MASS TERMS AND MASS RENORMALIZATION FOR SUSSKIND FERMIONS

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We discuss the symmetry properties of a geometrically motivated mass term giving different masses to the four flavours of Susskind fermions. Using this mass term we calculate the fermion self-energy in weak coupling perturbation theory at the one-loop level as well as the relation between the fermion masses on the lattice and in the continuum.

Among the different lattice fermion schemes, Susskind's method [1,2] is distinguished by the fact that it has a deep geometrical foundation: It can be interpreted as the geometrically natural way of putting the Dirac–Kähler equation on a lattice [3–5]. However, as compared to the Wilson formulation [6] or the SLAC fermions [7], it has the disadvantage of retaining a part of the species doubling. The Susskind action describes four fermions of equal mass. But this degeneracy is no lattice artifact, it is already present in the continuum Dirac–Kähler equation. Therefore it seems possible to interpret the additional degree of freedom as some kind of flavour, as was originally proposed by Susskind. In order to formulate a more realistic lattice model of QCD, based on the Susskind fermions, it is, however, necessary to lift the mass degeneracy of the different flavours. Several degeneracy breaking mass terms have already been discussed in the literature [8–16]. Important questions concerning such mass terms are: Will they be stable under renormalization and how do they modify the symmetries of the Susskind action?

We want to study these problems for a geometrically motivated mass term [14] and thereby substantiate the remarks concerning this point which were made in ref. [16]. We use the notations of ref. [3] and write the action for the free euclidean Dirac–Kähler equation in the four-dimensional continuum as

$$\int \{\tfrac{1}{4}(\bar{\phi}, (\mathrm{d} - \delta)\phi)_0 + \tfrac{1}{4}m(\bar{\phi}, \phi)_0\} . \tag{1}$$

Here ϕ is an inhomogeneous differential form:

$$\phi = \sum_H \varphi(x, H)\, \mathrm{d}x^H .$$

The sum over H runs over all ordered sets of indices (multi-indices). Projection on flavour b is given by $\phi \to \phi \vee P^{(b)}$, where $\vee$ denotes the Clifford product and

$$P^{(b)} = \frac{1}{4} \sum_{K \in \mathcal{K}} \epsilon_K \eta_K^{(b)}\, \mathrm{d}x^K . \tag{2}$$

Here we have introduced the set of multi-indices $\mathcal{K} = \{\emptyset, \{12\}, \{34\}, \{1234\}\}$ related to the reduction group of ref. [3]. This choice of the reduction group corresponds to the use of the Weyl representation of the γ-matrices. Furthermore

$$\epsilon_\emptyset = \epsilon_{1234} = 1 , \quad \epsilon_{12} = -\epsilon_{34} = \mathrm{i} ,$$

$$\eta_\emptyset^{(1)} = -\eta_{12}^{(1)} = \eta_{34}^{(1)} = -\eta_{1234}^{(1)} = 1 ,$$

$$\eta_\emptyset^{(2)} = \eta_{12}^{(2)} = -\eta_{34}^{(2)} = -\eta_{1234}^{(2)} = 1,$$

$$\eta_K^{(3)} = \eta_K^{(1)}\eta_K^{(2)} , \quad \eta_K^{(4)} = 1 \quad \forall K \in \mathcal{K} . \tag{3}$$

Now it is straightforward to write down an action which gives a mass m_b to flavour b:

$$\int \left\{ \tfrac{1}{4}(\bar{\phi}, (\mathrm{d} - \delta)\phi)_0 + \frac{1}{4} \sum_{b=1}^{4} m_b (\bar{\phi}, \phi \vee P^{(b)})_0 \right\} . \tag{4}$$

By transition to Dirac components according to the formulas given in ref. [3] this action becomes

$$\int d^4x \sum_{b=1}^{4} \bar{\psi}^{(b)}(x)(\gamma^\mu \partial_\mu + m_b)\psi^{(b)}(x) . \qquad (5)$$

Using (2) we get for the mass term

$$\int \frac{1}{4} \sum_{b=1}^{4} m_b(\bar{\phi}, \phi \vee P^{(b)})_0$$

$$= \frac{1}{16} \sum_{K \in \mathcal{K}} \epsilon_K M_K \int (\bar{\phi}, \phi \vee dx^K)_0 , \qquad (6)$$

with

$$M_K = \sum_{b=1}^{4} \eta_K^{(b)} m_b . \qquad (7)$$

Note that (4) is invariant under the replacement

$$\phi \to \phi \vee dx^K , \quad \bar{\phi} \to \bar{\phi} \vee dx^K \quad (K \in \mathcal{K}) , \qquad (8)$$

i.e. under the action of the reduction group.

We introduce in euclidean space a hypercubic lattice with lattice spacing a and lattice unit vectors e_μ: $(e_\mu)_\nu = a\delta_{\mu\nu}$. The cells of this lattice are denoted by (x, H), where x is a lattice point and H a multi-index. A general cochain (the lattice analogue of a differential form) can be written as $\phi = \Sigma_{x,H}\, \varphi(x, H)\, d^{x,H}$ in terms of the elementary cochains $d^{x,H}$ defined by

$$d^{x,H}((x', H')) = a^h \delta_{x,x'} \delta_{H,H'}$$

$$(h = \text{number of elements in } H) . \qquad (9)$$

Furthermore, let $\mathcal{V}$ be the volume four-chain $= \Sigma_x (x, 1234)$. Applying the rules for the translation of continuum expressions into their lattice equivalents given in ref. [3], we get for the lattice analogue of the action (1):

$$\tfrac{1}{4}(\bar{\phi}, (\widetilde{\Delta} - \widetilde{\nabla} + m)\phi)_0(\mathcal{V}) . \qquad (10)$$

Here $\widetilde{\Delta}$ and $\widetilde{\nabla}$ denote the dual boundary operator and the dual coboundary operator, respectively.

What are the symmetries of this action? (See refs. [11,13,16] for related discussions.) Apart from the well-known U(1) (U(1) × U(1) for $m = 0$) symmetry of Susskind fermions, we have invariance under the discrete flavour transformations, the lattice translations, and the lattice point group. The discrete flavour transformations are given by

$$\phi \to \phi \vee d^H =: F_H \phi , \quad \bar{\phi} \to \bar{\phi} \vee d^H$$

$$\text{with } d^H = \sum_x d^{x,H} . \qquad (11)$$

Here we used the lattice Clifford product as defined in ref. [3]. An element S of the lattice point group acts on a cochain ϕ according to the formula

$$(S^* \phi)((x, H)) = \phi(S^{-1}(x, H)). \qquad (12)$$

The continuum analogue of (12) transforms the components $\varphi(x, H)$ of ϕ as antisymmetric tensors. The corresponding spinor transformation can be written as a product of this tensor transformation and an appropriate flavour transformation [3,4,16]. A rotation about the angle α in the μ–ν-plane, for example, has to be accompanied by the flavour transformation

$$\phi \to \phi \vee [\cos(\alpha/2) + \sin(\alpha/2)\, dx^{\mu\nu}] . \qquad (13)$$

Comparing with the lattice flavour transformations (11) we see that (13) is available on the lattice only for $\alpha = \pi$. Therefore we can define spinor rotations $\bar{D}_{\mu\nu}$ on the lattice only for rotation angle π:

$$\bar{D}_{\mu\nu} := F_{\mu\nu} \circ D^*_{\mu\nu} , \qquad (14)$$

where

$$(D_{\mu\nu} x)_\lambda = x_\lambda , \quad \text{if } \lambda \neq \mu, \nu$$
$$\qquad\qquad (\mu < \nu) .$$
$$= -x_\lambda , \quad \text{if } \lambda = \mu, \nu . \qquad (15)$$

Analogous considerations may be applied to S_μ, the reflection with respect to a lattice hyperplane orthogonal to the μ-direction:

$$(S_\mu x)_\lambda = (-1)^{\delta_{\lambda\mu}} x_\lambda . \qquad (16)$$

They lead to the following spinor transformation $\bar{S}_\mu$ on the lattice:

$$\bar{S}_\mu := F_{C\{\mu\}} \circ S^*_\mu , \quad C\{\mu\} = \{1234\} \backslash \{\mu\} . \qquad (17)$$

A mass term which gives different masses to the flavours and therefore breaks flavour invariance is no longer invariant under the tensor transformations, because they contain a flavour transforming piece. It should, however, be invariant under spinor transformations. Hence we require that the lattice analogue of (6) be invariant under $\bar{D}_{\mu\nu}$, $\bar{S}_\mu$ as well as with respect to the flavour transformations F_{12}, F_{34}, F_{1234}, which correspond to the reduction group (compare (8)). It is straightforward to check that the mass term

$$\frac{1}{16}\sum_{K\in\mathcal{K}} \epsilon_K M_K 2^{-k} \sum_{L\subset K} (\bar{\phi}, (T_{e_L}\phi) \vee d^K)_0(\mathcal{V}) \,, \quad (18)$$

which is hermitian and reproduces (6) in the naive continuum limit, fulfills these requirements. Here k denotes the number of elements in K, and the translation operator T_{e_L} is defined by

$$T_{e_L}\phi = \sum_{x,H} \varphi(x+e_L, H) d^{x,H} \,, \quad e_L = \sum_{\mu\in L} e_\mu \,. \quad (19)$$

The introduction of the T's is necessary in order to achieve invariance under $\bar{D}_{\mu\nu}, \bar{S}_\mu$. Note that

$$\sum_{L\subset K} (\bar{\phi}, (T_{e_L}\phi) \vee d^K)_0(\mathcal{V}) \quad (20)$$

is invariant with respect to the tensor transformation $R^*_{\mu\nu}$, where $R_{\mu\nu}$ is a rotation about $\pi/2$ in the μ–ν-plane, provided $K \cap \{\mu,\nu\} = \emptyset$ or $K\cap\{\mu,\nu\} = \{\mu,\nu\}$. Consequently, (18) is invariant under $\pi/2$ rotations in the 1–2- and in the 3–4-plane, although corresponding spinor transformations cannot be defined. In the special case $m_1 = m_2, m_3 = m_4$, where $M_{12} = M_{34} = 0$, we even have invariance with respect to all $R^*_{\mu\nu}$.

What happens to these symmetries upon introducing interaction with a gauge field? We couple the gauge field in such a way that the lattice gauge transformations act at the center of the lattice cells. This coupling corresponds to the original Susskind formulation. The resulting theory with interaction has the same invariances as in the free case, provided the transformations of the fermion field are accompanied by suitable transformations of the gauge field (see refs. [16,17] for details). Nevertheless, the restoration of Lorentz invariance in the continuum limit might be questionable, since (18) with four different masses is not invariant under all lattice rotations. Furthermore, one would like to know the connection between the bare masses m_j on the lattice and the continuum masses.

In order to study these problems, we have calculated the fermion self-energy in weak coupling perturbation theory at the one-loop level. It turns out to be advantageous to use Susskind's one-component formulation. Therefore we put

$$\varphi(x,H) = \chi(x+\tfrac{1}{2}e_H) \,, \quad \bar{\varphi}(x,H) = \bar{\chi}(x+\tfrac{1}{2}e_H) \,, \quad (21)$$

so that the χ's live on a lattice with lattice spacing $a/2$. They transform according to the fundamental representation of the gauge group SU(N). For convenience we replace a by $2a$, but retain the conventions that x denotes a point with coordinates $x_\mu = an_\mu$ ($n_\mu \in \mathbf{Z}$) and that $(e_\mu)_\nu = a\delta_{\mu\nu}$. In this way we get for the action of the fermions coupled to the lattice gauge field $U_\mu(x) \in \mathrm{SU}(N)$

$$2a^3 \sum_{x,\mu} c_\mu(x)\bar{\chi}(x)[U^+_\mu(x)\chi(x+e_\mu) - U_\mu(x-e_\mu)\chi(x-e_\mu)]$$

$$+a^4 \sum_{K\in\mathcal{K}} \epsilon_K M_K 2^{-k} \sum_{L\subset K}\sum_x \sigma_K(x)\bar{\chi}(x) U^+_L(x+e_L)$$

$$\times\, U_{K\setminus L}(x+e_L)\chi(x-e_K+2e_L) \,. \quad (22)$$

The sign factors $c_\mu(x)$, $\sigma_K(x)$ are defined by

$$c_\mu(x) = (-1)^{(x_1+x_2+\ldots+x_{\mu-1})/a} \,, \quad (23)$$

$$\sigma_\emptyset(x) = 1 \,, \quad \sigma_{12}(x) = -(-1)^{x_2/a} \,,$$

$$\sigma_{34}(x) = -(-1)^{x_4/a} \,, \quad \sigma_{1234}(x) = (-1)^{(x_2+x_4)/a} \,. \quad (24)$$

Moreover

$$U_H(x) := U_{\mu_1}(x-e_{\mu_1})U_{\mu_2}(x-e_{\mu_1}-e_{\mu_2})\ldots U_{\mu_h}(x-e_H) \quad (25)$$

for $H = \{\mu_1, \mu_2, \ldots, \mu_h\}$, $\mu_1 < \mu_2 < \ldots < \mu_h$. The action (22) is invariant under the gauge transformations

$$\chi(x) \to g(x)\chi(x) \,, \quad \bar{\chi}(x) \to \bar{\chi}(x) g^{-1}(x) \,,$$

$$U_\mu(x) \to g(x+e_\mu) U_\mu(x) g^{-1}(x) \,, \quad g(x) \in \mathrm{SU}(N) \,. \quad (26)$$

For the gauge field we take the standard Wilson action

$$\frac{1}{g^2}\sum_{x,\mu,\nu} \mathrm{Tr}(1 - U^+_\nu(x)U^+_\mu(x+e_\nu)U_\nu(x+e_\mu)U_\mu(x)) \,, \quad (27)$$

with the gauge fixing term

$$\frac{a^2}{2\xi}\sum_{x,a}\left(\sum_\mu [A^a_\mu(x) - A^a_\mu(x-e_\mu)]\right)^2 \,. \quad (28)$$

We have set $U_\mu(x) = \exp(\mathrm{i}gaA^a_\mu(x)T^a)$, where the T^a are the hermitian generators of SU(N) in the fundamental representation.

Calculating $\langle\chi_i(x)\bar{\chi}_j(x')\rangle$ (i,j = SU(N)-indices) at the one-loop level we get in the spinor basis

$$\sum_{H,H'} (\gamma_H)_{ab} \langle \chi_i(2x+e_H)\bar{\chi}_j(2x'+e_{H'})\rangle (\gamma^*_{H'})_{cd}$$

$$= \pi^{-4} \int_{-\pi/(2a)}^{\pi/(2a)} d^4p \exp[ip\cdot(2x-2x')]$$

$$\times \left(G(p)^{bd}_{ac} + \sum_{\substack{a',b'\\c',d'}} G(p)^{bb'}_{aa'} \Sigma(p)^{b'd'}_{a'c'} G(p)^{d'd}_{c'c} + ... \right), \tag{29}$$

where as $a \to 0$

$$G(p)^{bd}_{ac} = \sum_{j=1}^{4} ((i\not{p} + m_j)^{-1})_{ac}(\hat{P}_j)_{bd} + o(1), \tag{30}$$

$$\Sigma(p)^{bd}_{ac} = \sum_{j=1}^{4} \Sigma_j(p)_{ac}(\hat{P}_j)_{bd} + o(1), \tag{31}$$

and $\gamma_H = \gamma_{\mu_1}\gamma_{\mu_2}...\gamma_{\mu_h}$ for $H = \{\mu_1, \mu_2, ..., \mu_h\}$, $\mu_1 < \mu_2 < ... < \mu_h$. The indices a, c are spinor indices, b, d are flavour indices, and $\hat{P}_j$ is the matrix in flavour space which projects onto flavour j. Under the assumption $m_j > 0$ for all j, we find for the self-energy Σ_j in the Feynman gauge $\xi = 1$:

$$\Sigma_j(p) = (g^2/8\pi^2)[(N^2-1)/2N]$$

$$\times [i\not{p}\Sigma^{(1)}_j(p^2) + m_j\Sigma^{(2)}_j(p^2) + \Delta_j], \tag{32}$$

with

$$\Sigma^{(1)}_j(p^2) = \int_0^1 dx\, x \ln[a^2(1-x)(p^2x + m_j^2)] + A_1, \tag{33}$$

$$\Sigma^{(2)}_j(p^2) = 2\int_0^1 dx \ln[a^2(1-x)(p^2x+m_j^2)] + A_2, \tag{34}$$

$$\Delta_1 = m_2B_1 + (m_3+m_4)B_2,$$

$$\Delta_2 = m_1B_1 + (m_3+m_4)B_2,$$

$$\Delta_3 = (m_1+m_2)B_2 + m_4B_1,$$

$$\Delta_4 = (m_1+m_2)B_2 + m_3B_1. \tag{35}$$

The constants A_1, A_2, B_1, B_2 are combinations of certain integrals. Numerical computation yields the values $A_1 = 3.53, A_2 = 6.52, B_1 = -2.72, B_2 = -9.70$. In the expression (32) Lorentz invariance is recovered. But some remnant of the flavour mixing on the lattice is still present in the continuum: The self-energy for flavour j depends on the bare masses of all other flavours through the Δ_j-terms.

Comparing (32) with the continuum expression for the fermion self-energy one can derive the relation between the bare mass on the lattice and the renormalization group invariant mass in the continuum [18–20]. The latter is usually defined as

$$\tilde{m} = [\ln(\mu/\Lambda)]^{2\gamma_0/\beta_0} m(\mu). \tag{36}$$

Here $m(\mu)$ is the running mass at a scale μ, Λ is the continuum coupling constant Λ-parameter in some renormalization scheme and γ_0, β_0 are the coefficients of the first term of the γ_m and β function, respectively. We have

$$\gamma_0 = \tfrac{3}{2}(N^2-1)/2N, \tag{37}$$

and in the quenched approximation β_0 takes the value $11\,N/3$. One finds

$$\tilde{m}_j = m_j[(8\pi^2/\beta_0)g^{-2} + \ln(\Lambda_L/\Lambda)$$

$$+ \ln C - \Delta_j/3m_j]^{2\gamma_0/\beta_0}. \tag{38}$$

In this formula, Λ_L denotes the lattice Λ-parameter. In the MS scheme we obtain

$$\ln C = \tfrac{1}{3}[A_1 - A_2 + \tfrac{1}{2} + \tfrac{3}{2}(\gamma_E - \ln 4\pi)] = -1.81, \tag{39}$$

where γ_E is the Euler–Mascheroni constant. In the $\overline{\text{MS}}$ scheme we get

$$\ln C = \tfrac{1}{3}(A_1 - A_2 + \tfrac{1}{2}) = -0.83. \tag{40}$$

Note that Δ_j/m_j depends only on the ratios of the bare fermion masses.

Summarizing we can say that the mass term (18) does not generate new mass counterterms. This is a consequence of its symmetry properties. Furthermore, eq. (38) shows how the bare masses m_j have to vary with a, if the bare coupling constant g depends on the lattice spacing in the usual way and $\tilde{m}_j$ is kept fixed. But it is, of course, an open question, whether this one loop result is sufficient in connection with present Monte Carlo simulations.

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