Nuclear Physics B239 (1984) 395–409

MULTIPLE BREMSSTRAHLUNG IN GAUGE THEORIES AT HIGH ENERGIES
(IV). The process $e^+e^- \to \gamma\gamma\gamma\gamma$*

CALKUL Collaboration

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Received 2 January 1984

We calculate the helicity amplitudes and the cross section for the process $e^+e^- \to 4\gamma$ in the high-energy limit. The resulting expressions are presented in a form which allows an easy numerical evaluation. They are valid for the kinematical configurations where at most two photons are emitted in directions nearly parallel to the lepton directions.

1. Introduction

Over two years ago, some of us developed a formalism [1, 2] for evaluating transition amplitudes of bremsstrahlung processes at high energies. That formalism was applied to $e^+e^- \to 4\gamma$ [1], which may be considered to be one of the simplest examples of double bremsstrahlung. It is the purpose of the present paper to study this process in more detail including, in particular, the effects of the electron mass for nearly collinear photons.

Experimentally, since 193 events of the type $e^+e^- \to 3\gamma$ have been observed at high energies [3], we expect the four-photon production to become observable in the near future. From the experimental point of view, the radiative Bhabha process $e^+e^- \to e^+e^-\gamma\gamma$, to be studied in paper (VI), is perhaps of more direct interest. In particular, this process contributes a correction to luminosity measurements by small-angle Bhabha scattering [4].

* Work supported in part by NATO research grant no. RG 079.80.
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¶ Work supported in part by the United States Department of Energy under contract no. DE-AS02-76-ER03227.

Another motivation for studying the process $e^+e^- \to 4\gamma$ is that due to its symmetrical structure it is the simplest double bremsstrahlung process from the theoretical point of view. Its study should provide us with more insight into more complicated double bremsstrahlung processes like $e^+e^- \to \mu^+\mu^-\gamma\gamma$ and $e^+e^- \to e^+e^-\gamma\gamma$.

As shown before [1], certain helicity amplitudes for $e^+e^- \to 4\gamma$ have factorization properties which are straightforward generalizations of results obtained in $e^+e^- \to 3\gamma$ [5]. The other helicity amplitudes, however, have a more complicated structure. Nevertheless, the result can be expressed in such a way that a numerical evaluation is rather easy. This is to be contrasted with the standard approach, which meets considerable computational difficulties, as the process is described by 24 Feynman diagrams, each containing a string of seven gamma-matrices!

This paper is organized as follows. In sect. 2, we use our formalism to obtain the helicity amplitudes. By introducing an explicit representation for the spinors, we then express these amplitudes in terms of components of the various four-momenta in the process. This is worked out in sect. 3.

In sects. 2 and 3, we assumed that the electron mass could be neglected. This approximation fails when photons are emitted in directions nearly parallel to the incoming lepton directions. In sect. 4, it is shown how the finite mass effects can be taken into account when there is only one collinear photon in the process, and, in sect. 5, we analyze the case of two collinear photons. In sect. 6 we present a discussion of the different formulae to be used when different regions of phase space are examined. Sect. 7 gives our conclusions.

Finally, in the appendix, we derive formulae for certain helicity amplitudes for the annihilation of e^+e^- into an arbitrary number of photons.

2. Helicity amplitudes

We consider the massless fermion limit first. Four-photon production is described by Feynman diagrams of the type shown in fig. 1, of which there are 24. Denoting by p_-^μ, λ and by p_+^μ, λ' the four-momentum and the helicity of the electron and of the positron, and by k_i^μ, ε_i^μ $(i=1,2,3,4)$ the photon four-momenta and polarizations, we can write the matrix element as follows:

$$M = -ie^4\bar{v}_{\lambda'}(p_+)\not{\varepsilon}_4\frac{\not{k}_4-\not{p}_+}{-2(p_+k_4)}\not{\varepsilon}_3\frac{\not{p}_- - \not{k}_1-\not{k}_2}{(p_- - k_1 - k_2)^2}\not{\varepsilon}_2\frac{\not{p}_- - \not{k}_1}{-2(p_-k_1)}\not{\varepsilon}_1 u_\lambda(p_-)$$

$$+23 \text{ other permutations of } (1,2,3,4). \tag{2.1}$$

Following ref. [1], we define the various helicities and polarizations by

$$\begin{aligned} u_\pm(p_-) &= \tfrac{1}{2}(1\pm\gamma^5)u_\pm(p_-), \\ \bar{v}_\pm(p_+) &= \tfrac{1}{2}\bar{v}_\pm(p_+)(1\pm\gamma^5), \\ \not{\varepsilon}_i^\pm &= -N_i\left[\not{k}_i\not{p}_+\not{p}_-(1\pm\gamma^5) - \not{p}_+\not{p}_-\not{k}_i(1\mp\gamma^5)\right], \\ N_i^{-2} &= 16(p_+p_-)(p_+k_i)(p_-k_i), \quad i=1,2,3,4. \end{aligned} \tag{2.2}$$

Let us denote the helicity amplitudes by $M(\lambda', \lambda; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, with λ_i the helicity of the photon i. Clearly, for $\lambda' = \lambda$, the helicity amplitudes vanish. Moreover, for $\lambda' = -\lambda$, only one of the two terms in $\not{\epsilon}_i^{\lambda_i}$ (eq. (2.2)) gives a nonvanishing contribution. As was already shown in ref. [1], it follows that, for all λ,

$$M(\lambda, -\lambda; +, +, +, +) = M(\lambda, -\lambda; -, -, -, -) = 0. \tag{2.3}$$

Finally, any two amplitudes for which only the λ_i differ by a permutation can be obtained from one another by applying the same permutation to the k_i^μ.

There are thus only six distinct nonzero helicity amplitudes, which we shall present in this section. They are obtained by writing out a given amplitude using eqs. (2.1) and (2.2), and then simplifying the expression by applying anticommutation relations and explicitly summing all contributions to the amplitude.

In this way, we obtain the following expressions:

$$M(-,+;-,+,+,+) = -\tfrac{1}{2}ie^4A(p_+p_-)(p_-k_1)\bar{v}(p_+)\not{k}_1(1+\gamma^5)u(p_-),$$

$$M(+,-;+,-,-,-) = -\tfrac{1}{2}ie^4A(p_+p_-)(p_-k_1)\bar{v}(p_+)\not{k}_1(1-\gamma^5)u(p_-),$$

$$M(-,+;+,-,-,-) = \tfrac{1}{2}ie^4A(p_+p_-)(p_+k_1)\bar{v}(p_+)\not{k}_1(1+\gamma^5)u(p_-),$$

$$M(+,-;-,+,+,+) = \tfrac{1}{2}ie^4A(p_+p_-)(p_+k_1)\bar{v}(p_+)\not{k}_1(1-\gamma^5)u(p_-),$$

$$\begin{aligned}
&M(-,+;-,-,+,+)\\
&\quad= \tfrac{1}{4}ie^4A(p_+p_-)(p_- - k_1 - k_2)^2\bar{v}(p_+)(\not{k}_1+\not{k}_2)(1+\gamma^5)u(p_-) + \tfrac{1}{8}ie^4A\bar{v}(p_+)\\
&\quad\times\Big\{\frac{1}{\Delta_{13}}\not{k}_4\not{p}_-\not{k}_2(\not{p}_- - \not{k}_1)\not{k}_3\not{p}_+\not{k}_1 + \frac{1}{\Delta_{14}}\not{k}_3\not{p}_-\not{k}_2(\not{p}_- - \not{k}_1)\not{k}_4\not{p}_+\not{k}_1\\
&\qquad+ \frac{1}{\Delta_{23}}\not{k}_4\not{p}_-\not{k}_1(\not{p}_- - \not{k}_2)\not{k}_3\not{p}_+\not{k}_2 + \frac{1}{\Delta_{24}}\not{k}_3\not{p}_-\not{k}_1(\not{p}_- - \not{k}_2)\not{k}_4\not{p}_+\not{k}_2\Big\}\\
&\quad\times(1+\gamma^5)u(p_-),\\
&M(+,-;+,+,-,-)\\
&\quad= \tfrac{1}{4}ie^4A(p_+p_-)(p_- - k_1 - k_2)^2\bar{v}(p_+)(\not{k}_1+\not{k}_2)(1-\gamma^5)u(p_-) + \tfrac{1}{8}ie^4A\bar{v}(p_+)\\
&\quad\times\Big\{\frac{1}{\Delta_{13}}\not{k}_4\not{p}_-\not{k}_2(\not{p}_- - \not{k}_1)\not{k}_3\not{p}_+\not{k}_1 + \frac{1}{\Delta_{14}}\not{k}_3\not{p}_-\not{k}_2(\not{p}_- - \not{k}_1)\not{k}_4\not{p}_+\not{k}_1\\
&\qquad+ \frac{1}{\Delta_{23}}\not{k}_4\not{p}_-\not{k}_1(\not{p}_- - \not{k}_2)\not{k}_3\not{p}_+\not{k}_2 + \frac{1}{\Delta_{24}}\not{k}_3\not{p}_-\not{k}_1(\not{p}_- - \not{k}_2)\not{k}_4\not{p}_+\not{k}_2\Big\}\\
&\quad\times(1-\gamma^5)u(p_-),
\end{aligned}\tag{2.4}$$

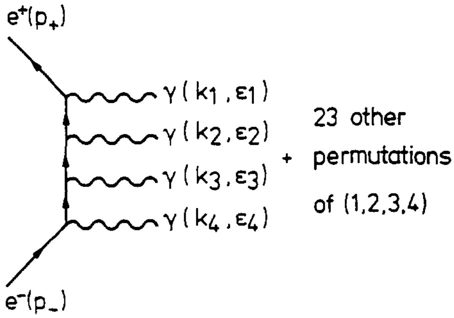

Fig. 1. Feynman diagrams for $e^+e^- \to 4\gamma$.

where

$$A = \left[(p_+k_1)(p_-k_1)(p_+k_2)(p_-k_2)(p_+k_3)(p_-k_3)(p_+k_4)(p_-k_4)\right]^{-1/2},$$

$$\Delta_{ij} = -2(p_-k_i) - 2(p_-k_j) + 2(k_ik_j), \qquad i, j = 1,2,3,4. \tag{2.5}$$

It should be noted that the sign in front of the first term of $\mathrm{M}(-,+;-,-,+,+)$ is given incorrectly in ref. [1].

3. Evaluation of helicity amplitudes

To obtain the cross section for $e^+e^- \to 4\gamma$, we must know the squared absolute values of the helicity amplitudes given by eqs. (2.4). The standard procedure, however, leads to very lengthy traces for the last two amplitudes and hardly any simplifications occur in the result. A much more convenient method consists in evaluating the helicity amplitudes directly as complex numbers for a given point in phase space. To this end, we introduce explicit representations for the γ matrices and the spinors. This procedure was already applied in the study of $e^+e^- \to 4$ jets [6].

Suppose we go to the e^+e^- c.m. frame, with the z-direction along $\boldsymbol{p}_+$, and that we introduce the notation

$$k_{\pm} = k_0 \pm k_z, \qquad k_{\perp} = k_x + ik_y, \tag{3.1}$$

for any vector k^{μ}. Defining the quantities

$$Z_{ij} = k_{i_+}k_{j_-} - k^*_{i_\perp}k_{j_\perp}, \qquad i, j = 1,2,3,4, \tag{3.2}$$

we can rewrite the helicity amplitudes of eqs. (2.4) as

$$\begin{aligned}
M(-,+;-,+,+,+) &= 4ie^4Bk_{1+}k^*_{1\perp}\,,\\
M(+,-;+,-,-,-) &= 4ie^4Bk_{1+}k_{1\perp}\,,\\
M(-,+;+,-,-,-) &= -4ie^4Bk_{1-}k^*_{1\perp}\,,\\
M(+,-;-,+,+,+) &= -4ie^4Bk_{1-}k_{1\perp}\,,\\
M(-,+;-,-,+,+) &= -2ie^4BE^{-1}F^*(1,2,3,4)\,,\\
M(+,-;+,+,-,-) &= -2ie^4BE^{-1}F(1,2,3,4)\,,
\end{aligned} \tag{3.3}$$

where E denotes the beam energy and

$$\begin{aligned}
B &= [k_{1+}k_{1-}k_{2+}k_{2-}k_{3+}k_{3-}k_{4+}k_{4-}]^{-1/2}\,,\\
F(1,2,3,4) &= (k_{1\perp}+k_{2\perp})\Delta_{12}\\
&\quad+\frac{k_{4\perp}}{\Delta_{13}}(2Ek_{2+}k^*_{3\perp}k_{1\perp}+Z_{21}Z^*_{13})\\
&\quad+\frac{k_{3\perp}}{\Delta_{14}}(2Ek_{2+}k^*_{4\perp}k_{1\perp}+Z_{21}Z^*_{14})+\frac{k_{4\perp}}{\Delta_{23}}(2Ek_{1+}k^*_{3\perp}k_{2\perp}+Z_{12}Z^*_{23})\\
&\quad+\frac{k_{3\perp}}{\Delta_{24}}(2Ek_{1+}k^*_{4\perp}k_{2\perp}+Z_{12}Z^*_{24})\,.
\end{aligned} \tag{3.4}$$

By adding the squared absolute values of all helicity amplitudes and by averaging over the initial spins, we obtain the unpolarized squared matrix element:

$$\begin{aligned}
\overline{|M|^2} = 2e^8B^2\Bigg\{&4\sum_{i=1}^{4}(k_{i+}^2+k_{i-}^2)k_{i+}k_{i-}\\
&+E^{-2}\big[|F(1,2,3,4)|^2+|F(1,3,2,4)|^2+|F(1,4,2,3)|^2\\
&+|F(2,3,1,4)|^2+|F(2,4,1,3)|^2+|F(3,4,1,2)|^2\big]\Bigg\}\,,
\end{aligned} \tag{3.5}$$

and the unpolarized cross section is given by

$$\mathrm{d}\sigma = \frac{\delta^4(p_+ + p_- - k_1 - k_2 - k_3 - k_4)}{128(2\pi)^8E^2}\,\overline{|M|^2}\,\frac{\mathrm{d}^3\boldsymbol{k}_1\,\mathrm{d}^3\boldsymbol{k}_2\,\mathrm{d}^3\boldsymbol{k}_3\,\mathrm{d}^3\boldsymbol{k}_4}{k_{1_0}k_{2_0}k_{3_0}k_{4_0}}\,. \tag{3.6}$$

4. Single collinear bremsstrahlung

The helicity amplitudes derived in the previous section were obtained by neglecting the electron mass m. When the photons are not emitted parallel to the beam direction, the approximation $m = 0$ is justified because the beam energy E is large.

In collinear situations, however, some terms proportional to m^2 should not be neglected. The reason is that in propagators terms like $(p_{\pm}k_i)$ are then of order m^2. This gives in the square of the amplitude a peaking factor of order E^2/m^2 and, obviously, terms of the form $m^2E^2/(p_{\pm}\cdot k_i)^2$ give contributions of the same size, but they have been neglected so far.

For the case where only one photon is emitted in a direction close to the beam axis, we have shown in ref. [5] that the finite mass corrections are given by the cross section of the lower-order process (here, $e^+e^- \to 3\gamma$) expressed in the appropriate variables.

For $e^+e^- \to 3\gamma$, the spin averaged squared matrix element is [5]:

$$F_3(p_+, p_-; k_1, k_2, k_3) = 2e^6(p_+p_-)\frac{\sum_{i=1}^3\left[(p_+k_i)^2+(p_-k_i)^2\right](p_+k_i)(p_-k_i)}{\prod_{i=1}^3(p_+k_i)(p_-k_i)}. \tag{4.1}$$

If k_4 is the four-momentum of the one photon which travels along the beam direction, then we must add to $\overline{|M|^2}$ from eq. (3.5) the finite mass correction

$$\begin{aligned} F_m = &-\frac{e^2m^2}{(p_+k_4)^2}F_3(p_+ - k_4, p_-; k_1, k_2, k_3) \\ &-\frac{e^2m^2}{(p_-k_4)^2}F_3(p_+, p_- - k_4; k_1, k_2, k_3). \end{aligned} \tag{4.2}$$

Formulae (3.5) and (4.2) now describe correctly the kinematical configuration where $\boldsymbol{k}_4$ is nearly parallel to $\boldsymbol{p}_+$ or $\boldsymbol{p}_-$. Evaluated in one of these limits, they take on a particularly simple form. For $\boldsymbol{k}_4$ along $\boldsymbol{p}_-$, we have

$$\begin{aligned} \overline{|M|^2} + F_m \simeq\ & 2\frac{e^8B^2k_{4+}^2}{(p_-k_4)^2}\left[(2E)^2+(2E-k_{4-})^2+\frac{m^2k_{4-}^3}{4E^2k_{4+}}\right] \\ &\times\sum_{i=1}^3 k_{i+}k_{i-}\left[k_{i+}^2+\left(\frac{2E}{2E-k_{4-}}\right)^2k_{i-}^2\right], \end{aligned} \tag{4.3}$$

and, for $\boldsymbol{k}_4$ along $\boldsymbol{p}_+$,

$$\begin{aligned} \overline{|M|^2} + F_m \simeq\ & 2\frac{e^8B^2k_{4-}^2}{(p_+k_4)^2}\left[(2E)^2+(2E-k_{4+})^2+\frac{m^2k_{4+}^3}{4E^2k_{4-}}\right] \\ &\times\sum_{i=1}^3 k_{i+}k_{i-}\left[k_{i-}^2+\left(\frac{2E}{2E-k_{4+}}\right)^2k_{i+}^2\right]. \end{aligned} \tag{4.4}$$

The case where two photons are emitted along the beam axis is more complicated and shall be treated in the next section.

5. Double collinear bremsstrahlung

The most convenient way to calculate the cross section in the case of collinear radiation of two photons is to calculate all the helicity amplitudes in this limit. The general method for this purpose was explained in ref. [7]. The central idea was to take a representation for $\not\epsilon$ for the collinear photons for which we could show that only the Feynman diagrams with the collinear photons next to the appropriate spinors had to be considered.

We have to examine two separate cases: two photons travelling together nearly parallel to a fermion direction, and two photons separately parallel to the e^+ and e^- directions. The former case is somewhat simpler and shall be treated first.

Suppose that $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ make small angles with $\boldsymbol{p}_-$, i.e.

$$(p_-k_i) = \mathrm{O}(m^2), \qquad k_{i+} = \mathrm{O}(m^2), \qquad k_{i\perp} = \mathrm{O}(m), \qquad i = 3,4. \tag{5.1}$$

Following the recipe of ref. [7], we know that all helicity amplitudes can be written as a product of two factors: an helicity amplitude for $e^+e^- \to 2\gamma$ and a proportionality factor which was called F_2, eqs. (5.1)–(5.3) in ref. [7]. These quantities F_2 depend on the momenta of the collinear photons, whereas the helicity amplitude for $e^+e^- \to 2\gamma$ does not.

It is a simple matter to evaluate the functions F_2 in the e^+e^- c.m. frame. In ref. [7], they contain an arbitrary four-vector q, which can conveniently be chosen equal to p_+. We then find

$$\begin{aligned}
F_2(+,+;+,+) &= 128CE^5k_{3+}\left(2Ek_{4+} - k_{3-}k_{4+} + k_{3\perp}k^*_{4\perp}\right),\\
F_2(+,+;-,-) &= 64CE^4k_{3+}(2E - k_{3-} - k_{4-})\left(2Ek_{4+} - k_{3-}k_{4+} + k_{3\perp}k^*_{4\perp}\right),\\
F_2(+,-;+,+) &= 32CE^4\left[4Ek_{3+}\left(2Ek_{4+} - k_{3-}k_{4+} - k_{4+}k_{4-} + k^*_{3\perp}k_{4\perp}\right)\right.\\
&\qquad\left. + \Delta_3 k^*_{3\perp}k_{4\perp}\right],\\
F_2(+,-;-,-) &= 32CE^4\left[2k_{3+}(2E - k_{3-})\left(2Ek_{4+} - k_{3-}k_{4+} + k^*_{3\perp}k_{4\perp}\right)\right.\\
&\qquad\left. + \Delta_3 k^*_{3\perp}k_{4\perp}\right],\\
F_2(+,+;+,-) &= 0,\\
F_2(+,+;-,+) &= 32CmE^3\left[-2Ek_{3-}k_{4+}k^*_{3\perp} - 2Ek_{3+}k_{4-}k^*_{4\perp} - k_{3-}(k_{3-} + k_{4-})\right.\\
&\qquad\left.\times\left(k_{3+}k^*_{4\perp} - k_{4+}k^*_{3\perp}\right)\right],\\
F_2(+,-;+,-) &= -64CmE^4k_{3+}k_{4-}k_{4\perp},\\
F_2(+,-;-,+) &= -32CmE^3k_{3-}k^*_{3\perp}\left(2Ek_{4+} - k_{3-}k_{4+} + k^*_{3\perp}k_{4\perp}\right),
\end{aligned} \tag{5.2}$$

with

$$C = N_3 N_4 / \Delta_3 \Delta_{34}, \qquad \Delta_i = -2(p_- k_i), \qquad i = 1,2,3,4. \tag{5.3}$$

The first two labels of F_2 denote the helicities of photons 3 and 4, photon 3 being the one closest to $u(p_-)$ in the Feynman diagrams. The third label is minus the helicity of the e^+, and the last one denotes the e^- helicity. This time, the phase choice for the fermion spinors, which was made in ref. [6], determines the phases of the last three quantities F_2 in eqs. (5.2).

For the so-called "allowed" amplitudes, in which the e^- helicity is not flipped, we then have in the collinear limit,

$$M(-\lambda, \lambda; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = e^2 [F_2(\lambda_3, \lambda_4; \lambda, \lambda) + (3 \leftrightarrow 4)] M^0(-\lambda, \lambda; \lambda_1, \lambda_2), \tag{5.4}$$

and for the "forbidden" amplitudes, which are proportional to m,

$$M(\lambda, \lambda; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = e^2 [F_2(\lambda_3, \lambda_4; -\lambda, \lambda) + (3 \leftrightarrow 4)] M^0(\lambda, -\lambda; \lambda_1, \lambda_2), \tag{5.5}$$

where the quantities M^0 are defined by

$$M^0(\lambda, -\lambda; \lambda_1, \lambda_2) = -ie^2 \bar{v}(p_+) \left[\not{\varepsilon}_1^{\lambda_1} \frac{\not{k}_1 - \not{p}_+}{\Delta'_1} \not{\varepsilon}_2^{\lambda_2} + \not{\varepsilon}_2^{\lambda_2} \frac{\not{k}_2 - \not{p}_+}{\Delta'_2} \not{\varepsilon}_1^{\lambda_1} \right] \frac{1 - \lambda\gamma^5}{2} u(p_-), \tag{5.6}$$

with

$$\Delta'_i = -2(p_+ k_i), \qquad i = 1,2,3,4. \tag{5.7}$$

These expressions M^0 are the remainders of the relevant Feynman diagrams, once the reference to the collinear photons is removed, but with the proper fermion helicity projection operators. They can be viewed as helicity amplitudes for $e^+e^- \to 2\gamma$ in the massless limit, but written in the appropriate variables. They are easily evaluated with the method of refs. [1, 2], yielding

$$\begin{aligned}
M^0(+,-;+,-) &= -2ie^2 k_{1+} k_{2\perp} [k_{1+} k_{1-} k_{2+} k_{2-}]^{-1/2}, \\
M^0(+,-;-,+) &= -2ie^2 k_{2+} k_{1\perp} [k_{1+} k_{1-} k_{2+} k_{2-}]^{-1/2}, \\
M^0(-,+;+,-) &= -2ie^2 k_{2+} k_{1\perp}^* [k_{1+} k_{1-} k_{2+} k_{2-}]^{-1/2}, \\
M^0(-,+;-,+) &= -2ie^2 k_{1+} k_{2\perp}^* [k_{1+} k_{1-} k_{2+} k_{2-}]^{-1/2},
\end{aligned} \tag{5.8}$$

all other helicity amplitudes being zero in this limit.

Combining eqs. (5.2)–(5.5) and (5.8), it is now trivial to derive the helicity amplitudes in the double collinear limit. We merely give the result for the unpolarized squared matrix element:

$$\overline{|M|^2} = 2e^8\left(\frac{k_{1+}}{k_{1-}} + \frac{k_{2+}}{k_{2-}}\right)(k_{3+}k_{3-}k_{4+}k_{4-})^{-1}\left\{4\left[1+\left(\frac{2E-k_{3-}-k_{4-}}{2E}\right)^2\right]|A_1|^2 \right.$$
$$\left. + |A_2(3,4)|^2 + |A_2(4,3)|^2 + |A_3|^2 + |A_4(3,4)|^2 + |A_4(4,3)|^2\right\} \tag{5.9}$$

where B is given by eq. (3.4) and

$$A_1 = \frac{E^2 k_{3+}k_{4+}}{(p_-k_3)(p_-k_4)}\left[1 + m^2 Z_{34}Z_{43}/4E^2 k_{3+}k_{4+}\Delta_{34}\right],$$

$$A_2(3,4) = \frac{1}{\Delta_{34}}\left\{k_{3+}(2E-k_{3-})(2Ek_{4+} - Z_{43})/(p_-k_3)\right.$$
$$\left. + \left[2Ek_{3+}k_{4+}(2E-k_{3-}-k_{4-}) - m^2 k_{4-}k_{3\perp}k^*_{4\perp}/2E\right]/(p_-k_4)\right\},$$

$$A_3 = \frac{m}{2(p_-k_3)(p_-k_4)}\left[k_{3-}k_{4+}k^*_{3\perp} + k_{3+}k_{4-}k^*_{4\perp} - (k^*_{3\perp} + k^*_{4\perp})Z_{34}Z_{43}/\Delta_{34}\right],$$

$$A_4(3,4) = \frac{mk_{3-}k_{3\perp}}{2E\Delta_{34}}\left[(2Ek_{4+} - Z_{43})/(p_-k_3) + 2Ek_{4+}/(p_-k_4)\right]. \tag{5.10}$$

In obtaining eq. (5.9), we used the relations

$$k_{1\perp} = -k_{2\perp} + \mathrm{O}(m), \qquad k_{1+}k_{1-} = k_{2+}k_{2-} + \mathrm{O}(m), \tag{5.11}$$

as well as the relation of the type

$$k_{2+}(2E - k_{3-} - k_{4-}) = 2Ek_{1-} + \mathrm{O}(m). \tag{5.12}$$

In eq. (5.9), the terms proportional to A_1 and A_2 are the contributions of the allowed amplitudes, whereas the terms with A_3 and A_4 arise from the forbidden amplitudes.

To obtain the unpolarized squared matrix element for the collinear limit when $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ are nearly parallel to $\boldsymbol{p}_+$, it suffices to interchange p_+ and p_- in eqs. (5.9),(5.10). This amounts to interchanging the subscripts + and − as well as replacing Δ_{34} by Δ'_{34}, where, in general,

$$\Delta'_{ij} = -2(p_+k_i) - 2(p_+k_j) + 2(k_ik_j), \qquad i, j = 1,2,3,4. \tag{5.13}$$

For the mixed double collinear limit, we take $\boldsymbol{k}_3$ along $\boldsymbol{p}_+$ and $\boldsymbol{k}_4$ along $\boldsymbol{p}_-$. This case can be viewed as a simultaneous occurrence of two single collinear limits. We know from sect. 4 in ref. [7] that we need to know the quantities F_1 describing the single collinear emission. For photon 4 along $\boldsymbol{p}_-$, we have

$$
\begin{aligned}
&F_1(+;+,+)=16N_4E^3k_{4+}/\Delta_4,\\
&F_1(+;-,-)=8N_4E^2k_{4+}(2E-k_{4-})/\Delta_4,\\
&F_1(+;-,+)=-4N_4mEk_{4-}k_{4\perp}^*/\Delta_4,\\
&F_1(+;+,-)=0,\\
&F_1(\lambda_4;\lambda',\lambda)=[F_1(-\lambda_4;-\lambda',-\lambda)]^*.
\end{aligned}
\tag{5.14}
$$

We used the same notation as in ref. [7], i.e. the two last labels of F_1 depend on the helicities of the e^+ and e^-, respectively.

Of course, for photon 3 along $\boldsymbol{p}_+$, we have analogous expressions, which we denote by G_1. They are

$$
\begin{aligned}
&G_1(+;+,+)=16N_3E^3k_{3-}/\Delta'_3,\\
&G_1(+;-,-)=8N_3E^2k_{3-}(2E-k_{3+})/\Delta'_3,\\
&G_1(+;-,+)=-4N_3mEk_{3+}k_{3\perp}/\Delta'_3,\\
&G_1(+;+,-)=0,\\
&G_1(\lambda_3;\lambda,\lambda')=[G_1(-\lambda_3;-\lambda,-\lambda')]^*.
\end{aligned}
\tag{5.15}
$$

This time, the second (third) label of G_1 depends on the e^- (e^+) helicity.

In this mixed double collinear limit, one finds that every helicity amplitude can be written as a sum of two terms. For an allowed amplitude, $\lambda'=-\lambda$, one obtains a contribution from the amplitude where the e^+ and e^- retain their helicities, and a contribution in which both helicities are flipped. More precisely:

$$
\begin{aligned}
M(-\lambda,\lambda;\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\,&e^2F_1(\lambda_4;\lambda,\lambda)G_1(\lambda_3;-\lambda,-\lambda)\tilde{M}^0(-\lambda,\lambda;\lambda_1,\lambda_2)\\
&+e^2F_1(\lambda_4;-\lambda,\lambda)G_1(\lambda_3;\lambda,-\lambda)\tilde{M}^0(\lambda,-\lambda;\lambda_1,\lambda_2).
\end{aligned}
\tag{5.16}
$$

Similarly, for the forbidden amplitudes, $\lambda'=\lambda$, one and only one fermion helicity must be flipped. Thus

$$
\begin{aligned}
M(\lambda,\lambda;\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\,&e^2F_1(\lambda_4;\lambda,\lambda)G_1(\lambda_3;-\lambda,\lambda)\tilde{M}^0(-\lambda,\lambda;\lambda_1,\lambda_2)\\
&+e^2F_1(\lambda_4;-\lambda,\lambda)G_1(\lambda_3;\lambda,\lambda)\tilde{M}^0(\lambda,-\lambda;\lambda_1,\lambda_2).
\end{aligned}
\tag{5.17}
$$

The expressions $\tilde{M}^0$ in eqs. (5.16),(5.17) are again remainders of the relevant Feynman diagrams after removal of the collinear photon parts. This time, they read

$$\tilde{M}^0(\lambda, -\lambda; \lambda_1, \lambda_2) = -ie^2\bar{v}(p_+)\left[\not{\varepsilon}_1^{\lambda_1}\frac{\not{p}_- - \not{k}_2 - \not{k}_4}{\Delta_{24}}\not{\varepsilon}_2^{\lambda_2} + (1 \leftrightarrow 2)\right]\frac{1-\lambda\gamma^5}{2}u(p_-). \tag{5.18}$$

In this way, all helicity amplitudes can be readily calculated, and we find for $\boldsymbol{k}_3$ along $\boldsymbol{p}_+$, $\boldsymbol{k}_4$ along $\boldsymbol{p}_-$,

$$\overline{|M|^2} = 2e^8B^2\left[\frac{Ek_{3-}k_{4+}}{(p_+k_3)(p_-k_4)}\right]^2\left[(2E)^2 + (2E - k_{3+})^2 + \frac{m^2k_{3+}^3}{4E^2k_{3-}}\right]$$
$$\times\left[(2E)^2 + (2E - k_{4-})^2 + \frac{m^2k_{4-}^3}{4E^2k_{4+}}\right]\frac{k_{1+}k_{1-}(k_{1+}^2 + k_{2+}^2)}{(2E - k_{3+})^2}. \tag{5.19}$$

This time, we used the relations

$$k_{1+}(2E - k_{4-}) = k_{2-}(2E - k_{3+}) + \mathrm{O}(m),$$
$$k_{2+}(2E - k_{4-}) = k_{1-}(2E - k_{3+}) + \mathrm{O}(m). \tag{5.20}$$

Interchanging 3 and 4 in eq. (5.19) then gives the formula for $\overline{|M|^2}$ when $\boldsymbol{k}_3$ is along $\boldsymbol{p}_-$ and $\boldsymbol{k}_4$ is along $\boldsymbol{p}_+$. This then concludes the derivation of the formulae for the cross section in all double collinear configurations.

6. Discussion

The formulae we derived in the preceding sections can be used for different purposes. Either one studies the reaction $e^+e^- \to 4\gamma$ when all four photons are observed, or one misses one or two photons, in which case this reaction is a background for $e^+e^- \to 3\gamma$ or 2γ. The undetected photons can escape along the beam directions, and one will want to integrate over these regions in phase space. In this section, we summarize which formulae have to be used in the different regions of phase space.

We can distinguish seven regions in phase space:

(i) $|k_{3\perp}|, |k_{4\perp}| \gg m$;

(ii) $|k_{3\perp}| \gg m$, $|k_{4\perp}| \ll E$, $k_{4z} < 0$;

(iii) $|k_{3\perp}| \gg m$, $|k_{4\perp}| \ll E$, $k_{4z} > 0$;

(iv) $|k_{3\perp}|, |k_{4\perp}| \ll E$, $k_{3z} < 0$, $k_{4z} < 0$;

(v) $|k_{3\perp}|, |k_{4\perp}| \ll E$, $k_{3z} > 0$, $k_{4z} < 0$;

(vi) $|k_{3\perp}|, |k_{4\perp}| \ll E$, $k_{3z} < 0$, $k_{4z} > 0$;

(vii) $|k_{3\perp}|, |k_{4\perp}| \ll E$, $k_{3z} > 0$, $k_{4z} > 0$.

Note that the positive z-axis was taken along the $\boldsymbol{p}_+$ direction.

The scale at which the transition occurs from $|k_{3\perp}|$ or $|k_{4\perp}| \ll E$ to $\gg m$ is rather arbitrary and can, e.g. be taken to be $|k_{3\perp}|$ or $|k_{4\perp}| = \sqrt{mE}$. In other words, $\ll E$ is to be interpreted as $< \sqrt{mE}$, and $\gg m$ as $> \sqrt{mE}$.

In region (i), none of the photons are collinear ones, and one should use the zero-mass formula (3.5). In region (ii), only photon 4 is collinear with e^-, and formula (4.3) should be used. For region (iii), one should similarly use formula (4.4). In the regions (iv)–(vii), both photons are collinear. For region (iv), one uses formula (5.9), and for region (vii) one uses again eq. (5.9), but with p_+ and p_- interchanged. This amounts to interchanging the subscripts $+$ and $-$, as well as replacing Δ_{34} by Δ'_{34} (see eq. (5.13)). In the regions (iv) and (vii), photons 3 and 4 are indistinguishable, and one should include a statistical factor $\frac{1}{2}$ in the cross section formula (3.6). Finally, for region (v), one should take formula (5.19), and for region (vi), formula (5.19) again, but with 3 and 4 interchanged.

7. Conclusions

By introducing explicit polarization vectors for the photons in a covariant way, we were able to obtain relatively simple expressions for the helicity amplitudes for $e^+e^- \to 4\gamma$ in the massless fermion limit. Using a specific representation for the spinors, it was then possible to express these amplitudes as complex functions of the components of the four-momenta. With eq. (3.6), it becomes easy and straightforward to evaluate numerically the unpolarized cross section for a given point in phase space.

When one or more photons are radiated along the beam directions, finite mass corrections have to be taken into account, however. We showed how this could be accomplished for the cases where one or two photons are in this collinear configuration. A summary of the formulae to be used in the different regions of phase space was presented in sect. 6.

We hope that the simplicity of our formulae will stimulate more precise analyses of experiments where this reaction is observed.

One of us (TTW) thanks Professor Hans Joos, Professor Erich Lohrmann, Professor Paul Söding, Professor Volker Soergel and Professor Thomas Walsh for their kind hospitality at DESY. Also discussions with Professor H. Takeda and Professor S. Yamada are gratefully acknowledged.

Appendix

The simplicity of the helicity amplitudes (eqs. (2.4)) in which three photon helicities are equal is quite remarkable. It is easy to show that this is a general feature of helicity amplitudes for $e^+e^- \to n$ photons, where $n-1$ photon helicities are the same.

Consider the case of $M(-,+;-,+,+,\dots,+,+)$. With our choice of $\varepsilon_i^{\pm}$ (eq. (2.2)) only $(n-1)!$ Feynman diagrams contribute, i.e. the ones where $\not\varepsilon_1^-$ is inserted next to $u_+(p_-)$. A typical diagram is

$$M_1 = i(-e)^n 2^{n-1}\left(\prod_{i=1}^{n} N_i\right)\bar{v}(p_+)\not k_n \not p_+ \not p_- \frac{\not k_n - \not p_+}{-2(p_+k_n)} \not k_{n-1}\not p_+ \not p_- \frac{\not k_n + \not k_{n-1} - \not p_+}{(p_+ - k_{n-1} - k_n)^2}$$

$$\times \not k_{n-2}\not p_+ \not p_- \frac{\not k_n + \not k_{n-1} + \not k_{n-2} - \not p_+}{(p_+ - k_{n-2} - k_{n-1} - k_n)^2} \dots \not k_2 \not p_+ \not p_-$$

$$\times \frac{\not p_- - \not k_1}{-2(p_-k_1)} \not p_+ \not p_- \not k_1 (1+\gamma^5) u(p_-)$$

$$= -i(-e)^n 2^{n-1}\left(\prod_{i=1}^{n} N_i\right)\bar{v}(p_+)\not p_-(\not k_n + \not k_{n-1} - \not p_+)(\not k_{n-1} - \not p_+)\not p_+ \not p_-$$

$$\times \frac{\not k_n + \not k_{n-1} - \not p_+}{(p_+ - k_{n-1} - k_n)^2} \not k_{n-2} \not p_+ \not p_- \frac{\not k_n + \not k_{n-1} + \not k_{n-2} - \not p_+}{(p_+ - k_{n-2} - k_{n-1} - k_n)^2} \dots \not k_2 \not p_+ \not p_-$$

$$\times \not k_1 \not p_+ (1+\gamma^5) u(p_-). \tag{A.1}$$

The complete helicity amplitude is obtained by adding to (A.1) the remaining $(n-1)! - 1$ permutations of $(2,3,\dots,n)$.

Adding to M_1 the contribution with n and $(n-1)$ interchanged, we obtain

$$M_2 = -i(-e)^n 2^{n-1}\left(\prod_{i=1}^{n} N_i\right)\bar{v}(p_+)\not p_- \not p_+ \not p_- (\not k_n + \not k_{n-1} - \not p_+)\not k_{n-2} \not p_+ \not p_-$$

$$\times \frac{\not k_n + \not k_{n-1} + \not k_{n-2} - \not p_+}{(p_+ - k_{n-2} - k_{n-1} - k_n)^2} \dots \not k_2 \not p_+ \not p_- \not k_1 \not p_+ (1+\gamma^5) u(p_-)$$

$$= -i(-e)^n 2^{n}\left(\prod_{i=1}^{n} N_i\right)\bar{v}(p_+)\not p_-(\not k_n + \not k_{n-1} + \not k_{n-2} - \not p_+)(\not k_{n-2} - \not p_+)$$

$$\times \not p_+ \not p_- \frac{\not k_n + \not k_{n-1} + \not k_{n-2} - \not p_+}{(p_+ - k_{n-2} - k_{n-1} - k_n)^2} \dots \not k_2 \not p_+ \not p_- \not k_1 \not p_+ (1+\gamma^5) u(p_-)(p_+p_-).$$

$$\tag{A.2}$$

Note that the symmetrization in n and $(n-1)$ had the net effect of cancelling the denominator $(p_+ - k_n - k_{n-1})^2$ and bringing out a factor $2(p_+p_-)$. A similar phenomenon occurs when we symmetrize M_2 in n, $(n-1)$ and $(n-2)$. We obtain

$$M_3 = -i(-e)^n 2^{n+1}\left(\prod_{i=1}^{n} N_i\right)(p_+p_-)^2 \bar{v}(p_+)\not{p}_-(\not{k}_n+\not{k}_{n-1}+\not{k}_{n-2}-\not{p}_+)$$

$$\times \not{k}_{n-3}\not{p}_+\not{p}_- \ldots \not{k}_2\not{p}_+\not{p}_-\not{k}_1\not{p}_+(1+\gamma^5)u(p_-). \tag{A.3}$$

This process of symmetrization can be continued. All denominators are cancelled and the last symmetrization brings out a factor

$$2(p_+p_-)(k_n+k_{n-1}+\cdots+k_2-p_+)^2 = -4(p_+p_-)(p_-k_1). \tag{A.4}$$

By now, all $(n-1)!$ Feynman diagrams have been added and the helicity amplitude reads

$$M(-,+;-,+,+,\ldots,+)$$

$$= i(-e)^n 2^{2n-2}\left(\prod_{i=1}^{n} N_i\right)(p_+p_-)^{n-2}(p_-k_1)\bar{v}(p_+)\not{p}_-\not{k}_1\not{p}_+(1+\gamma^5)u(p_-)$$

$$= -\tfrac{1}{2}i(-e)^n A_n(p_+p_-)^{n/2-1}(p_-k_1)\bar{v}(p_+)\not{k}_1(1+\gamma^5)u(p_-), \tag{A.5}$$

with

$$A_n = \left[\prod_{i=1}^{n}(p_+k_i)(p_-k_i)\right]^{-1/2}. \tag{A.6}$$

Similarly, one finds

$$M(+,-;+,-,-,\ldots,-)$$

$$= -\tfrac{1}{2}i(-e)^n A_n(p_+p_-)^{n/2-1}(p_-k_1)\bar{v}(p_+)\not{k}_1(1-\gamma^5)u(p_-),$$

$$M(-,+;+,-,-,\ldots,-)$$

$$= \tfrac{1}{2}i(-e)^n A_n(p_+p_-)^{n/2-1}(p_+k_1)\bar{v}(p_+)\not{k}_1(1+\gamma^5)u(p_-),$$

$$M(+,-;-,+,+,\ldots,+)$$

$$= \tfrac{1}{2}i(-e)^4 A_n(p_+p_-)^{n/2-1}(p_+k_1)\bar{v}(p_+)\not{k}_1(1-\gamma^5)u(p_-). \tag{A.7}$$

For $n = 3$, the formulae (A.5)–(A.7) reproduce the results of ref. [3] for $e^+e^- \to 3\gamma$, and for $n = 4$, we recover the expressions of eqs. (2.4). It should be noted that the amplitudes (A.5) and (A.7) consist of a product of $n - 2$ "infrared factors"

$$-e\left[\frac{(p_+p_-)}{(p_+k_i)(p_-k_i)}\right]^{1/2} \tag{A.8}$$

times the corresponding helicity amplitude for $e^+e^- \to 2\gamma$.

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