Nuclear Physics B246 (1984) 143–156

GLUEBALL SPECTROSCOPY FROM STRONG COUPLING EXPANSIONS IN HAMILTONIAN LATTICE QCD

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Received 15 February 1984

Masses of all the glueballs which are created by 6- or 7-link operators are calculated to order g^{-8} in pure SU(3) hamiltonian lattice gauge theory. Several low-lying states are found with masses $m(0^{++*}) \sim 1.4\ m_s$, $m(0^{++**}) \sim 1.7\ m_s$ (* and ** stand for radial excitations and m_s is the mass of the lowest 0^{++} state), $m(0^{--}) \sim 2.2\ m_s$, $m(1^{+-}) \sim m(1^{+-*}) \sim 1.6\ m_s$, $m(1^{-+}) \sim 1.8\ m_s$, $m(1^{--}) \sim 2.2\ m_s$ and $m(2^{++}) \sim 1.3\ m_s$. These values are obtained at the point $g^{-2} \simeq 0.8$, which lies near the scaling region.

1. Introduction

Since the pioneering work on glueballs in strong coupling (SC) hamiltonian lattice gauge theory was carried out by Kogut, Sinclair and Susskind [1], many authors have calculated masses of glueballs in lattice theories using various methods [2–9, 13]. Among them, only the authors of refs. [3–6] have actually made efforts to study the *whole* SU(3) spectrum. They used Monte Carlo (MC) variational methods. In these computations, however, only a few masses have shown clear asymptotic freedom behaviour, and the spectrum is not pinned down yet. There are also strong coupling expansion results (up to order g^{-16}) for the lowest three states that are made of simple plaquette operators, namely $J^{PC} = 0^{++}$, 1^{+-} and 2^{++}, in the hamiltonian [1] and the euclidean [7–9] lattice gauge theories. They are consistent with each other ($m(1^{+-}) \sim (1.6\text{–}2.0)m(0)^{++}$ and $m(2^{++}) \sim (1.0\text{–}1.2)m(0^{++})$)*, but disagree with the MC results ($m(1^{+-}) \sim (2.8\text{–}4.0)m(0^{++})$ and $m(2^{++}) \sim (1.5\text{–}2.8)m(0^{++})$) [4, 6]. In our opinion, more precise and improved calculation [10–12] should be done by MC methods [13]. In addition, other possible approaches such as strong and weak coupling (WC) expansions [14, 15] need to be exhausted for the calculation of *all*

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* Even if we start from the same series, we will obtain different mass ratios depending on the methods of analysis. We quote the values from the extrapolation methods. The tangential method, which is regularly used in the analyses of MC results, is dangerous to apply to SC expansions when the series are short.

excited masses. In this paper we try to get a rough impression of the *whole* SU(3) spectrum with the aid of ordinary hamiltonian SC perturbation theory. Although the series we have obtained are rather short, they present many interesting first non-MC results.

We will work in an SU(3) pure gauge system on a regular lattice with the standard hamiltonian [16]

$$H = g^2(2a)^{-1}\sum_{\ell,c} E^c(\ell)E^c(\ell) - (ag^2)^{-1}\sum_P \mathrm{Tr}\{U(P)+U^\dagger(P)\}$$

$$= g^2(2a)^{-1}\{W^{(0)} - yV\}, \qquad y = 2g^{-4}, \tag{1}$$

where a is the lattice spacing, g the bare gauge coupling, $U(P)$ the ordered product of link operators, $U(\ell)$, around the boundary of a spatial plaquette P and the chromoelectric field E^c $(c=1,\ldots,8)$ is normalized to satisfy the commutation relation $[E^c(\ell), U(\ell')] = \frac{1}{2}\lambda^c U(\ell)\delta_{\ell\ell'}$ (λ^c is Gell-Mann's matrix). The perturbation calculation will be done for the dimensionless energy $w = 2ag^{-2}$, and we will deal with the dimensionless hamiltonian $W = 2ag^{-2}H$. The first term $W^{(0)} = \sum_\ell E^2(\ell)$ is regarded as the unperturbed hamiltonian ($E^2(\ell)$ becomes the quadratic Casimir operator when it acts on state vectors), while the second term $-yV = -2g^{-4}\sum_P \mathrm{Tr}(U(P)+U^+(P))$ is regarded as the perturbation with the expansion parameter $y = 2g^{-4}$. First of all, we have to produce the zeroth order wave functions of glueballs. Since the hamiltonian has a cubic symmetry, more precisely the full octahedral one $\mathrm{O}_h = \mathrm{O}\times i$ (O is the octahedral or proper cubic group, i denotes inversion) [17], every energy eigenstate must be in one of the irreducible representations (rep.'s) of the symmetry group. The group O has only 5 irreducible rep.'s, namely the trivial rep. A_1, another one-dimensional rep. A_2, 2-dim. rep. E, 3-dim. vector rep. T_1 and another 3-dim. rep. T_2. There are simple relations between the irreducible rep.'s of the group O and integer spin J rep.'s, D_J, of the proper rotation group SO(3) restricted to the group O. How many times an irreducible rep. R of O appears in a rep. D_J can be determined easily with the aid of an orthogonality relation for the irreducible characters $\chi(R)$. We reproduce the results of ref. [17] for the first few cases

$$D_0 = A_1, \qquad D_1 = T_1,$$

$$D_2 = E \oplus T_2, \qquad D_3 = A_2 \oplus T_1 \oplus T_2,$$

$$D_4 = A_1 \oplus E \oplus T_1 \oplus T_2. \tag{2}$$

The extension of these formulae for the full symmetry groups O_h and O(3) is trivial. One should only assign the parity of the representations, namely change the representation symbol R into $R^{P=\pm 1}$. The cubic symmetry O_h of the hamiltonian

system is expected to turn to the approximate, rotational symmetry O(3) in the weak coupling region [18]. Accordingly the cubic irreducible multiplets in the strong coupling region will turn into the ordinary integer-spin multiplets in correspondence with the decomposition formulae (2). The confluence or splitting of the levels for higher spin cases ($J \geqq 2$) will occur in the intermediate coupling region $0.4 \lesssim y \lesssim 1.65$, which has been determined by the behaviour of the β-function $a \times \mathrm{d}g/\mathrm{d}a$ deduced from the SC expansion series of the string tension up to the order of y^6 [19]. A discussion of the actual cases will be given later.

2. Wave functions

The irreducible sets of zeroth-order state vectors are constructed as follows [3]. First, we apply the various types of gauge invariant operators at a lattice site n (for simple examples see fig. 1) to the vacuum $|\Omega\rangle$ in such combinations as to have definite charge conjugation parities ($C = \pm 1$):

$$|\alpha(n), C = \pm 1, i\rangle = \sqrt{2}^{-1}\{O_i^\alpha(n) \pm \overline{O}_i^\alpha(n)\}|\Omega\rangle, \tag{3}$$

where α denotes the type of the operator (#1, #2,... in fig. 1), index i ($= 1, \ldots, N_\alpha$) distinguishes operators within sets of operators of the same type (for #2* and #3*, see fig. 2) and $\overline{O}_i^\alpha$ is the complex conjugate of O_i^α. Notice that for the operators depicted in fig. 1 the charge conjugation is equivalent to taking the complex

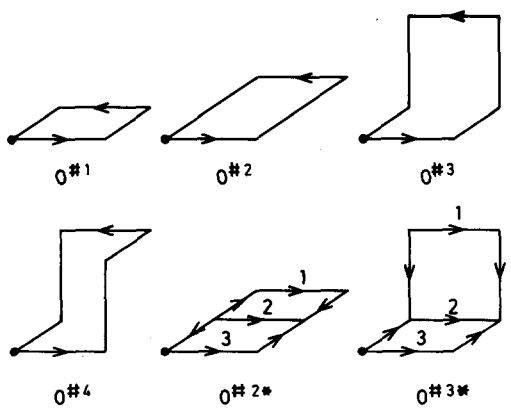

Fig. 1. $O^{\#1}$–$O^{\#4}$ are traces of the products of link operators $U(\ell)$ along the loops. $O^{\#2^*}$ and $O^{\#3^*}$ are defined as $O^\alpha = \frac{1}{2}\sqrt{3}\,\varepsilon_{ijk}\varepsilon_{\ell mn} \times U(1)_{i\ell}U(2)_{jm}U(3)_{kn}$, where $U(2)$ is an ordinary operator on link 2, $U(1)$ and $U(3)$ are the products of operators along bended lines 1 and 3, respectively. The numerical factor $\frac{1}{2}\sqrt{3}$ is necessary in order to get unit vectors ($\|O^\alpha|\Omega\rangle\|^2 = 1$).

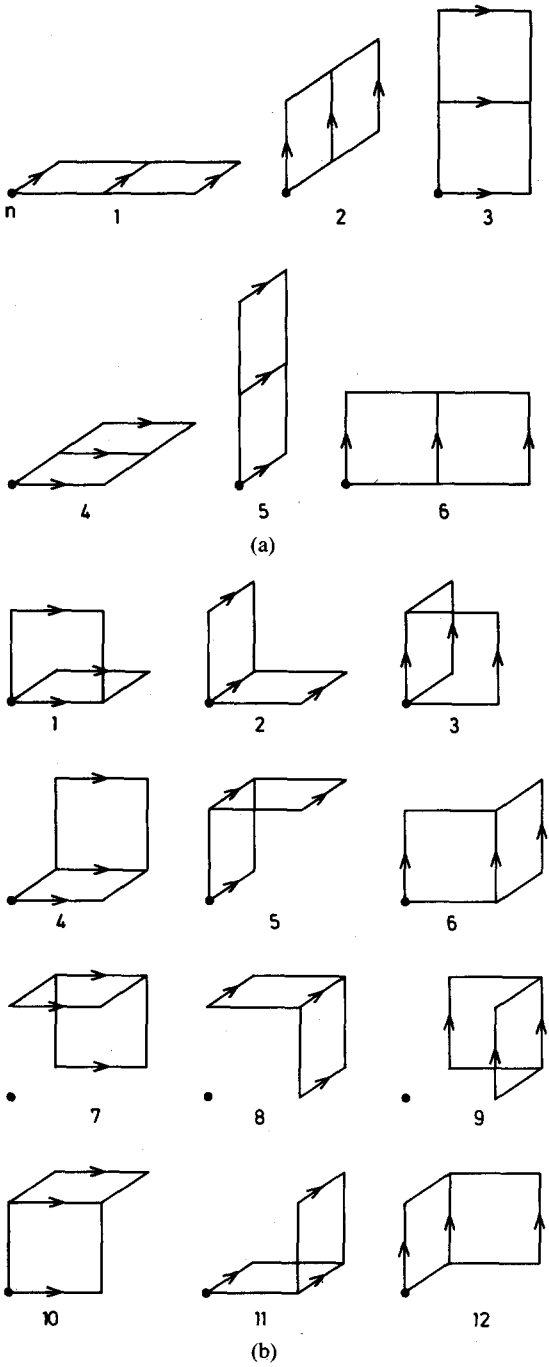

Fig. 2. (a) Operators $O_i^{\#2^*}(n)$ $(i=1,\ldots,6)$. Zero-momentum states $|\#2^*, C=\pm 1, i\rangle$ transform into $\pm|2\#^*, C=\pm 1, i\rangle$ under the space inversion. Therefore $P=C$ for all $|\#2^*\rangle$ states. (b) Operators $O_i^{\#3^*}(n)$ $(i=1,\ldots,12)$. Zero-momentum states transforms as $|\#3^*, C=\pm 1, i\rangle \leftrightarrow \pm|\#3^*, C=\pm 1, i+6\rangle$ $(i=1,\ldots,6)$ under the space inversion.

conjugate, because $U_{n\mu}$ transforms into $U_{n\mu}{}^{+}$ under the charge conjugation ($U_{n\mu} \sim \exp[ig\int_n^{n+\hat{\mu}} \mathrm{d}x A_\mu(x)]$, and $A_\mu(x) \to -A_\mu(x)$). Next, we construct the zero-momentum states, averaging over the whole space (we are interested only in the masses):

$$|\alpha, C = \pm 1, i\rangle = \sqrt{N_s}^{-1} \sum_n |\alpha(n), C = \pm 1, i\rangle , \tag{4}$$

where N_s is the number of lattice sites. Each zero-momentum vector space (dimension $= N_\alpha$) distinguished by α and C is a representation space of the cubic group (not irreducible in general), because each vector $|\alpha, C, i\rangle$ in such a space transforms into another one $|\alpha, C, j\rangle$ in the same space under a cubic rotation. The $N_\alpha \times N_\alpha$ matrix representation $\Gamma^{\alpha C}(G)$ of the group element G can be easily obtained by simple geometrical consideration (the transformation property of the vectors under the space inversion $G = i$ can be deduced from that of the link operators, $U_{n,\mu} \to U^{+}_{-n-\hat{\mu},\mu}$. See fig. 2). We have to decompose each space further into irreducible representation spaces using the orthogonality relation for the irreducible characters of O_h [17]. For the spaces of 4-, 6- and 8-link states, the decomposition formulae have been obtained in ref. [3]. Here we obtain the results for the spaces of 7-link states:

$$\begin{aligned}
\Gamma^{\#2^*+} &= A_1^{++} \oplus A_2^{++} \oplus 2E^{++} , \\
\Gamma^{\#2^*-} &= T_1^{--} \oplus T_2^{--} , \\
\Gamma^{\#3^*+} &= A_1^{++} \oplus E^{++} \oplus T_1^{-+} \oplus T_2^{++} \oplus T_2^{-+} , \\
\Gamma^{\#3^*-} &= A_2^{--} \oplus E^{--} \oplus T_1^{+-} \oplus T_1^{--} \oplus T_2^{+-} .
\end{aligned} \tag{5}$$

The symbols on the right-hand side stand for R^{PC}. Finally, we have to construct orthonormal state vectors explicitly for each irreducible representation. This has also been done in ref. [3] by the use of Schur's lemma for 4-, 6- and 8-link states. We have obtained 7-link irreducible states as linear combinations:

$$|R^{PC}, r, \alpha\rangle = \sum_i b_i(R^{PC}, r, \alpha)|\alpha, C, i\rangle , \qquad r = 1, \ldots, N_R , \tag{6}$$

where only the index i is summed over, and r distinguishes the vectors in the representation space. The coefficients $b_i(R^{PC}, r, \alpha)$'s are given in table 1.

3. Mass calculation

We have calculated the masses of all these 6- and 7-link states up to second order in y. The results are given in table 2 together with the results for 4-link states that

TABLE 1a
Irreducible wave functions $|R^{PC}, r, \#2^*\rangle$ as linear combinations of $|\#2^*, C, i\rangle$'s

R^{PC} \ i	1	2	3	4	5	6
A_1^{++}	1	1	1	1	1	1
A_2^{++}	1	1	1	−1	−1	−1
E^{++}	0	1	−1	−1	0	1
	2	−1	−1	−1	2	−1
E^{++}	0	1	−1	1	0	−1
	2	−1	−1	1	−2	1
	1	0	0	0	1	0
T_1^{--}	0	1	0	0	0	1
	0	0	1	1	0	0
	1	0	0	0	−1	0
T_2^{--}	0	1	0	0	0	−1
	0	0	1	−1	0	0

According to $C = \pm 1$ the vectors $|\#2^*, C = \pm 1, i\rangle$ have to be chosen in eq. (6). Suitable overall normalization factors should be understood (e.g. $b_i(A_1^{++}, \#2^*) = \sqrt{6}^{-1}$ for all i, $b_{1 \text{ or } 5}(T_1^{--}, 1, \#2^*) = \sqrt{2}^{-1}$).

TABLE 1b
Irreducible wave functions $|R^{PC}, r, \#3^*\rangle$

R^{PC} \ i	1	2	3	$\overline{4}$	$\overline{5}$	$\overline{6}$	7	8	9	$\overline{10}$	$\overline{11}$	$\overline{12}$
A_1^{++}, A_2^{--}	1	1	1	1	1	1	1	1	1	1	1	1
E^{++}, E^{--}	−1	1	0	−1	1	0	−1	1	0	−1	1	0
	−1	−1	2	−1	−1	2	−1	−1	2	−1	−1	2
	−1	−1	0	−1	1	0	1	1	0	1	−1	0
T_1^{-+}, T_2^{+-}	0	−1	−1	0	−1	1	0	1	1	0	1	−1
	−1	0	−1	1	0	−1	1	0	1	−1	0	1
	0	0	1	0	0	−1	0	0	1	0	0	−1
T_2^{++}, T_1^{--}	1	0	0	−1	0	0	1	0	0	−1	0	0
	0	1	0	0	−1	0	0	1	0	0	−1	0
	1	−1	0	1	1	0	−1	1	0	−1	−1	0
T_2^{-+}, T_1^{+-}	0	1	−1	0	1	1	0	−1	1	0	−1	−1
	−1	0	1	1	0	1	1	0	−1	−1	0	−1

For $i = \overline{4}, \overline{5}, \overline{6}, \overline{10}, \overline{11}$ and $\overline{12}$ the vectors $(-1)\,|\#3^*, C = -1, i\rangle$ have to be chosen corresponding to $C = -1$ (e.g. $b_1(A_2^{--}, \#3^*) = -b_4\,(A_2^{--}, \#3^*) = \sqrt{12}^{-1}$). For the rest, the rule is the same as in table 1a.

TABLE 2
Series for dimensionless masses $w = 2ag^{-2}m$ of the states $|R^{PC}, r, \alpha\rangle$ up to second order in $y = 2g^{-4}$

α	R^{PC}	const	y	y^2
#1	A_1^{++}	$\frac{16}{3}$	−1	−0.1064
	E^{++}		−1	0.1093
	T_1^{+-}		1	0.0180
#2	A_1^{++}	8	0	−0.9527
	A_2^{++}			−0.9527
	E^{++}			−0.9527
	$E^{++\prime}$			−0.9772
	T_1^{+-}			0.0473
	$T_2^{+-\prime}$			0.0288
#3	A_1^{--}	8	0	0.4146
	$E^{++\prime}$			−0.8110
	E^{--}			−0.0855
	T_1^{-+}			−0.7105
	T_2^{--}			0.1646
	T_2^{-+}			−1.0438
	$T_2^{+-\prime}$			0.1413
#(3 + 4)	A_1^{++}	8	−1.1547	−0.9530
	T_1^{+-}		−0.9428	−0.3944
	T_2^{++}		−0.6667	−0.8117
#(3 − 4)	A_1^{++}	8	1.1547	−0.0870
	T_1^{+-}		0.9428	0.3127
	T_2^{++}		0.6667	−0.3117
#4	A_2^{+-}	8	0	−0.2046
#2*	$A_1^{++\prime}$	$\frac{28}{3}$	0	0.8279
	A_2^{++}			0.6806
	E^{++}			0.6806
	$E^{++\prime}$			0.8138
	$T_1^{--\prime}$			0.2708
	T_2^{--}			0.0806
#3*	$A_1^{++\prime}$	$\frac{28}{3}$	0	−0.0399
	A_2^{--}			0.3785
	$E^{++\prime}$			0.2820
	E^{--}			0.0708
	T_1^{+-}			−0.1596
	T_1^{-+}			0.4404
	$T_1^{--\prime}$			−0.4776
	T_2^{++}			0.7734
	T_2^{+-}			0.0455
	T_2^{-+}			0.6455

have been obtained in ref. [1], where a detailed explanation of the calculational method is presented. We will mention a few important points which are peculiar to the calculation for 6- and 7-link states. At zeroth order, all the 4-link states are degenerate with the dimensionless energy $w^{(0)}(4) = 2ag^{-2}E = \frac{16}{3}$, likewise the 6-link states with $w^{(0)}(6) = 8$ and 7-link states with $w^{(0)}(7) = \frac{28}{3}$. To first order, all the diagonal matrix elements of V among 6- or 7-link states vanish, while the off-diagonal elements between the 6-link states $|R^{PC}, r, \#3\rangle$ and $|R^{PC}, r', \#4\rangle$, both of which belong to the same representations ($R^{PC} = A_1^{++}$, T_1^{+-} or T_2^{++}), have non-zero values through the processes depicted in fig. 3. Mass splitting occurs by the diagonalization procedure, as is well known from degenerate perturbation theory, and the corresponding zeroth-order eigenstates are determined as

$$|R^{PC}, r, \#(3+4)\rangle = \sqrt{2}^{-1}\{\,|R^{PC}, r, \#3\rangle \pm |R^{PC}, r, \#4\rangle\}\,, \tag{7}$$

with suitable relabeling of r's within the space of $|R^{PC}, r, \#3\rangle$'s. We chose a phase convention such that the states $|R^{PC}, r, \#(3+4)\rangle$ would correspond to the lower mass states. Next, to second order, we calculate the matrix elements of the operator

$$W^{(2)} = V\frac{1-P_k}{w^{(0)}(k) - W^{(0)}}V, \qquad P_k = \sum_{\text{all}} |k\text{-link}\rangle\langle k\text{-link}|\,, \tag{8}$$

when we are interested in the masses of k-link states. Off-diagonal elements between $|R^{PC}, r, \#2\rangle$ and $|R^{PC}, r', \#3\rangle$ ($R^{PC} = E^{++}$ or T_2^{+-}) and those between $|R^{PC}, r, \#2^*\rangle$ and $|R^{PC}, r', \#3^*\rangle$ ($R^{PC} = A_1^{++}$, E^{++} or T_1^{--}) are nonzero in general.

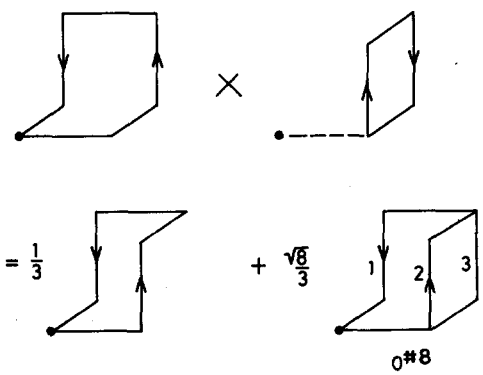

Fig. 3. An operator $O^{\#3}$ grows into $\frac{1}{3}\,O^{\#4} + \frac{1}{3}\sqrt{8}\,O^{\#8}$ by one application of the perturbation operator V. The operator $O^{\#8}$ is defined as $O^{\#8} = \frac{3}{8}\sqrt{2}\,U(1)_{ij}U(2)_{k\ell}U^8(3)_{\alpha\beta}\lambda^{\alpha}_{kj}\lambda^{\beta}_{i\ell}$, where $U(1)$ and $U(2)$ are the products of ordinary operators along bended lines 1 and 2 respectively. $U^8(3)$ is the product of octet representation operators along 3 and λ^{α} ($\alpha = 1, \ldots, 8$) is Gell-Mann's matrix. A numerical factor $\frac{3}{8}\sqrt{2}$ is necessary in order to get a unit vector ($\|O^{\#8}|\Omega\rangle\|^2 = 1$).

For these states, by the diagonalization of $W^{(2)}$, the mass splitting occurs and the zeroth-order wave functions are determined as the linear combinations

$$|R^{PC'}, r, \alpha\rangle = \cos\theta_{\alpha\beta}|R^{PC}, r, \alpha\rangle + \sin\theta_{\alpha\beta}|R^{PC}, r, \beta\rangle,$$

$$|R^{PC'}, r, \beta\rangle = -\sin\theta_{\alpha\beta}|R^{PC}, r, \alpha\rangle + \cos\theta_{\alpha\beta}|R^{PC}, r, \beta\rangle, \tag{9}$$

where α and β stand for #2 and #3, or #2* and #3* respectively. The mixing angles $\theta_{\alpha\beta}$ depend on both the values of diagonal and off-diagonal matrix elements, and for most cases $|\theta_{\alpha\beta}| = (20\text{–}30)°$. Attention should be paid to E^{++} cases. There are two equivalent E^{++} representations in the spaces $|\#2\rangle$ and also in $|\#2^*\rangle$. By a suitable linear transformation within each space, one E^{++} representation (we denote it as $|E^{++}, r, \#2\rangle$ or $|E^{++}, r, \#2^*\rangle$ respectively) decouples from the others, namely the off-diagonal elements $\langle E^{++}, \#2|W^{(2)}|E^{++}, \#3\rangle$ or $\langle E^{++}, \#2^*|W^{(2)}|E^{++}, \#3^*\rangle$ vanish, and then becomes a zeroth-order eigenstate. For the remaining E^{++} states, the regular diagonalization procedure was used and the notation (9) was used for the resulting zeroth-order order states. The calculation for all other states in table 2 is straightforward, because there is no possibility of degeneracy lifting (no off-diagonal element).

4. Mass ratios

From these series, we produced the series for mass ratios of all states to the lowest scalar ($|A_1^{++}, \#1\rangle$) mass m_s, and compared the values at $y = 1.3$ ($g^{-2} \simeq 0.8$) in order to get a rough spectrum pattern (see fig. 5). The Padé extrapolation method is inapplicable for these low-order series, because each [1,1] Padé approximant has a pole in the $y > 0$ region in general, except for a few cases. The point $y = 1.3$ was chosen as a typical value, because it lies in the upper half of the rather wide crossover region, $0.4 \lesssim y \lesssim 1.65$ [19]. The whole spectrum does not change drastically when the y-value is changed from 1.0 to 1.6. In most cases the change in the ratio m/m_s of a higher mass is larger than that of a lower mass (several typical ratios as functions of y are drawn in fig. 4). Moreover the fourth-order series of the mass ratios for 4-link states [1] give, at this point, the consistent values $(m(T_1^{+-})/m_s \sim 1.4$ and $m(E^{++})/m_s \sim 0.9)$ with those for the second-order ones $(m(T_1^{+-})/m_s \sim 1.6$ and $m(E^{++})/m_s \sim 1.1$. In addition, [2,2] Padé approximants of these fourth-order series also give consistent values in the continuum limit $(m(T_1^{+-})/m_s \sim 1.6$ and $m(E^{++})/m_s \sim 1.0$ as $y \to \infty)$. Therefore we hope that our second-order results at $y = 1.3$ will give rather good estimates of the masses for other excited states. At $y = 1.3$, the series for the ratio of m_s to the square root of the string tension σ [19] up to second order and fourth order give $m_s/\sqrt{\sigma} \sim 2.8$ and 2.6 respectively. This means that the mass of the lowest scalar is fixed around $m_s \sim 1.1$ GeV, provided that $\sqrt{\sigma}$ is about 400 MeV. This value should, however, be understood to have a large systematic error. Let us now discuss the results for each case in turn.

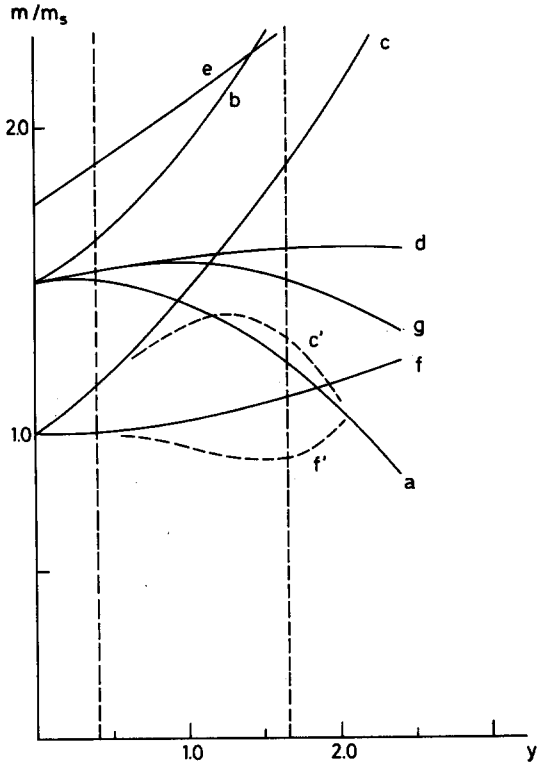

Fig. 4. Ratios of masses to the lowest scalar mass m_s (second order in $y = 2g^{-4}$). Curves a, b, c, d, e, f and g are for states $|A_1^{++}, \#(3+4)\rangle$, $|A_1^{--}, \#3\rangle$, $|T_1^{+-}, r, \#1\rangle$, $|T_1^{+-}, r, \#(3+4)\rangle$, $|T_1^{--\prime}, r, \#3^*\rangle$, $|E^{++}, r, \#1\rangle$ and $|T_2^{++}, r, \#(3+4)\rangle$ respectively. Dotted curves c' and f' are fourth-order results for c and f from ref. [1]. The crossover region is $0.4 \lesssim y \lesssim 1.65$ [19].

(i) A_1^{++}. There are two stable ($m < 2m_s$) excited states, $|A_1^{++}, \#(3+4)\rangle$ and $|A_1^{++}, \#2\rangle$. According to the decomposition formulae (2), the A_1 state can appear only in spin-0, -4 and higher spin states. From the naive expectation that the masses of higher spin states are rather heavy, we can suppose that both of these low-lying states belong to the radial excitations of the lowest scalar. Using the notations * or ** for the radial excitations, we get

$$m(0^{++*}) \sim 1.4 m_s, \qquad m(0^{++**}) \sim 1.7 m_s. \tag{10a}$$

Series for these two mass ratios have [1,1] Padé approximants which are regular in the positive half-space $y > 0$. They give in the continuum limit $m(0^{++*}) \sim 1.5\ m_s$ and $m(0^{++**}) \sim 2.3\ m_s$. These results might be used for the estimation of systematic errors.

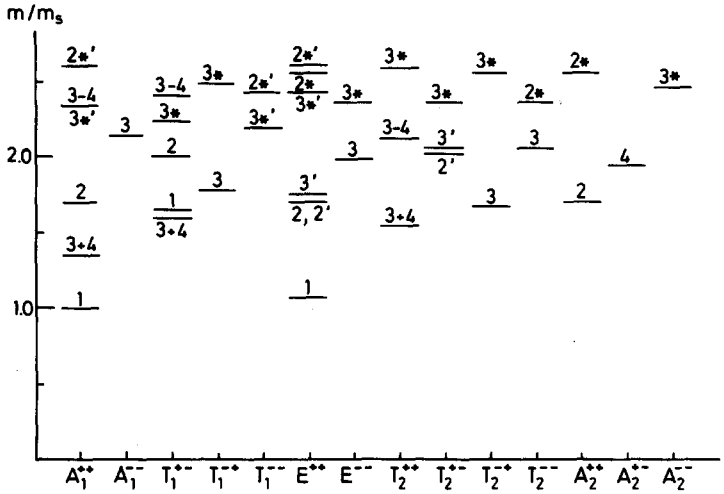

Fig. 5. Mass ratios at $y = 1.3$ ($g^{-2} \simeq 0.8$) from the second-order series.

(ii) A_1^{--}. The lowest state $|A_1^{--}, \#3\rangle$ can be regarded as 0^{--}. This is an "odd ball", i.e. a state which cannot appear in the usual quark model states. The prediction is

$$m(0^{--}) \sim 2.2 m_s . \tag{10b}$$

MC results give $m(0^{--}) \sim (2.9\text{–}4.0) m_s$ [4,6], omitting error bars.

(iii) T_1^{+-}. The two ($\times 3$) lowest states, $|T_1^{+-}, r, \#1\rangle$, which have been considered in ref. [1], and $|T_1^{+-}, r, \#(3+4)\rangle$, are practically degenerate. From formulae (2), we can regard these states as axial vector. The prediction is

$$m(1^{+-}) \sim m(1^{+-}{}^*) \sim 1.6 m_s . \tag{10c}$$

A quasi-level crossing or repelling will occur near the point $y = 1.3$ (see (c) and (d) in fig. 4), but the change of the masses will be small compared to the rather large errors which we are taking into account implicitly. [1,1] Padé for $m_{\#(3+4)}(T_1^{+-})/m_s$ goes to 1.9 in the continuum limit and [2,2] Padé for $m_{\#1}(T_1^{+-})/m_s$ goes to 1.6 [1]. MC results give $m(1^{+-}) \sim (2.8\text{–}4.0) m_s$ [4,6].

(iv) T_1^{-+}. The lowest state $|T_1^{-+}, r, \#3\rangle$ may be 1^{-+}, an odd ball. The prediction is

$$m(1^{-+}) \sim 1.8 m_s . \tag{10d}$$

This is the lowest odd ball we have obtained. This result is consistent with ref. [6]. MC results are $m(1^{-+}) \sim 2.3 m_s$ [6] and $m(1^{-+}) \sim (3.1\text{–}3.9) m_s$ [4].

(v) T_1^{--}. The lowest $|T_1^{--\prime}, r, \#3^*\rangle$ may be 1^{--}, the screened gluon

$$m(1^{--}) \sim 2.2 m_s. \tag{10e}$$

MC results are $m(1^{--}) \sim (3.1\text{–}4.4) m_s$ [6].

(vi) E^{++} and T_2^{++}. We naively expect that the lowest states in both representations, $|E^{++}, r, \#1\rangle$ and $|T_2^{++}, r', \#(3+4)\rangle$, will form a degenerate 5-dimensional tensor multiplet in the weak coupling region. Although the series are too short to manifest this degeneracy, we can see the tendency of these two masses to meet at some point ($y \sim 2.5$) beyond the crossover region (see (f) and (g) in fig. 4), where the predictive power of the series may be weak. We point out here that it is another way of checking the restoration of O(3) invariance to see whether these states become degenerate in the weak coupling region (the splitting in the strong coupling region is such that the ratio $m(T_2^{++})/m(E^{++}) \sim 1.5$). The situation will be the same in the euclidean theory, and the MC simulation may be convenient for this purpose. The actual mass of the 2^{++} state will be the mean value of $m(E^{++})$ and $m(T_2^{++})$. So

$$m(2^{++}) \sim 1.3 m_s. \tag{10f}$$

[1,1] Padé for $m_{\#(3+4)}(T_2^{++})/m_s$ goes to 1.8 as $y \to \infty$ and [2,2] Padé for $m_{\#1}(E^{++})/m_s$ goes to 1.0 [1]. The MC results are $m(2^{++}) \sim (1.5\text{–}2.8) m_s$ [4,6].

(vii) E^{--}, T_1^{--}, T_2^{--} and A_2^{--}. All the lowest states are practically degenerate. It is very difficult to assign their spins correctly, because there are many possibilities ($2^{--} = E^{--} + T_2^{--}$, $3^{--} = A_2^{--} + T_1^{--} + T_2^{--}$ and $4^{--} = A_1^{--} + E^{--} + T_1^{--} + T_2^{--}$). We can only say that 2^{--}, 3^{--} or 4^{--} will be found with mass $m \sim (2.0\text{–}2.5) m_s$.

We will not say anything about other states because of the above mentioned difficulty.

5. Conclusions and discussion

We obtained several crude predictions (10a)–(10b) for low-lying glueball masses from the strong coupling expansions up to second order. In most cases, they are rather low compared with the MC results [4,6]. In particular for 0^{++} and 1^{+-} cases, there appeared radial excitations with masses lying near the lowest ones. If this is the case, the analyses of MC variational calculations should be done more carefully for these cases. The validity of the simple exponential decay assumption for the correlation functions $\langle R^{PC}|e^{-atH}|R^{PC}\rangle \sim e^{-atm(R^{PC})}$ crucially depends on the precision of the trial wave functions. In the MCV calculations only such states which are made of simple loop operators have been taken into account as basis vectors. On the other hand, in the hamiltonian SC expansions, all possible states, which appear as the intermediate states in the calculation, are taken into account systematically. Typical non-simple-loop states which have appeared in our calculation are states

that are created by operators *like* $O^{\#2^*}$ and $O^{\#3^*}$ but whose link variables are suitably replaced by others belonging to *sextet* or *octet* rep.'s (see also $O^{\#8}$ in fig. 3). The exact eigenstates have the components of all these states, and the contribution (absolute value) of each component to the masses is comparable to that of a simple-loop state, at least in a second-order calculation.

We also proposed a method of checking the restoration of rotational invariance. That is to examine the degeneracy of the lowest E^{++} and T_2^{++} states in the weak coupling region by some calculational methods. For the SU(2) case, the weak coupling expansion [15] has consistently shown this degeneracy. And it also suggests the existence of stable low-lying states ($m(0^{++*}) \sim 1.7m_s$, $m(2^{++}) \sim 0.8m_s$ and $m(2^{++*}) \sim 1.6m_s$). For the SU(3) case, there is no WC expansion result at present because of the practical complexity. The higher-order SC expansions for the low-lying states in SU(3) case will be promising.

We have not calculated the masses of 8-link states. Since there are many states which belong to the same rep.'s (see table in ref. [3]), the diagonalization of the hamiltonian is a very hard task. However, there are some interesting states among them, for example 0^{-+}, 0^{+-} and 1^{++}, which do not exist in 4-, 6- and 7-link states. Therefore, the second-order calculation for 8-link states will also be interesting.

In the euclidean lattice gauge theory (LGT), it is difficult to extract an excited mass from SC expansion series for a correlation function between general (e.g. 6-link) operators. Such a function generally has contributions from the lowest mass states ($ma \sim 4 \ln g^2$) in addition to the expected one from an excited mass state ($ma \sim 6 \ln g^2$) (There is an attempt to estimate the masses of excited states in the SC limit in 3-dim. euclidean LGTs [20]). In order to get a simple exponentially decaying function, we need a considerably precise definition of the operator corresponding to the excited mass eigenstate. This is the reason why we worked in the hamiltonian LGT instead of working in the euclidean LGT.

The author would like to thank Prof. G. Mack, G. Münster, B. Berg and G. Schierholz for useful discussions, especially G. Mack and G. Münster for reading the manuscript carefully. He also acknowledges support from the Alexander von Humboldt Foundation.

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