig. 3. SU(3) glueball masses $m(\beta)a(\beta = 5.7)$ as a function of β: (a) the scalar; (b) the tensor.

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