MONTE CARLO STUDY OF GLUEBALL MASSES IN SU(2)

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The multi-hit technique proposed by Parisi, Petronzio and Rapuano for the measurement of correlations is applied to the calculation of glueball masses in SU(2) lattice gauge theory. The value of the mass gap (0^+ glueball mass) turnes out to be $m_{0+} = (166 \pm 15)\ \Lambda_{\mathrm{latt}}$ or (1090 ± 170) MeV, if recent string tension determinations are taken for the scale.

One of the interesting predictions of QCD is the existence of hadronic states made of pure gauge field quanta, so called "glueballs". After the first qualitative hints obtained in some phenomenological models (like e.g. the bag model), the first quantitative results about the glueball spectrum were obtained recently in lattice Monte Carlo investigations [1–3]. (For a review and an extensive list of references see ref. [4].) These first calculations have several problems. Some of them are more serious, like the neglect of light dynamical quarks (hence no mixing with quark states), some of them less serious, like too small lattices or statistics. In the long run, however, there is a definite hope that lattice Monte Carlo calculations will provide us with reliable predictions about the glueball spectrum.

The first difficulty in the glueball mass calculations is that the typical value of correlations from which the glueball masses are extracted is small. Therefore, a large number of Monte Carlo sweeps (typically 10^4–10^5) is required to separate the signal from the noise, even at such moderate distances like $d = 2, 3$ (in lattice units).

As we shall demonstrate in the present letter, the "multihit-technique" [5] reduces the fluctuations in the correlations used to extract glueball masses, hence it is conceivable that it allows for a better determination of the glueball spectrum.

The idea of this method is to substitute observables Ω' for Ω, such that $\langle\Omega'\rangle = \langle\Omega\rangle$ but

$$\langle\Omega'^2\rangle \ll \langle\Omega^2\rangle\,, \tag{1}$$

where, in SU(N) gauge field theory,

$$\langle\Omega\rangle \equiv \frac{\int \mathrm{D}U \exp(-S)\,\Omega(U)}{\int \mathrm{D}U \exp(-S)}\,. \tag{2}$$

is the expectation value of Ω with SU(N) lattice action S and SU(N) Haar-measure $\int \mathrm{D}U$. The expectation values $\langle\Omega'^2\rangle$ and $\langle\Omega^2\rangle$ are measures for the fluctuations of Ω' and Ω. The multihit technique amounts to replace the gauge field variables $U(b)$ on link b by variables $\bar{U}(b)$, where $\bar{U}(b)$ is given by

$$\bar{U}(b) = \frac{\int \mathrm{d}U(b) \exp\{(2/g^2)\,\mathrm{Tr}[U(b)\,U_{\mathrm{int}}]\}\,U(b)}{\int \mathrm{d}U(b) \exp\{(2/g^2)\,\mathrm{Tr}[U(b)\,U_{\mathrm{int}}]\}}\,. \tag{3}$$

g is the bare coupling constant, U_{int} is the sum of the products of link matrices $\tilde{U}$ which interact with $U(b)$. $\bar{U}(b)$ equals the average of $U(b)$ at fixed interacting variables $\tilde{U}$. For SU(2), the integral in (3) can be done explicitly. The result is

$$\bar{U}(b) = [I_2((4/g^2)K)/I_1((4/g^2)K)]\,K \cdot U_{\mathrm{int}}^{-1}\,, \tag{4}$$

where I_1 and I_2 are modified Bessel-functions and $K = |\det U_{\mathrm{int}}(b)|^{1/2}$.

The multihit procedure may be simultaneously applied to gauge field variables $U(b_1), \ldots, U(b_n)$ ($b_1, \ldots, b_n, b_{n+1} \ldots, b_N$ are links of the lattice), for which

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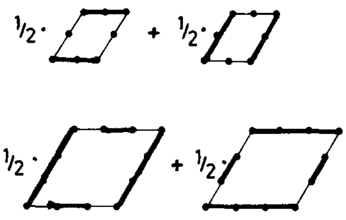

Fig. 1. 2 × 2 and 3 × 3 loops with "multihit links" b (thick links) where $\bar{U}(b)$ may be substituted for $U(b)$.

$$\langle \Omega[U(b_1), ..., U(b_n), U(b_{n+1}), ..., U(b_N)] \rangle$$
$$= \langle \Omega[\bar{U}(b_1), ..., \bar{U}(b_n), U(b_{n+1}), ..., U(b_N)] \rangle . \quad (5)$$

In the case of the standard (one-plaquette) Wilson-action this is satisfied if

$$S_{b_i} \cap S_{b_j} = \emptyset \quad \forall i \neq j \;;\; i,j \in \{1, ..., n\} . \qquad (6)$$

Here S_{b_i} denotes the set of all plaquettes containing the link b_i.

If Ω is the trace of products of U's along a single plaquette, only one $U(b)$ may be replaced by $\bar{U}(b)$. In the case of products along closed 2 × 2 or 3 × 3 loops, for instance, the combinations of fig. 1 are allowed by (6).

For later use let us introduce

$$O = \mathrm{Tr}[U(\partial \mathcal{L})] , \qquad (7)$$

where $\partial\mathcal{L}$ is the boundary of a general closed planar loop. Such functionals are the building blocks for wavefunctions representing glueball states with specific parity, momentum and spin. The planar loops can be characterized by a four-dimensional position vector $x = (\boldsymbol{x}, t)$ ($\boldsymbol{x}$ = space-, t = time-coordinate) and by a plane with space-like orientation j ($j \in \{1,2,3\}$). The glueball operator with definite three-momentum $\boldsymbol{p}$ is given by

$$O(\boldsymbol{p}, t) = \sum_{\boldsymbol{x}, j} \alpha_j O_{\boldsymbol{x}, t, j} \exp[\mathrm{i}(\boldsymbol{p}, \boldsymbol{x})] . \qquad (8)$$

The weight factors α_j depend on spin and parity [4], and $(\boldsymbol{p}, \boldsymbol{x})$ denotes the scalar product

$$(\boldsymbol{p}, \boldsymbol{x}) = 2\pi(k_1 x_1/N_1 + k_2 x_2/N_2 + k_3 x_3/N_3) . \qquad (9)$$

$N_{1,2,3}$ are the lattice sizes in space-like directions, $x_{1,2,3}$ the lattice coordinates and $k_{1,2,3}$ are the integer numbers characterizing momenta. (Here periodic boundary conditions are assumed in all directions.)

Let us denote by N_4 the lattice size in the time direction, and by a the lattice constant. Then the correlation function

$$\mathcal{C}(\boldsymbol{p}, d) = \langle \tilde{O}(\boldsymbol{p}, t_1) \tilde{O}(\boldsymbol{p}, t_2) \rangle - \langle \tilde{O}(\boldsymbol{p}, t_1) \rangle \langle \tilde{O}(\boldsymbol{p}, t_2) \rangle$$

$$\xrightarrow[|t_1 - t_2| \gg 1]{} \exp(-|t_1 - t_2| a E_p)$$
$$+ \exp[-(N_4 - |t_1 - t_2|) a E_p] \qquad (10)$$

has (for large time distances $d \equiv |t_1 - t_2| \leqslant N_4/2$) an exponential behaviour characteristic for the lowest energy E_p in the channels coupled to the operator O in question. (The second term is due to the periodicity in time.) In the Monte Carlo simulation the left-hand side of (10) is calculated as the following average:

$$\mathcal{C}(\boldsymbol{p}, d; N_m) = \frac{1}{N_m N_4} \sum_{m=1}^{N_m} \sum_{t=1}^{N_4} \mathrm{SL}(m, \boldsymbol{p}, t)$$
$$\times \tfrac{1}{2} [\mathrm{SL}(m, \boldsymbol{p}, t+d) + \mathrm{SL}(m, \boldsymbol{p}, t-d)]$$
$$- \left(\frac{1}{N_m N_4} \sum_{m=1}^{N_m} \sum_{t=1}^{N_4} \mathrm{SL}(m, \boldsymbol{p}, t) \right)^2 . \qquad (11)$$

N_m is the total number of measurements and, for instance, for the 0^+ state

$$\mathrm{SL}(m, \boldsymbol{p}, t) = \frac{1}{3} \sum_{j=1}^{3} \sum_{\boldsymbol{x}} \exp[\mathrm{i}(\boldsymbol{p}, \boldsymbol{x})] O_{\boldsymbol{x}, t, j}(m) . \qquad (12)$$

Here $O_{\boldsymbol{x},t,j}(m)$ is the value of the glueball operator $O_{\boldsymbol{x},t,j}$ measured on the given gauge field configuration.

In order to be able to replace more links in the glueball operator by multihit links, generally, large planar rectangular loops are the best. Very large loops are, however, weakly coupled to the lowest glueball states therefore, at a given $\beta \equiv 2/g^2$-value, one has to find an optimum. In the present paper we use, in most cases, the planar 2 × 2 loop. At larger β-values we also made some shorter test runs with planar 3 × 3 loops which looked, at such β-values, even better than 2 × 2. On the basis of e.g. eq. (4) it can be generally expected that the multihit technique works better for smaller β. The question, whether in the intermediate β-range eq.

Table 1
The measured correlations at β = 2.3.

d	J^σ, p			
	$0^+, 0$	$0^+, 1$	$2^+, 0$	$2^+, 1$
0	$(5.8 \pm 0.2) \times 10^{-4}$	$(2.04 \pm 0.04) \times 10^{-4}$	$(3.70 \pm 0.03) \times 10^{-4}$	$(1.84 \pm 0.04) \times 10^{-4}$
1	$(2.0 \pm 0.1) \times 10^{-4}$	$(5.9 \pm 0.3) \times 10^{-5}$	$(6.6 \pm 0.6) \times 10^{-5}$	$(3.4 \pm 0.2) \times 10^{-5}$
2	$(6.3 \pm 0.3) \times 10^{-5}$	$(1.33 \pm 0.06) \times 10^{-5}$	$(6.8 \pm 3.0) \times 10^{-6}$	$(4.5 \pm 0.5) \times 10^{-6}$
3	$(2.2 \pm 0.2) \times 10^{-5}$	$(4.1 \pm 0.3) \times 10^{-6}$	–	$(1.1 \pm 0.2) \times 10^{-6}$
4	$(1.4 \pm 0.2) \times 10^{-5}$	$(2.2 \pm 0.4) \times 10^{-6}$	–	$(5.0 \pm 3.0) \times 10^{-7}$

(1) really holds for the glueball operators, can only be decided by a Monte Carlo experiment.

The main part of our Monte Carlo data was taken on 8^4 lattice at $\beta = 2.30$ for the standard Wilson action. We took the continuous SU(2) gauge group and used the heat bath updating [6]. After every 4th sweep measurements of the 2×2 loop correlations were performed taking subsequently all four directions as the time. The final statistics amounts to 30000 measurements. (It took about 60 h on the IBM 3081D at DESY.) Besides the $\boldsymbol{p} = 0$ time-slices also the three lowest momenta were projected out, belonging to $(k_1, k_2, k_3) = (1,0,0); (0,1,0)$ and $(0,0,1)$, respectively. Besides the symmetric combination of the three orientations j in eq. (12) (corresponding to the 0^+ state) we also considered the combination with weights (2, −1, −1) giving the spin–parity 2^+ state [4]. This is presumably the lowest excited glueball state [7]. The final results for the correlation functions obtained by the multihit procedure are summarized in table 1.

For the error estimate we divided the data, as usual, in bins of variable length and determined the correlation (10), (11) within the bins. The result in one bin was considered as one measured value for the correlation. The error was then estimated from the set of these numbers. In case of sufficiently large numbers of bins we also used the programs of Whitmer [8] for the error estimate, although the simple way based on the standard deviation estimate gave, in general, similar errors.

An important point is, that for a given time-distance there is always a minimum bin length which is needed for a reasonable result. Below this the statistics within the bin is insufficient for the accurate cancellation of the two large terms in (11). It turns out that for too short bins the obtained value of the correlation, which has to be finally positive because of the positivity of the transfer matrix [9], has a tendency to be negative. This is, in fact, to be expected as the expectation value of the time-slices occurring in the second term of (11) is averaged over the time before taking the product (i.e. the square of the average). Once the bin length approaches the minimum acceptable value, the negative correlation values disappear and the result tends to the true positive value from below. Since the saturation occurs earlier for the time slices with smaller distances, the result is an apparently higher mass if the statistics is insufficient. The actual behaviour for different time distances is shown in fig. 2, where the absolute values of the measured correlations are given as a function of the bin length. (For large bin length this does, of course, not matter since all the values are already positive.) As can be seen, the slope of the initial pieces of the curves is roughly consistent with an $1/\sqrt{N}$ decrease, although there are also deviations. Fig. 2 also shows, that our 30000 sweeps are roughly the minimum on statistics which is needed to obtain the values of the 0^+ correlation at all time distances on the 8^4 lattice.

Since the multihit procedure was used also for $d = 0$ and 1, the corresponding values in table 1 are not the true correlations. Multihit links can only be used for non-interacting links and some of the loops with $d = 0$ and 1 do interact. In the glueball mass calculations we are, however, interested anyway in large distance correlations, where the use of multihit links is not restricted. Taking the values with $d = 2,3,4$ in table 1, our estimate for the 0^+ glueball mass is, from the $\boldsymbol{p} = 0$ correlations, $am_{0^+} = 1.05 \pm 0.09$. The energy value from the $\boldsymbol{p} = 1$ correlation is $aE_1 = 1.20 \pm 0.07$.

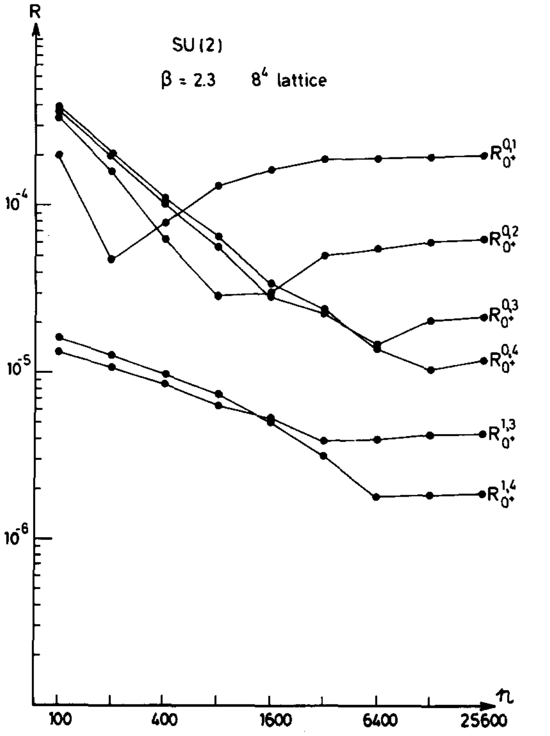

Fig. 2. The absolute value R of the 2 × 2 loop correlation (11) measured with variable bin length n on an 8^4 lattice at $\beta = 2.3$ $R^{p,d}_{J^\sigma}$ denotes the absolute value of the correlation for momentum p and spin–parity J^σ at distance d.

Assuming Lorentz-invariant energy–momentum dispersion, this gives $am_{0^+} = 0.91 \pm 0.09$, and the combined value from $\boldsymbol{p} = 0$ and 1 is

$$am_{0^+} = 0.98 \pm 0.09 \, . \tag{13}$$

Using the two-loop β-function, this gives

$$m_{0^+} = (166 \pm 15)\Lambda_{\mathrm{latt}} = (1090 \pm 170)\ \mathrm{MeV} \, . \tag{14}$$

In the last step we used a recent high precision string tension measurement [10], which gave at $\beta = 2.3$, $\sqrt{\kappa} = (63 \pm 4)\ \Lambda_{\mathrm{latt}}$. For the value of $\sqrt{\kappa}$ we took $\sqrt{\kappa} = 420$ MeV. (Note that the number extracted at $\beta = 2.6$ in ref. [10] is lower, namely $\sqrt{\kappa} = (56 \pm 3)\Lambda_{\mathrm{latt}}$. This is due to the overshooting of asymptotic scaling between $\beta = 2.3$ and $\beta = 2.6$.)

The value for am_{0^+} in eq. (13) is somewhat (15–20%) lower than the latest results [7,11] which were obtained on similar lattices and from similar statistics but not using the multihit technique. As can be seen from table 1, for the $2^+, \boldsymbol{p} = 0$ state we were unable to determine the correlation function beyond distance $d = 2$. A subjective estimate from the $2^+, \boldsymbol{p} = 0$ data could, however, be $am_{2^+} = 1.7$–2.3. In the case of $\boldsymbol{p} = 1$ the correlation in the 2^+ channel is well determined, but the energy is getting at larger distances obviously too low. In fact, the lattice symmetry at $\boldsymbol{p} \neq 0$ is smaller than for $\boldsymbol{p} = 0$, therefore the different spins are mixed up [12]. This can be well observed in our results since the energies obtained from the largest distances are already almost as small as for $0^+, \boldsymbol{p} = 1$. Therefore we think at present $\boldsymbol{p} \neq 0$ states cannot be used for the excited glueballs. This casts doubt on the reliability of excited glueball mass values calculated, for instance, in ref. [11].

Besides the $\beta = 2,3$ data on 8^4 we also obtained, by the same method, data at $\beta = 2.5$ on a 12^4 lattice. This has nearly the same physical size as the 8^4 lattice at $\beta = 2,3$. We collected altogether 12000 measurements (in about 150 CPU hours on the Siemens 7.882 computer of the University of Hamburg), but at the end it turned out that this statistics is perhaps a factor 2–4 too low. In spite of this, we briefly present also these results, because we think that it could reflect some of the problems of the earlier calculations. In this case a direct comparison of the multihit technique with the conventional method was also done by performing 6400 measurements without multihit. The obtained values of the 0^+ correlation functions are given in table 2. The mass value one can extract from distances $d = 2$ and 3 is, say $am_{0^+} = 1.30 \pm 0.20$. Within the errors, this is consistent with the numbers given in refs. [7,11], but it is obviously inconsistent with the $\beta = 2.3$ result eq. (13) and scaling. In fact, using the scale factor $\xi = 1.74$ determined in ref. [10] from the quark–antiquark potential, scaling would re-

Table 2
The measured correlations at $\beta = 2.5$.

d	$J^\sigma, \boldsymbol{p}$	
	$0^+, 0$	$0^+, 1$
0	$(1.47 \pm 0.01) \times 10^{-4}$	$(6.33 \pm 0.01) \times 10^{-5}$
1	$(4.27 \pm 0.01) \times 10^{-5}$	$(1.73 \pm 0.01) \times 10^{-5}$
2	$(1.1 \pm 0.1) \times 10^{-5}$	$(3.82 \pm 0.02) \times 10^{-6}$
3	$(3.1 \pm 0.5) \times 10^{-6}$	$(8.2 \pm 0.6) \times 10^{-7}$
4	$(2.6 \pm 5.2) \times 10^{-7}$	$(1.5 \pm 3.9) \times 10^{-7}$

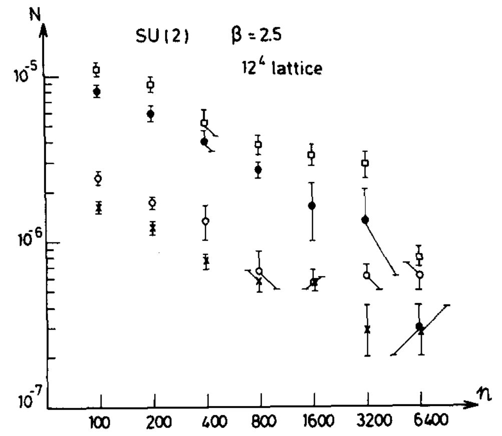

Fig. 3. The statistical noise N for different bin lengths n, as measured at $\beta = 2.5$ by the average absolute value of the apparent correlation at distance 4 and 5. Open squares: $p = 0$ without multihit, full circles: $p = 0$ with multihit, open circles: $p = 1$ without multihit, and crosses: $p = 1$ with multihit.

quire at $\beta = 2.5$ $am_{0+} \approx 0.56$. There are two possible explanations of this failure. First, the physical distance corresponding to $d = 3$ at $\beta = 2.5$ is substantially (by a factor of 2.3) smaller than $d = 4$ at $\beta = 2.3$. The mass determined at the same physical distance at $\beta = 2.5$ can be considerably lower. Second, as explained in detail in connection with fig. 2, insufficient statistics leads also to a considerably higher mass. (As an example, if at $\beta = 2.3$ a bin length of 6000 is chosen, the mass extracted form $d = 3$ and 2 is $am_{0+} = 1.4$, with a bin length of 4000 $am_{0+} = 1.8$!) In our opinion, at such high β-values substantially higher statistics is needed in order to have the appropriate bin length for the determination of large distance correlations.

For the comparison of the statistical noise with and without multihit we plotted in fig. 3 the average at distance $d = 4$ and 5 of the absolute value of the apparent 0^+, $p = 0$ and 1 correlations for $\beta = 2.5$. As it can be seen, the noise with multihit is roughly a factor 2 smaller (corresponding to a gain of about 4 in statistics). A similar comparison at $\beta = 2.3$, with smaller statistics, gave at $\beta = 2.3$ an even larger factor of about 3–4 (corresponding to about a factor 10 gain in computer time).

In conclusion, the multihit technique is rather useful for the determination of glueball correlations, but still further efforts are needed in order to improve the glueball mass values obtained from lattice Monte Carlo calculations.

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References

[1] B. Berg, A. Billoire and C. Rebbi, Ann. Phys. (NY) 142 (1982) 185; 146 (1983) 470 (Addendum).
[2] M. Falcioni, E. Marinari, M.L. Paciello, G. Parisi, F. Rapuano, B. Taglienti and Zhang Yi Chang, Phys. Lett. 110B (1982) 295.
[3] K. Ishikawa, G. Schierholz and M. Teper, Phys. Lett. 110B (1982) 399.
[4] B. Berg, Cargèse lecture (1983); preprint DESY 84-012 (1984).
[5] G. Parisi, Les Houches lecture (1983); Frascati preprint, LNF-83-36(P) (1983);
G. Parisi, R. Petronzio and F. Rapuano, Phys. Lett. 128B (1983) 418.
[6] M. Creutz, Phys. Rev. D21 (1980) 2308.
[7] B. Berg, A. Billoire, S. Meyer and C. Panagiotakopoulos, Saclay preprint PhT 84-24 (1984).
[8] C. Whitmer, Princeton preprint (1983).
[9] M. Lüscher, Commun. Math. Phys. 54 (1977) 283.
[10] F. Gutbrod and I. Montvay, Phys. Lett. 136B (1984) 411.
[11] G. Schierholz and M. Teper, Phys. Lett. 136B (1984) 64; 69.
[12] N. Kimura and A. Ukawa, Nucl. Phys. B205 [FS5] (1982) 637.