Nuclear Physics B240 [FS12] (1984) 543-558

STRONG COUPLING EXPANSION FOR THE MASS GAP IN SU(2) LATTICE GAUGE THEORY WITH MIXED ACTION

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Received 14 March 1984
(Revised 7 May 1984)

We perform a strong coupling expansion up to $O(\beta^7)$ for the mass-gap in SU(2) lattice gauge theory with mixed action A novel feature of the strong coupling expansion is discussed The strong coupling series appears to approach the scaling region more smoothly and Padé approximants become more stable than in the case with simple Wilson action The region of validity of a recently proposed resummation of perturbation theory as applied to the determination of the asymptotic scaling behavior is investigated Results of a strong coupling calculation for the heat kernel action, which is related to the mixed action for a special choice of parameters, are also reported

1. Introduction

In recent years a lot of effort has been spent on the calculation of the mass spectrum of pure Yang–Mills theory in the lattice formulation The lattice as a non-perturbative cut-off admits, in particular, the analytic calculation of the mass spectrum in the strong coupling regime [1–3]

The difficulty with lattice gauge theories is to obtain the continuum limit This is especially true if one is concerned with strong coupling computations, because in this case one is far from the physical region which is the weak coupling region in an asymptotically free theory So far, however, the available strong coupling series for the Wilson action are much too short to apply series extrapolation techniques reliably

There are in principal two ways to proceed

(i) The brute-force method would be to make the available strong coupling series for the simple Wilson action much longer Then there is some hope that in this way one may get information about continuum physics Unfortunately, this is expected to be impossible by means of a "hand calculation", due to the enormous amount of time estimated And computer algorithms for a "machine calculation" are very difficult to design

(ii) A more skilful way is to improve the simple Wilson action such that, starting from the strong coupling region, the weak coupling region is attained *faster* Following this line one can check lattice action universality as a by-product

Adoptıng thıs second poınt of vıew, ın thıs paper we wıll modıfy the usual Wılson actıon mınımally by the ınclusıon of the trace over the boundary of a sıngle plaquette ın the adjoınt representatıon of the gauge group SU(2)

The organızatıon of thıs artıcle ıs as follows Sect 2 introduces the model, sect 3 ıs devoted to the strong couplıng expansıon In sect 4, the contınuum lımıt ıs dıscussed ın detaıl Sect 5 provides the summary and the conclusıons The two appendices provıde the detaıled dıscussıon of a sımple tube model ıntroduced ın sect 3, and some results on the strong couplıng expansıon for the generalızed heat kernel action whıch ıs related to our model

2. The model

We consıder euclıdean pure Yang–Mılls lattıce gauge theory on a hypercubıcal lattıce wıth gauge group SU(2) ın $d=4$ dımensıons The actıon is gıven by

$$S=\sum_{p} S_{p}=\sum_{p}\left\{\frac{\beta_{f}}{d_{f}}\operatorname{Re}\chi_{f}(U_{p})+\frac{\beta_{a}}{d_{f}^{2}}|\chi_{f}(U_{p})|^{2}\right\}, \tag{2.1}$$

where the sum runs over all unorıented plaquettes U_p ıs as usual the ordered product of the group-valued gauge fields attached to the lınks ın the boundary of p χ_f ıs the character of the fundamental representatıon of SU(2) and β_f, β_a are couplıng parameters $d_f=\chi_f(1)$ The first term ın (2 1) ıs the well-known Wılson actıon whıle the second term represents the admıxture of the adjoınt representatıon, where we have made use of the Clebsch–Gordan decomposıtıon

$$\tfrac{1}{2}\otimes\tfrac{1}{2}=1\oplus 0 \tag{2 2}$$

If we perform the naıve contınuum lımıt of (2.1), we agree wıth the contınuum theory ıf we ıdentıfy

$$\frac{\beta_{f}}{2d_{f}}+\frac{\beta_{a}}{d_{f}}=\frac{1}{g_{0}^{2}}, \tag{2.3}$$

where g_0 ıs the bare couplıng constant For later convenıence we parametrıze the couplıngs as

$$\beta_{a}=\kappa\beta_{f}. \tag{2 4}$$

3. Strong coupling expansion

The mass-gap m (correspondıng to the lowest-lyıng glueball ın the $J^P=0^+$ sector) can be determined vıa the asymptotıc decay of the connected correlatıon functıon [2] for two local operators O_1 and O_2 we get after rotatıon ınto the (ımagınary) tıme directıon x_4

$$\Gamma(x_{4})=\langle O_{1}(x_{4})O_{2}(0)\rangle-\langle O_{1}(x_{4})\rangle\langle O_{2}(0)\rangle\xrightarrow[x_{4}\to\infty]{}\sim e^{-mx_{4}}, \tag{3 1}$$

hence

$$m = -\lim_{x_4\to\infty} \frac{1}{x_4} \ln \Gamma(x_4) \tag{3 2}$$

In order to make an optimal projection on the glueball eigenstate, we choose $O_{1,2}$ as linear combinations of space-like loops transforming trivially under the symmetry group of the spatial sublattice We choose these loops to be plaquettes for simplicity In addition we sum over all positions in the spatial sublattice in order to project on $\underline{p}=0$ Hence $O_{1,2}$ can be written as

$$O_{1,2}(x_4) = \frac{1}{\sqrt{3N_s}} \sum_{\underline{x}} \sum_{\substack{\mu,\nu=1,2\,3\\ \mu<\nu}} \chi_f(U_{\underline{x},x_4\ \hat{\mu},\hat{\nu}}) , \tag{3 3}$$

where N_s is the number of sites of the spatial sublattice

Making use of the cluster expansion of the generating functional of Γ, we obtain a cluster expansion of the glueball mass m [2]

$$\begin{aligned} m &= -\lim_{x_4\to\infty} \frac{1}{x_4} \ln \left\{ \frac{1}{3N_s} \sum_{\underline{x},\mu<\nu} \sum_{\underline{y},\mu'<\nu'} d_f^2 \partial_{\beta_{f_1}} \partial_{\beta_{f_2}} \ln Z(\beta_f, \beta_{f_1}, \beta_{f_2}, \beta_a) \Big|_{\beta_{f_1}=\beta_f=\beta_{f_2}} \right\} \\ &= -\lim_{x_4\to\infty} \frac{1}{x_4} \ln \left\{ \frac{1}{3N_s} \sum_{\underline{x},\mu<\nu} \sum_{\underline{y},\mu'<\nu'} d_f^2 \partial_{\beta_{f_1}} \partial_{\beta_{f_2}} \sum_C a(C)\phi(C) \Big|_{\beta_{f_1}=\beta_f=\beta_{f_2}} \right\} , \end{aligned} \tag{3 4}$$

where we have chosen $\beta_f = \beta_{f_1}$ ($\beta_f = \beta_{f_2}$) for $U_{p_1} = U_{\underline{x},x_4\ \hat{\mu},\hat{\nu}} \in O_1$ ($U_{p_2} = U_{\underline{y},0\ \hat{\mu}',\hat{\nu}'} \in O_2$) in the lattice action (2 1) $\phi(C)$ is the activity of the (multi-) polymer cluster $C=(X_1^{n_1}, X_2^{n_2},\ \ )$ and the combinatorial coefficients $a(C)$ take into account the multiplicities $n_1, n_2,\ \ $, of the polymers $X_1, X_2,\ \ .$, and how they are connected [4]. The properties of $a(C)$ ensure that the cluster C is link-wise connected In terms of the character expansion coefficients $c_{r_p}(\beta_f, \beta_a)$ defined by

$$c_{r_p}(\beta_f, \beta_a) = \int_G \mathrm{d}U_p\, \chi_{r_p}(U_p^{-1}) e^{S_p} \tag{3 5}$$

$\phi(C)$ reads

$$\begin{aligned} \phi(C) &= \prod_i \phi(X_i) \\ &= \prod_i \int_G \prod_b \mathrm{d}U_b \prod_{p\in X_i} \sum_{r_p} \frac{c_{r_p}(\beta_f, \beta_a)}{c_0(\beta_f, \beta_a)} \chi_{r_p}(U_p) \end{aligned} \tag{3.6}$$

The polymers X_i are closed surfaces composed of the plaquettes of the lattice To each plaquette p of X_i we attach an irreducible representation r_p of the gauge group G Finally we have to sum over all non-equivalent irreducible representations of G

For later convenience we define in addition the expansion coefficients u, v, w and x·

$$u(\beta_f, \kappa) = d_{1/2}^{-1} \frac{c_{1/2}(\beta_f, \beta_a = \kappa\beta_f)}{c_0(\beta_f, \beta_a = \kappa\beta_f)},$$

$$v(\beta_f, \kappa) = d_1^{-1} \frac{c_1(\beta_f, \beta_a = \kappa\beta_f)}{c_0(\beta_f, \beta_a = \kappa\beta_f)},$$

$$w(\beta_f, \kappa) = d_{3/2}^{-1} \frac{c_{3/2}(\beta_f, \beta_a = \kappa\beta_f)}{c_0(\beta_f, \beta_a = \kappa\beta_f)},$$

$$x(\beta_f, \kappa) = d_2^{-1} \frac{c_2(\beta_f, \beta_a = \kappa\beta_f)}{c_0(\beta_f, \beta_a = \kappa\beta_f)} \tag{3 7}$$

In general, starting from a polymer X with fixed geometry which may be as well considered as a graph on the underlying lattice, we can get several polymer activities $\phi(X)$ depending on how the irreducible representations are distributed on X Clearly, the number of different $\phi(X)$ and therefore the total number of nontrivial coefficients in the strong coupling expansion of m depends on the order of the computation *and* on the leading order of the character expansion coefficients $c_r(\beta_f, \beta_a)$ as a function of r For our model we have

$$c_r(\beta_f, \beta_a) \sim O(\beta^{[r+1/2]}), \quad r = 0, \tfrac{1}{2}, 1, \quad , \tag{3.8}$$

compared to the leading order behavior of the character expansion coefficients of pure Wilson action

$$c_r(\beta_f) \sim O(\beta^{2r}), \quad r = 0, \tfrac{1}{2}, 1, \ . \ . \tag{3 9}$$

Hence, qualitatively up to fixed order, we expect *more nontrivial* coefficients in our model than in the model with simple Wilson action. However, the strong coupling expansion in our model cannot be obtained by a naive application of the method used in [2]

To illustrate this, consider the leading order polymer X_0 in the strong coupling expansion of m, the long straight tube connecting p_1, p_2 with $U_{p_1} \in O_1$, $U_{p_2} \in O_2$, which has the activity

$$\phi(X_0) = d_f^2 u(\beta_{f_1}, \kappa) u(\beta_{f_2}, \kappa) u(\beta_f, \kappa)^{4t}, \tag{3 10}$$

(fig. 1). Then the leading order term $m^{(0)}$ of the mass $m = m^{(0)} + \Delta m$ is

$$m^{(0)} = -4 \ln u \tag{3.11}$$

Provided O_1, O_2 project on a state with definite mass m, the cluster expansion exponentiates in the form

$$e^{-mt} = e^{-m^{(0)}t} e^{-\Delta mt} = e^{-m^{(0)}t} \left\{ 1 - \Delta mt + \frac{1}{2!} \Delta m^2 t^2 \quad \right\}, \tag{3 12}$$

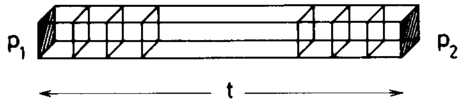

Fıg 1 The leadıng order polymer X_0 ın the strong couplıng expansıon of the mass-gap m stretchıng ın the (ımagınary) tıme dırectıon x_4 $p_{1\,2} \in O_{1\,2}$, O beıng the operator generatıng a glueball when actıng on the vacuum

and correctıons Δm to the mass $m^{(0)}$ can be calculated by keeping the t-lınear terms ın the cluster expansion of Γ

Now, a certaın class of hıgher-order correctıons consist of those 1-polymer clusters X obtaınable by the removal of n rıngs u^4 from X_0 and subsequent insertıon of n rıngs v^4 together wıth ınner walls u at entrance and exıt sıde of the v^4 rıngs (fig 2) The relatıve activity $\hat{\phi} = \phi(X)\phi(X_0)^{-1}$ whıch ıs responsıble for the correctıon, is gıven by

$$4u^2\left(\frac{v^4}{u^4}\right)^n \tag{3 13}$$

In the case of the Wilson actıon each rıng contrıbutes $O(\beta^4)$, which can be read off from the leading order behavior (3 9)

But ıf we compute the same quantity for our model, we see from (3.8) that the contrıbutıon of a sıngle rıng v^4/u^4 is of $O(1)$!

Thus, all graphs of the type considered above, however large theır extensıon, contribute to order β^2! Clearly, thıs ıs due to the geometry of X_0 and ıs therefore a pecuharıty of the applıcation of strong couplıng expansıons to glueball masses In partıcular, a prıorı it ıs *not* clear whether these contrıbutions exponentiate ın the form (3 12)

In order to prove the exponentıatıon of the correctıons ın questıon, we follow a suggestıon due to Munster we consider a model defined on a lattıce whıch ıs a three-dımensıonal tube havıng sıngle plaquettes as space-like cross sectıons and

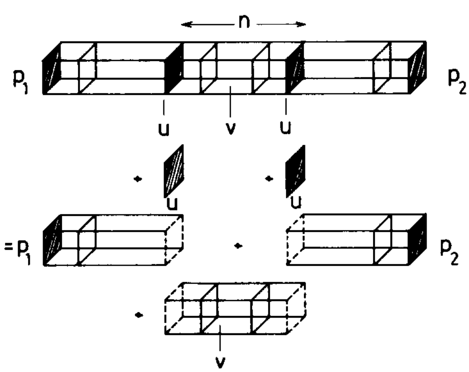

Fıg 2 Class of 1-polymer clusters wıth relatıve actıvıty $\hat{\phi} = 4u^2(v^4/u^4)^n$, whıch ıs ınvestıgated ın the study of the tube model

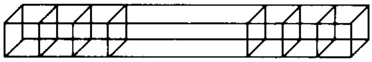

Fig 3 The tube model

extending infinitely in the time direction (fig 3) We choose an action such that the character expansion of a single plaquette Boltzmann factor contains the trivial, fundamental and adjoint representation only

The mass $m_{1/2}$ of the glueball whose wave-function consists of a single plaquette in the fundamental representation in the strong coupling limit can be determined as

$$e^{-m_{1/2}} = \lambda_{1/2}/\lambda_0 , \tag{3 14}$$

where λ_0 and $\lambda_{1/2}$ are the highest and next-to-highest eigenvalues of the transfer matrix T of the tube model We then compute $m_{1/2}$ independently by means of a strong coupling expansion summing all n-ring contributions We performed the calculation up to $O(\beta^7)$ and found complete equivalence of both results, concluding that the corrections (3.13) do exponentiate provided we sum over all n For details see appendix A

Let us now return to the full 4-dimensional model again Defining

$$R_v = \sum_{n=1}^{\infty} \left(\frac{v^4}{u^4}\right)^n = \frac{v^4}{u^4 - v^4} , \tag{3 15}$$

the strong coupling series becomes a power series in R_v However, for the numerical analysis, it is more convenient to expand v, w and x in terms of the expansion parameter $u(\beta_f, \kappa)$ Factorizing the leading behavior of R_v, the strong coupling series can be put into the alternative form

$$m = -4 \ln u + \sum_{k=1}^{7} m_k(\kappa) u^k + O(\beta^8) \tag{3 16}$$

The coefficients $m_k(\kappa)$ are of generic type

$$m_k(\kappa) = \left\{ \frac{1}{1 - (\frac{1}{3}\kappa)^4} \right\}^{k-1} \mathrm{pol}\,(\kappa) , \tag{3 17}$$

and are listed in table 1

Since the group theoretical part of the strong coupling expansion (distribution of irreducible representations on a graph with fixed geometry and matching of representations at links shared by more than 2 plaquettes) is much more extensive than in the case of Wilson action, it has been performed on a computer

4. The continuum limit

To obtain predictions for the glueball mass, the continuum limit $a \to 0$ has to be taken Due to dimensional arguments, on a lattice a physical mass is related to the

TABLE 1

Expansion coefficients of the mass-gap as defined in (3.17)

$$m_1(\kappa)=-\kappa$$

$$m_2(\kappa)=\frac{1}{1-(\frac{1}{3}\kappa)^4}\left\{-\frac{1}{81}\kappa^6-\frac{2}{27}\kappa^4+\kappa^2+2\right\}$$

$$m_3(\kappa)=\left\{\frac{1}{1-(\frac{1}{3}\kappa)^4}\right\}^2\left\{-\frac{4}{19683}\kappa^{11}-\frac{26}{19683}\kappa^9+\frac{8}{243}\kappa^7+\frac{52}{243}\kappa^5-\frac{140}{81}\kappa^3-\frac{2}{3}\kappa\right\}$$

$$m_4(\kappa)=\left\{\frac{1}{1-(\frac{1}{3}\kappa)^4}\right\}^3\left\{-\frac{49}{12754584}\kappa^{16}-\frac{43}{1594323}\kappa^{14}+\frac{12755}{12754584}\kappa^{12}+\frac{41}{6561}\kappa^{10}\right.$$
$$\left.-\frac{13939}{157464}\kappa^8-\frac{3643}{6561}\kappa^6+\frac{3131}{648}\kappa^4-\frac{23}{27}\kappa^2-\frac{98}{3}\right\}$$

$$m_5(\kappa)=\left\{\frac{1}{1-(\frac{1}{3}\kappa)^4}\right\}^4\left\{-\frac{949}{11622614670}\kappa^{21}-\frac{74}{129140163}\kappa^{19}+\frac{16772}{645700815}\kappa^{17}+\frac{286}{1594323}\kappa^{15}\right.$$
$$\left.-\frac{7981}{2657205}\kappa^{13}-\frac{31790}{1594323}\kappa^{11}+\frac{322474}{2657205}\kappa^9+\frac{9022}{6561}\kappa^7-\frac{407567}{65610}\kappa^5+\frac{1088}{243}\kappa^3-\frac{2882}{81}\kappa\right\}$$

$$m_6(\kappa)=\left\{\frac{1}{1-(\frac{1}{3}\kappa)^4}\right\}^5\left\{-\frac{32}{7625597484987}\kappa^{28}-\frac{1987}{1129718145924}\kappa^{26}-\frac{1021}{564859072962}\kappa^{24}\right.$$
$$+\frac{449299}{627621192180}\kappa^{22}+\frac{1349831}{627621192180}\kappa^{20}-\frac{89255}{774840978}\kappa^{18}-\frac{3826303}{7748409780}\kappa^{16}$$
$$+\frac{792335}{86093442}\kappa^{14}+\frac{36251183}{860934420}\kappa^{12}-\frac{509951}{2125764}\kappa^{10}-\frac{35872267}{10628820}\kappa^8$$
$$\left.+\frac{423779}{26244}\kappa^6-\frac{443621}{65610}\kappa^4+\frac{2752}{135}\kappa^2-\frac{20984}{405}\right\}$$

$$m_7(\kappa)=\left\{\frac{1}{1-(\frac{1}{3}\kappa)^4}\right\}^6\left\{-\frac{64}{617673396283947}\kappa^{33}-\frac{646223}{17294855095950516}\kappa^{31}+\frac{3065}{30502389939948}\kappa^{29}\right.$$
$$+\frac{5474477}{320275094369454}\kappa^{27}-\frac{216859}{5648590729620}\kappa^{25}-\frac{8091577}{2636009007156}\kappa^{23}+\frac{318601}{104603532030}\kappa^{21}$$
$$+\frac{102562366}{366112362105}\kappa^{19}+\frac{37497961}{104603532030}\kappa^{17}-\frac{7441824287}{488149816140}\kappa^{15}-\frac{28865041}{860934420}\kappa^{13}$$
$$\left.-\frac{25041217}{1004423490}\kappa^{11}+\frac{239715407}{31886460}\kappa^9-\frac{844079939}{24800580}\kappa^7+\frac{1971371}{98415}\kappa^5+\frac{13201}{32805}\kappa^3+\frac{704}{9}\kappa\right\}$$

For $\kappa=0$ we find the expansion coefficients for the simple Wilson action as expected.

lattice spacing a and the bare coupling constant g_0 via

$$m=\frac{1}{a}f(g_0)\,, \tag{4.1}$$

i.e. in order to keep the mass finite, g_0 has to be tuned simultaneously as $a\to 0$. In asymptotically free theories the continuum physics is expected to be recovered in the weak coupling region $g_0\to 0$. In this domain the simultaneous change $a\to 0$,

$g_0 \to 0$, while keeping m finite is controlled by the perturbative solution of the renormalization group equation and one obtains in a 2-loop calculation the well-known relation

$$m = C_m \frac{1}{a}(\beta_0 g_0^2)^{-\beta_1/2\beta_0^2} \exp\left(-\frac{1}{2\beta_0 g_0^2}\right)\{1+\mathrm{O}(g_0^2)\} = C_m \Lambda_{\mathrm{L}}, \qquad (4\ 2)$$

where β_0 and β_1 are the first 2 universal (renormalization scheme-independent) coefficients of the perturbative expansion of the β-function and Λ_{L} is the lattice Λ-parameter which sets the mass scale of the theory The constant C_m provides a non-perturbative relation between the physical mass m and Λ_{L}

Given this context, the following questions arise naturally which is the fastest way of connecting strong and weak coupling regions? Does the strong coupling series, once a suitable scheme of analytic continuation has been chosen, exhibit the scaling behavior (4 2) in the weak coupling region?

Consider the phase diagram of the class of 2-parameter actions specified by (2 1) (fig 4) As indicated by the dashed lines, lines of contant bare coupling constant g_0 run through the parameter space with a slope $-\frac{1}{2}$ The fastest way to reach small bare coupling constants is therefore achieved, if one runs along lines with slope +2 Then, however, one will pass at least one of the first-order transition lines indicated in the figure But this means that we will encounter non-analyticities on our way, which will give trouble for the strong coupling expansion In this case the situation would be even worse than in the case of simple Wilson action where even the well-separated critical endpoint can be traced in the glueball mass

Now, in order to cure this problem, one can choose a straight line with *negative* slope Doing this we stay away as far as possible from the transition line and its

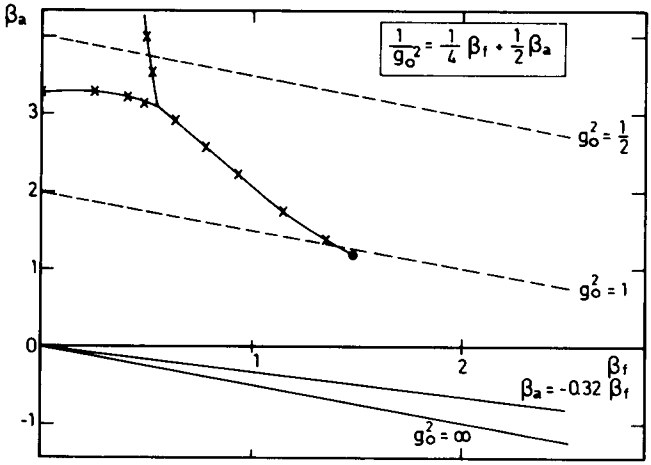

Fig 4 The phase diagram of our mixed action model The dashed lines are lines of constant bare coupling constant g_0, the + symbols are Monte Carlo data from [16]

critical endpoint However, this is dangerous too, because we are now staying in the vicinity of the strong coupling region $g_0 \to \infty$

The solution to this subtle problem is based on an analysis by Bitar, Gottlieb and Zachos [5] Their results have already been successfully applied to a Monte Carlo computation [6]

Bitar et al considered 1-plaquette, n-parameter actions ($n = 2, 3$) which coincide with our action for $n = 2$ up to the definition of the couplings β_f, β_a They then applied Migdal's approximate renormalization group transformations [7], i e they computed the sequence of effective large-distance actions projected to the space of 1-plaquette actions, obtained by successive upscaling of the lattice spacing by a factor $\lambda > 1$ Although this procedure is inexact for $d > 2$, it is believed to keep the essentials of the dynamics

Starting from various bare couplings β_f and β_a they found that all renormalization trajectories coalesce in one single line which becomes straight for $\beta_f \gtrsim 0\ 8$, with an approximate slope of $\kappa = -0\ 32$ with our conventions (to be precise, this is only true if β_f and β_a are chosen in the interesting region right of the line connecting the origin and the triple point. Trajectories starting from β_f and β_a left of this line unify in the β_a axis) Hence, actions which have $\beta_a/\beta_f \simeq -0\ 32$ are, in the sense of the Migdal approximate renormalization group transformations, the renormalized descendants of actions with larger bare couplings, or stated differently, they represent the same physics as actions which have a larger correlation length. Thus, this class of effective large-distance actions is closer to the continuum limit due to the relation $1/\xi a = m$ Once this line is reached, all actions stay invariant under the truncated Migdal renormalization group transformation (apart from the change of the coupling constant) until the infrared fixed point at $\beta_f = 0 = \beta_a$ is reached. This (fixed) line is an approximation to the projection of the *renormalized trajectory* of Kogut and Wilson [8] to the subspace of 1-plaquette actions Actions which approximately exhibit such a behavior are known to be of the generalized gaussian type [9], i e of heat kernel [10], Manton [11] or mixed action type with $\beta_a/\beta_f \cong -0\ 32$

As Lang et al [12] have demonstrated in a Monte Carlo simulation, heat kernel and Manton action show a smoother approach of the continuum limit and improved scaling For the heat kernel action we have confirmed these features within the framework of strong coupling expansions. Moreover, lattice action universality is ratified For a summary of the results see appendix B

In fig 5 we plot the strong coupling series over β_f for various negative values of κ In particular for larger negative κ, the strong coupling series seems to exhibit a smoother crossover to the weak coupling behavior, though it fails to show scaling behavior However, compared to the Wilson case, no qualitative improvement in the series can be observed This is believed to be due to the short series available In addition, we computed [4/3] Padé approximants for the same choice of κ, which appear to be more stable than in the Wilson case and mimic the strong coupling

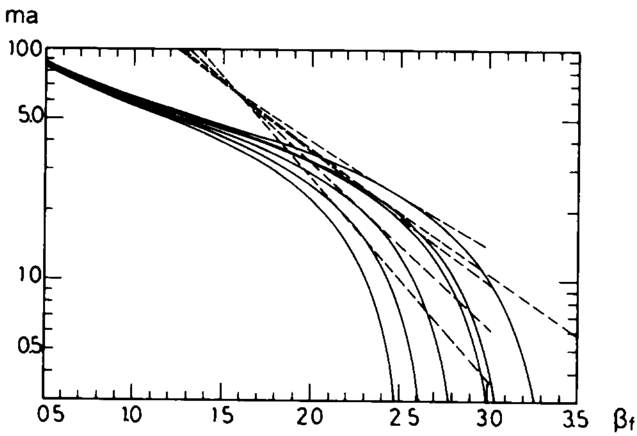

Fig 5 The strong coupling series (3 16) plotted over β_f From left to right we have $\kappa = 0, -0\ 1, -0\ 2, -0\ 3, -0\ 32, -0\ 4$ The expected scaling behavior according to (4 4) is indicated by the dashed lines

series to good accuracy in a large range of coupling constants, but also fail to indicate scaling behavior at weaker coupling (fig 6)

The problem still to be solved is how the constant C_m is to be determined. The usual procedure [2] is to assume a rapid crossover from strong to weak coupling behavior in the region of collapse of the strong coupling series. C_m is then determined by fitting Λ_L to the strong coupling curve, Λ_L being computed by some perturbative method However, if one performs an ordinary 2-loop perturbative calculation, as Otto and Randeria report [6], totally nonsensical results are obtained This was also found by Bhanot and Dashen [13] who computed the string tension in the β_f

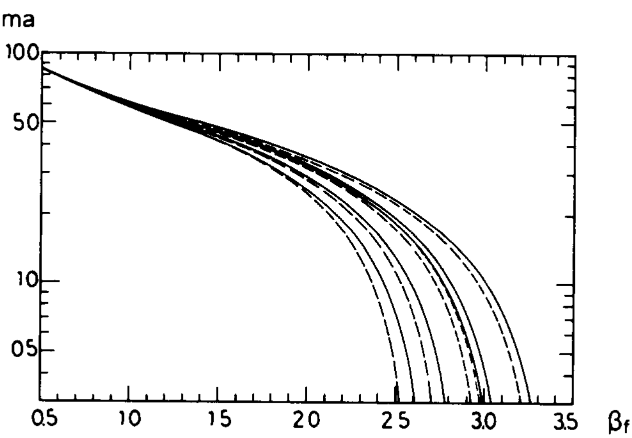

Fig 6 The strong coupling series of fig 5 together with the corresponding [4/3] Padé approximants Starting from the left, we have strong coupling series with $\kappa = -0\ 1$ and [4/3] Padé approximant (dashed line), strong coupling series with $\kappa = -0\ 2$ (full line) and [4/3] Padé approximant (dashed line) etc

and β_a plane by a Monte Carlo experiment For negative β_a they manifested large disagreements with scaling

This is by no means surprising· looking at (2.3) we conclude that a negative β_a results in a large bare coupling g_0 This can be seen, too, from the phase diagram fig 4 Staying in a region of large bare coupling, low-order perturbation theory cannot be expected to give correct results Hence, there is need for an improved scheme to compute Λ_L

In ref [14] Grossman and Samuel performed a resummation of perturbation theory They showed that in a large-N (gauge group SU(N)) approximation, the class of 1-plaquette, 2-parameter action under consideration is equivalent to the 1-plaquette, 1-parameter Wilson action, but with an *effective coupling* β_{eff}, which is given by

$$\beta_{eff} = \beta_f + 2\beta_a - \frac{5\beta_a}{2\beta_{eff}}, \tag{4.3}$$

at weak coupling By this method they then cured the problems of [13] and verified asymptotic scaling for the string tension Monte Carlo data Relying on (4 3), Otto and Randeria [6] also confirmed scaling of the mass-gap

Motivated by this success, our method is as follows for fixed κ, we express the asymptotic scaling behavior in terms of β_{eff}

$$\Lambda_L(\beta_{eff}) = \frac{1}{a}\left(\frac{\beta_{eff}}{4\beta_0}\right)^{\beta_1/2\beta_0^2} \exp\left(-\frac{1}{8\beta_0}\beta_{eff}\right), \tag{4 4}$$

and determine C_m such that this function fits the corresponding strong coupling curve The strong coupling curves stay unaffected by this procedure, hence they still fail to exhibit scaling behavior For the fits plotted over β_f, see fig 5. The κ dependence of the C_m parameter is shown in fig. 7 For decreasing κ, the κ dependence becomes weaker, i e the mass determination more stable. Hence we

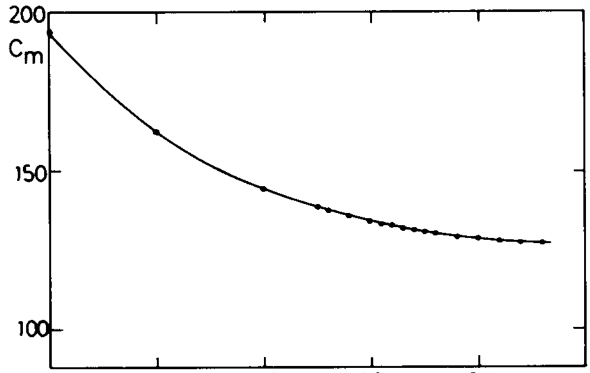

Fig 7 The κ-dependence of the C_m parameter as defined in (4 2)

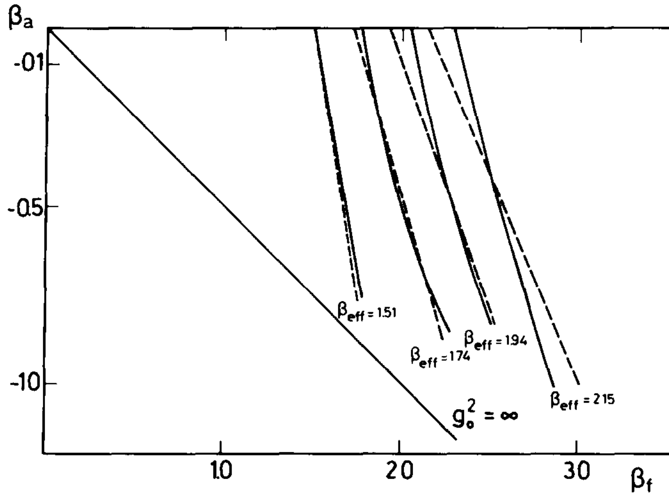

Fig 8 Lines of constant ma as determined from the strong coupling series (full lines) together with the predicted curves (dashed lines) which are lines of $\beta_{\rm eff}$ = const

conclude (approximate) universality for mixed action around the fixed line slope $\kappa = -0.32$ For comparison, recall the value $C_m = 127$ from the $O(\beta^8)$ Wilson action calculation

To get an impression whether the large-N resummation performed in [14] is applicable in the crossover region, we plot in fig 8 lines of constant ma together with the lines of constant $\beta_{\rm eff}$ according to (4 3), which should match in the region of validity From the figure it can be deduced

For $ma = 1$ the prediction fails, however, one does not have to worry about it because we are well *beyond* the crossover where we have no reason to trust the strong coupling series any more

For $ma = 2$ we have good agreement for the interesting region $-0.2 \geqslant \kappa \geqslant -0.32$ The predictive power even grows for increasing ma ($ma = 3, 4$), where, already being in a region of rather large coupling, the weak coupling result (4 3) is a priori not expected to be useful any more

5. Summary and conclusion

We have presented a strong coupling expansion of the mass-gap (which corresponds to the lowest-lying glueball in the $J^P = 0^+$ sector) up to $O(\beta^7)$ with mixed fundamental and adjoint action for the gauge group SU(2) We find low-order corrections due to graphs of arbitrary size as a novel feature in the strong coupling expansion These corrections are shown to exponentiate in the desired way The strong coupling series plotted for fixed ratios of adjoint and fundamental coupling approach the scaling region more smoothly than the corresponding series for simple Wilson action Padé approximants are more stable than in the case of Wilson action too However, as expected, series and approximants fail to exhibit scaling behavior qualitatively and quantitatively Nevertheless, using the tangent method we confirm universality for mixed action as well as for the heat kernel action, which are approximately equivalent for a special choice of parameters Finally, we independently checked the validity of the large-N resummation [14] already used in other calculations with mixed action [14, 6]

It is a great pleasure to thank Gernot Munster for many suggestions and illuminating discussions through all stages of this work. I would like to thank Prof. F. Gutbrod, Prof. H Joos, Prof G Kramer and Prof P Soding for supporting this work

Appendix A

THE TUBE MODEL

The model is defined on a straight tube of 1-plaquette cross section stretching in the (imaginary) time direction* (fig 3). It is geometrically equivalent to the leading order polymer X_0. The action is defined as

$$S = \sum_{p_s} S_{p_s} + \sum_{p_t} S_{p_t} , \tag{A.1}$$

where the first (second) sum runs over space-like (time-like) plaquettes only The distinction between space-like and time-like plaquettes permits, in general, different actions to be defined for space-like and time-like oriented plaquettes. As a by-product, it improved the visualization of the contributing graphs in the strong coupling expansion as mentioned in sect 3

The 1-plaquette actions S_{p_s} and S_{p_t} are defined as

$$\begin{aligned} e^{S_{p_s}(U_{p_s})} &= 1 + 2u_s\chi_{1/2}(U_{p_s}) + 3v_s\chi_1(U_{p_s}) , \\ e^{S_{p_t}(U_{p_t})} &= 1 + 2u_t\chi_{1/2}(U_{p_t}) + 3v_t\chi_1(U_{p_t}) , \end{aligned} \tag{A 2}$$

where we dropped an overall normalization factor [4]

To construct the transfer matrix T, we consider a single ring of fig 3 (fig 9) Recalling that T is the time translation operator for imaginary time, we get

$$\begin{aligned} T(x_i, y_i) &= \int_G \prod_{i=1}^{4} dU_{b_i}\, e^{S_{p_s}(X)/2} \prod_{l=1}^{4} e^{S_{p_t}(U_l(x_i, b_i, y_i))}\, e^{S_{p_s}(Y)/2} \\ &= \mathrm{T}(\chi(X), \chi(Y)) , \end{aligned} \tag{A 3}$$

$$\begin{aligned} X &= U_{x_4} U_{x_3}^{-1} U_{x_2}^{-1} U_{x_1} \\ Y &= U_{y_4} U_{y_3} U_{y_2}^{-1} U_{y_1}^{-1} \end{aligned} \tag{A.4}$$

where the second line of eq (A 3) follows from gauge invariance.

Now T is of the form $A^{1/2}BA^{1/2}$, A and B being symmetric matrices depending on $\chi(X)$, $\chi(Y)$, and since we aim at the determination of the eigenvalues of T, we

* The tube model has also been discussed in a variational approach by Patkos [17]

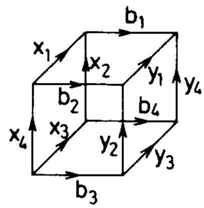

Fig 9 A single ring of the tube model (fig 3) together with the symbols referred to in the text

may consider as well the asymmetric transfer matrix

$$\hat{T} = BA\,, \tag{A 5}$$

which is easier to deal with Making use of the property (A 3), which holds of course for $\hat{T}$ too, we get

$$\hat{T}(\chi(X), \chi(Y)) = \sum_{r,s} t_{rs}\chi_r(X)\chi_s(Y)\,, \tag{A 6}$$

$$t_{rs} = \int_{\mathrm{G}} \mathrm{d}X\,\mathrm{d}Y\,\chi_r(X^{-1})\chi_s(Y^{-1})\hat{T}(\chi(X), \chi(Y)) \tag{A 7}$$

Performing the bond integrations $b_1, b_2, \ldots, b_4$ and subsequently the integrations over X and Y we get for the transfer matrix elements

$$(t_{rs})_{r,s=0,1/2,1} = \begin{pmatrix} 1 & 2u_s & 3v_s \\ 2u_s u_t^4 & u_t^4(1+3v_s) & 2u_s u_t^4 \\ 3v_s v_t^4 & 2u_s v_t^4 & v_t^4(1+3v_s) \end{pmatrix} \tag{A 8}$$

Application of ordinary non-degenerate Rayleigh–Schrodinger perturbation theory in the non-diagonal part of (A 8) up to 7th-order yields the eigenvalues λ_0 and $\lambda_{1/2}$ to the desired accuracy Then the mass can be computed using eq (3 14)

$$m_{1/2} = -\ln\frac{\lambda_{1/2}}{\lambda_0} \tag{A 9}$$

Appendix B

STRONG COUPLING EXPANSION OF THE MASS-GAP WITH GENERALIZED HEAT KERNEL ACTION FOR GAUGE GROUP SU(2)

Consider euclidean pure Yang–Mills lattice gauge theory on a hypercubical lattice in $d=4$ dimensions The SU(2) generalization of the heat kernel action [10]

$$S_{\mathrm{HK}} = \ln\prod_{\mathrm{p}} K(U_{\mathrm{p}}, \tfrac{1}{2}g^2)\,,$$

$$K(U_{\mathrm{p}}, \tfrac{1}{2}g^2) = \mathcal{N}\sum_{n=-\infty}^{+\infty}\frac{\varphi_{\mathrm{p}}+2\pi n}{\sin\varphi_{\mathrm{p}}}\exp\left[-\frac{2}{g^2}(\varphi_{\mathrm{p}}+2\pi n)^2\right], \tag{B 1}$$

($\mathcal{N}$ is the normalization constant; the coupling constant g is related to the bare coupling constant g_0 by $1/g^2 = 1/g_0^2 + \frac{1}{12}$) results in a particular simple character expansion of the 1-plaquette Boltzmann factor

$$e^{S_{\mathrm{HK}\,p}} = \sum_r d_r\, e^{-g^2 C_r/2} \chi_r(U_p)\,, \tag{B 2}$$

C_r being the eigenvalue of the SU(2) Casimir operator in the irreducible r-representation The usual SU(2) parameters u, v and w defined as in (3.7) read

$$\begin{aligned} u(q) &= q^{3/2}\,, \\ v(q) &= q^4\,, \\ w(q) &= q^{15/2}\,, \qquad q := e^{-g^2/4} \end{aligned} \tag{B 3}$$

Making use of the $O(\beta^8)$ result for Wilson action, the strong coupling expansion is computed as

$$\begin{aligned} m_{0^+} = &-4 \ln u(q) + 4q^3 - 3q^4 - 26q^6 - 12q^7 + \tfrac{9}{2}q^8 \\ &+ \tfrac{100}{3}q^9 - 150q^{10} + 36q^{11} - 619q^{12} - 72q^{13} + O(q^{15}) \end{aligned} \tag{B 4}$$

Using the tangent method of ref [2] and the correct expression for $\Lambda_{\mathrm{L}}^{\mathrm{HK}}$ we compute $C_m^{\mathrm{HK}} = 39$.

We say that we have universality, if for 2 different actions S and S' with corresponding lattice scale parameters Λ_{L} and Λ'_{L}, we have

$$\frac{C_m}{C'_m} = \frac{\Lambda'_{\mathrm{L}}}{\Lambda_{\mathrm{L}}}\,. \tag{B 5}$$

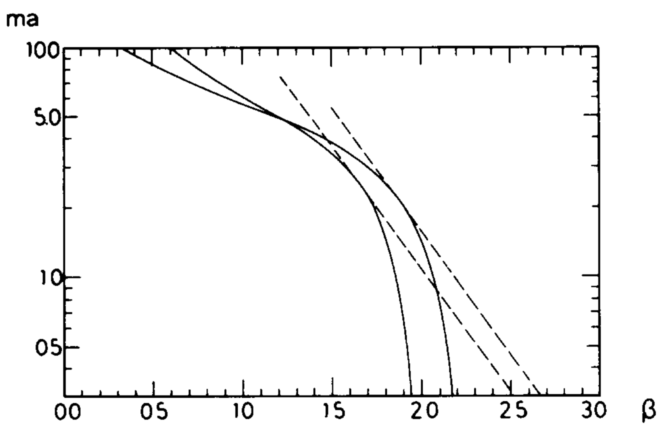

Fig 10 The strong coupling series for the Wilson and generalized heat kernel action (full lines) together with the expected scaling behavior (dashed lines) The curve which "scales earlier" corresponds to the heat kernel action

Recallıng $C_m^W = 127$ for the $O(\beta^8)$ calculatıon wıth the Wılson actıon, we get

$$\frac{\Lambda_L^{HK}}{\Lambda_L^W} = 3\ 26 \tag{B.6}$$

In [15], Gonzales et al determıne

$$\frac{\Lambda_L^{HK}}{\Lambda_L^W} = 3\ 07\,, \tag{B 7}$$

by applicatıon of the background field method, whıle they quote an experımental value of

$$\frac{\Lambda_L^{HK}}{\Lambda_L^W} = 4\ 20^{+0\ 9}_{-0\ 8} \tag{B 8}$$

from Monte Carlo measurements of the strıng tensıon.

Our result ıs ın good agreement wıth the theoretıcal value, thus unıversalıty of the mass-gap for the Wılson and heat kernel actıon may be concluded (fig 10)

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