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MONOPOLE-FERMION AND DYON-FERMION BOUND STATES (I). General properties and numerical results

Per OSLAND

Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg 52, Germany

Tai Tsun WU[1]

Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138, USA

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General properties of monopole-fermion and dyon-fermion bound states are investigated, for the case of Dirac monopoles and their dyon analogues. The attractive forces that lead to bound states are due to the interaction of a fermion anomalous magnetic moment with the monopole magnetic field and the additional Coulomb interaction in the dyon case. Detailed numerical results are presented.

1. Introduction

It was shown several years ago by Kazama and Yang [1, 2] that fermions can bind to magnetic monopoles. Because of rotational symmetry as expressed by monopole harmonics [3], the investigation of such bound states reduces to the study of the radial equations. These radial equations are especially simple for the states of lowest angular momentum; they consist of two coupled first-order differential equations instead of four. These states of lowest angular momentum have been investigated in some detail [1, 4], both analytically and numerically.

It is the purpose of the present papers to extend this work in two directions, both to higher angular momenta and to the case of a dyon instead of a monopole. The monopole-fermion system can be considered a special limit of a dyon-fermion system, which for some range of the parameters also possesses bound states. The hamiltonian we wish to study is thus [5]

$$H=\boldsymbol{\alpha}\cdot(\boldsymbol{p}-Ze\boldsymbol{A})+\beta M-\frac{\zeta}{r}-\kappa q\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}/(2Mr^3)\,. \tag{1.1}$$

The dyon is taken to be infinitely heavy, it has a magnetic charge g and an electric charge $Z_d e$. The fermion has a mass M, electric charge Ze, and an anomalous magnetic moment κ. The various charges only enter in the combinations (we use

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gaussian units, i.e., $E = Ze/r^2$, $B = g/r^2$, $e^2 = \alpha$)

$$q = Zeg\,,$$
$$-\zeta = ZZ_{\rm d}e^2 = ZZ_{\rm d}\alpha\,. \tag{1.2}$$

We note that the Dirac condition [6] restricts q:

$$q = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots\,. \tag{1.3}$$

With dyons, the corresponding condition is [7]

$$e_i g_j - e_j g_i = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots, \tag{1.4}$$

which does not restrict ζ, since the fermion is assumed to carry no magnetic charge.

In this paper we investigate the general case of dyon-fermion bound states through series expansions on the basis of the hamiltonian (1.1). These series expansions are used to obtain highly accurate numerical results, which give an overview of the properties of the bound states and are also used to ascertain the accuracy of the approximate results developed in papers II [8], III [9], IV [10], V [11] and VI [12].

In these five papers, two limiting cases are investigated where the eigenvalues and wave functions are determined approximately by analytic methods. Papers II, III and IV deal with the case of the lowest angular momentum $j = |q| - \frac{1}{2}$. For this lowest angular momentum, the wave functions do not depend on κ and $|q|$ separately, but only on the product $\kappa|q|$. For these states, the two limiting cases studied are those of weak binding (paper II) and of large values of $\kappa|q|$ (papers III and IV). The treatment of weak binding is then extended in papers V and VI to higher angular-momentum states.

Unless the Coulomb interaction is repulsive, there is an infinite number of states to which the results of each of the papers II–VI apply. The case of weak binding was previously treated in ref. [4] for the lowest angular momentum with $\zeta = 0$, where the wave function can be expressed in terms of Bessel functions. In paper II, the states of lowest angular momentum are studied with $\zeta \neq 0$, and it is found that confluent hypergeometric functions are needed. The further extension to higher angular momenta, with four coupled radial equations, is much more involved. Papers V and VI treat respectively the monopole case ($\zeta = 0$) and the dyon case ($\zeta \neq 0$). In addition to the Bessel and confluent hypergeometric functions, the solution of a fourth-order ordinary differential equation is needed. That solution can fortunately be expressed explicitly in terms of integrals of a product of a Bessel function and a hypergeometric function. For the dyon-fermion bound states (papers II and VI), the analytic methods yield a transcendental equation for the binding energy, whereas, for the monopole-fermion bound states (paper V), an explicit expression for the binding energy is obtained.

2. Eigenvalue equations

For the bound state wave functions, we basically follow the notation of Kazama and Yang [1];
type B, $j=|q|-\frac{1}{2}$:

$$\psi(\boldsymbol{r})=\frac{1}{r}\begin{bmatrix}\frac{\kappa q}{|\kappa q|}F(r)\eta_{jj_z}(\hat{\boldsymbol{r}})\\ -iG(r)\eta_{jj_z}(\hat{\boldsymbol{r}})\end{bmatrix}, \tag{2.1}$$

type A, $j\geq|q|+\frac{1}{2}$:

$$\psi(\boldsymbol{r})=\frac{1}{r}\begin{bmatrix}h_1(r)\xi^{(1)}_{jj_z}(\hat{\boldsymbol{r}})+\frac{q}{|q|}h_2(r)\xi^{(2)}_{jj_z}(\hat{\boldsymbol{r}})\\ -i\frac{\kappa}{|\kappa|}\left[h_3(r)\xi^{(2)}_{jj_z}(\hat{\boldsymbol{r}})+\frac{q}{|q|}h_4(r)\xi^{(1)}_{jj_z}(\hat{\boldsymbol{r}})\right]\end{bmatrix}. \tag{2.2}$$

It should be noted that, in the notation of (2.2),

$$\begin{bmatrix}h_1\\h_2\\h_4\\h_3\end{bmatrix}$$

is the h of Kazama and Yang [1].

The energy eigenvalues E are given by

$$H\psi=E\psi. \tag{2.3}$$

Since the Coulomb interaction part of the hamiltonian is radially symmetric, and proportional to the unit matrix, the derivation of the radial equations may be taken over from refs. [1, 13] by substituting

$$E\to E+\frac{\zeta}{r}. \tag{2.4}$$

In terms of the notation [1]

$$A=\tfrac{1}{2}\kappa|q|,\qquad B=\tfrac{1}{2}\kappa|q|\frac{E}{M}, \tag{2.5}$$

$$\rho=\frac{2M}{|\kappa q|}r, \tag{2.6}$$

the replacement (2.4) is

$$B\to B+\frac{\tilde{\zeta}}{\rho}, \tag{2.7}$$

where

$$\tilde{\zeta}=\frac{\kappa}{|\kappa|}\zeta. \tag{2.8}$$

With this replacement (2.7), the radial equations for the bound states of types B and A are respectively (see eqs. (20) and (9′) of [1])

$j=|q|-\frac{1}{2}$:

$$\frac{\mathrm{d}G}{\mathrm{d}\rho}=\left(A-B-\frac{\tilde{\zeta}}{\rho}-\frac{1}{\rho^2}\right)F,$$

$$\frac{\mathrm{d}F}{\mathrm{d}\rho}=\left(A+B+\frac{\tilde{\zeta}}{\rho}-\frac{1}{\rho^2}\right)G, \tag{2.9}$$

$j\geqslant|q|+\frac{1}{2}$:

$$\begin{bmatrix} \mathrm{d}/\mathrm{d}\rho-\mu/\rho & 0 & A+B+\tilde{\zeta}/\rho & 1/\rho^2 \\ 0 & \mathrm{d}/\mathrm{d}\rho+\mu/\rho & 1/\rho^2 & A+B+\tilde{\zeta}/\rho \\ A-B-\tilde{\zeta}/\rho & 1/\rho^2 & \mathrm{d}/\mathrm{d}\rho+\mu/\rho & 0 \\ 1/\rho^2 & A-B-\tilde{\zeta}/\rho & 0 & \mathrm{d}/\mathrm{d}\rho-\mu/\rho \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}=0, \tag{2.10}$$

where

$$\mu=[(j+\tfrac{1}{2})^2-q^2]^{1/2}. \tag{2.11}$$

In both (2.9) and (2.10), the sign of q does not appear. When the sign of κ is reversed, so are those of A, B and $\tilde{\zeta}$ by (2.5) and (2.8). Accordingly, eq. (2.10) has the further symmetry that it is invariant under

$$\kappa\to-\kappa,\quad h_2\to-h_2,\quad h_3\to-h_3. \tag{2.12}$$

Therefore, for states with $j\geqslant|q|+\frac{1}{2}$ but not for those with $j=|q|-\frac{1}{2}$, for any ζ the binding energy is independent of the sign of κ, and in fact the sign of κ does not enter in any essential way.

For $\zeta=0$, $E=0$ is an eigenvalue of (2.9) when $\kappa>0$ and also of (2.10) when $\kappa\neq0$ [1]. In the case of eq. (2.9) the eigenfunctions were given explicitly in ref. [1]:

$$F=-G=\mathrm{e}^{-|\kappa q|/2Mr-Mr}. \tag{2.13}$$

Similarly, for eq. (2.10), the eigenfunctions are

$$h_1=-h_4=\sqrt{r}\,\mathrm{e}^{-|\kappa q|/2Mr}K_{\mu-1/2}(Mr),$$

$$h_2=-h_3=\frac{-\kappa}{|\kappa|}\sqrt{r}\,\mathrm{e}^{-|\kappa q|/2Mr}K_{\mu+1/2}(Mr). \tag{2.14}$$

3. Series expansions for bound states of type B

For an arbitrary value of B, i.e., for E not necessarily equal to the bound-state energy, let v be the solution of (2.9) that is bounded at the origin, and similarly w

the one that is bounded at infinity. Their normalizations remain to be chosen. They are both two-component radial wave functions. Thus

$$Dv = 0, \qquad Dw = 0\,, \tag{3.1}$$

where

$$D = \begin{bmatrix} d/d\rho & -A-B-\tilde{\zeta}/\rho+1/\rho^2 \\ -A+B+\tilde{\zeta}/\rho+1/\rho^2 & d/d\rho \end{bmatrix}. \tag{3.2}$$

Let us first consider v. The behaviour at the origin is not affected in any essential way by the Coulomb interaction. Thus, up to possibly a power,

$$v \sim e^{-1/\rho} \quad \text{as} \quad \rho \to 0\,. \tag{3.3}$$

In order to work with a function that can be expanded as a series around the origin, let us define

$$v = e^{-1/\rho}\bar{v}, \qquad \bar{D} = e^{1/\rho} D\, e^{-1/\rho}\,. \tag{3.4}$$

The equation to be solved is then

$$\bar{D}\bar{v} = 0, \tag{3.5}$$

where

$$\bar{D} = \begin{bmatrix} d/d\rho + 1/\rho^2 & -A-B-\tilde{\zeta}/\rho+1/\rho^2 \\ -A+B+\tilde{\zeta}/\rho+1/\rho^2 & d/d\rho + 1/\rho^2 \end{bmatrix}. \tag{3.6}$$

We make the series ansatz

$$\bar{v} = \rho^{\alpha} \sum_{n=0} c^{(n)} \rho^n\,, \tag{3.7}$$

and note that the most singular terms in $\bar{D}$ annihilate

$$c^{(0)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{3.8}$$

This is just a rephrasing of the statement that [1]

$$\lim_{\rho \to 0} (F/G) = -1\,. \tag{3.9}$$

The second most singular terms in eq. (3.6) determine the leading exponent

$$\alpha = 0\,. \tag{3.10}$$

In order to find the recurrence formulas, it is convenient to perform a rotation,

$$c^{(n)} = R b^{(n)}\,, \tag{3.11}$$

with

$$R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{3.12}$$

The two-component vectors

$$b^{(n)}=\begin{bmatrix} b_1^{(n)} \\ b_2^{(n)} \end{bmatrix} \tag{3.13}$$

are given by

$$\begin{aligned} b_1^{(n)} &= \tfrac{1}{2}\{-(n-1)b_1^{(n-1)}+\tilde{\zeta}b_2^{(n-1)}+Ab_1^{(n-2)}+Bb_2^{(n-2)}\}\,, \\ b_2^{(n)} &= -\frac{1}{n}\{\tilde{\zeta}b_1^{(n)}+Bb_1^{(n-1)}+Ab_2^{(n-1)}\}\,, \end{aligned} \tag{3.14}$$

for $n=1,2,\dots$, and with the initial values

$$b^{(0)}=R^{-1}c^{(0)}=\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad b^{(-1)}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.15}$$

The solution v that is bounded at the origin may then be evaluated from eqs. (3.4), (3.7), and (3.11)–(3.15).

We now turn our attention to the solution w that is bounded as $\rho\to\infty$. In other words, we want to solve the radial equation $Dw=0$ for large values of ρ, where D is given by (3.2). Let

$$w=\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{3.16}$$

Then by the change of variables

$$x=\frac{1}{\sqrt{A^2-B^2}\rho}\,, \tag{3.17}$$

$$\hat{w}=\begin{bmatrix} |A+B|^{1/2}w_2 \\ (\kappa/|\kappa|)|A-B|^{1/2}w_1 \end{bmatrix}, \tag{3.18}$$

we find that eq. (3.1) is equivalent to

$$D_\infty\hat{w}=0\,, \tag{3.19}$$

where

$$D_\infty=\begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}x} & -A-B-\left(\dfrac{A+B}{A-B}\right)^{1/2}\dfrac{\zeta}{x}+\dfrac{1}{x^2} \\ -A+B+\left(\dfrac{A-B}{A+B}\right)^{1/2}\dfrac{\zeta}{x}+\dfrac{1}{x^2} & \dfrac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix}. \tag{3.20}$$

For the monopole-fermion case of lowest angular momentum there are no excited bound states unless $A>\frac{1}{8}$ [1]. Here, for the case of dyons, A can be positive or negative. In formulating eqs. (3.18)–(3.20) we have made use of the fact that

$$\frac{A+B}{|A+B|}=\frac{A-B}{|A-B|}=\frac{\kappa}{|\kappa|}\,. \tag{3.21}$$

We note that D_∞ involves ζ, not $\tilde{\zeta}$ (compare eq. (2.8)).

The operator (3.20) has the same form as (3.2), the only difference being that the two Coulomb terms have different coefficients. Therefore, with

$$\hat{w} = e^{-1/x}\hat{\hat{w}}, \tag{3.22}$$

$$\hat{\hat{w}} = x^{\alpha} \sum_{n=0} C^{(n)} x^n, \tag{3.23}$$

we find the leading power to be

$$\alpha = -\zeta \frac{B}{\sqrt{A^2 - B^2}}. \tag{3.24}$$

The power becomes very large when the binding gets weaker, $B \to A-$.

With the same rotation as previously,

$$C^{(n)} = RB^{(n)}, \tag{3.25}$$

we find the components of $B^{(n)}$ to be given by the recursion relations

$$B_1^{(n)} = \tfrac{1}{2}\{-(n-1+2\alpha)B_1^{(n-1)} + \bar{\zeta}B_2^{(n-1)} + AB_1^{(n-2)} + BB_2^{(n-2)}\},$$

$$B_2^{(n)} = -\frac{1}{n}\{\bar{\zeta}B_1^{(n)} + BB_1^{(n-1)} + AB_2^{(n-1)}\}, \tag{3.26}$$

where

$$\bar{\zeta} = \zeta \frac{A}{\sqrt{A^2 - B^2}}, \tag{3.27}$$

and with the initial values of eq. (3.15).

The desired solution w that is bounded as $\rho \to \infty$ is then given by eqs. (3.16), (3.18) and (3.22)–(3.26), with x defined by eq. (3.17).

For large n, the coefficients grow such that $|b_i^{(n)}|/|b_i^{(n-1)}|$ and $|B_i^{(n)}|/|B_i^{(n-1)}| \sim \frac{1}{2}n$. The series are thus asymptotic, not convergent. This is not surprising, since we expand around essential singularities.

4. Series expansions for bound states of type A

For $j \geq |q| + \frac{1}{2}$ there are four radial wave functions obeying the coupled differential equations (2.10). These equations have two linearly independent solutions that are bounded at small ρ; we denote them f and $\tilde{f}$. Likewise, there are two linearly independent solutions that are bounded as $\rho \to \infty$; we denote those g and $\tilde{g}$.

To avoid notation that is excessively cumbersome we shall use the same letter to designate corresponding quantities for the two cases of type A and type B. For example, D means the 2×2 differential operator of (3.2) in the preceding section but the 4×4 operator of (2.10) in this section. Some further such examples are $\bar{D}$, $c^{(n)}$, $b^{(n)}$, $C^{(n)}$, $B^{(n)}$, R, α, $b_1^{(n)}$, $b_2^{(n)}$, $B_1^{(n)}$, and $B_2^{(n)}$.

4.1. EXPANSIONS FOR SMALL ρ

We first consider f and $\tilde{f}$, which are four-component radial wave functions. They satisfy

$$Df=0\,, \qquad D\tilde{f}=0\,. \tag{4.1}$$

The behaviour at the origin is like in the previous case, $f\sim e^{-1/\rho}$, so we define

$$f=e^{-1/\rho}\bar{f}\,, \qquad \tilde{f}=e^{-1/\rho}\bar{\tilde{f}}\,,$$
$$\bar{D}=e^{1/\rho}D\,e^{-1/\rho}\,. \tag{4.2}$$

One thus has to solve

$$\bar{D}\bar{f}=0\,, \qquad \bar{D}\bar{\tilde{f}}=0\,, \tag{4.3}$$

with

$$\bar{D}=\begin{bmatrix} d/d\rho+1/\rho^2-\mu/\rho & 0 & A+B+\zeta/\rho & 1/\rho^2 \\ 0 & d/d\rho+1/\rho^2+\mu/\rho & 1/\rho^2 & A+B+\tilde{\zeta}/\rho \\ A-B-\tilde{\zeta}/\rho & 1/\rho^2 & d/d\rho+1/\rho^2+\mu/\rho & 0 \\ 1/\rho^2 & A-B-\zeta/\rho & 0 & d/d\rho+1/\rho^2-\mu/\rho \end{bmatrix}. \tag{4.4}$$

With the ansätze

$$\bar{f}=\rho^{\alpha}\sum_{n=0} c^{(n)}\rho^{n}\,, \qquad \bar{\tilde{f}}=\rho^{\tilde{\alpha}}\sum_{n=0}\tilde{c}^{(n)}\rho^{n}\,, \tag{4.5}$$

we find the most singular terms in $\bar{D}$ to annihilate

$$c^{(0)}=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \qquad \tilde{c}^{(0)}=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}. \tag{4.6}$$

We let f be the series starting with $c^{(0)}$, and $\tilde{f}$ be the one starting with $\tilde{c}^{(0)}$. In the two cases the coefficients of the next power in ρ determine the exponents α and $\tilde{\alpha}$ to be

$$\alpha=\mu\,, \qquad \tilde{\alpha}=-\mu\,, \tag{4.7}$$

for f and $\tilde{f}$, respectively. Even with a negative power, $\tilde{\alpha}<0$, the function $\tilde{f}$ is of course bounded as $\rho\to 0$, because of the exponential factor (cf. eq. (4.2)).

When $\alpha-\tilde{\alpha}=2\mu=$ integer, the ansätze (4.5) are not adequate. For $\bar{\tilde{f}}$ a logarithmic term is then required. We shall return to these special cases after having presented the solutions for the generic case where 2μ is not an integer.

Let us then proceed to determine the solution $\bar{f}$. Like in the two-equation case, it is convenient to rotate,

$$c^{(n)}=Rb^{(n)}\,, \tag{4.8}$$

with

$$R=\begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}. \tag{4.9}$$

The vectors $b^{(n)}$ are then found to be given by

$$\begin{bmatrix}b_1^{(n)}\\ b_2^{(n)}\end{bmatrix}=-\tfrac{1}{2}\left\{\begin{bmatrix}n-1 & 0\\ 0 & n-1+2\mu\end{bmatrix}\begin{bmatrix}b_1^{(n-1)}\\ b_2^{(n-1)}\end{bmatrix}+\tilde{\zeta}\begin{bmatrix}b_3^{(n-1)}\\ b_4^{(n-1)}\end{bmatrix}\right.$$
$$\left.+A\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}b_1^{(n-2)}\\ b_2^{(n-2)}\end{bmatrix}+B\begin{bmatrix}b_3^{(n-2)}\\ b_4^{(n-2)}\end{bmatrix}\right\}, \tag{4.10a}$$

$$\begin{bmatrix}n+2\mu & 0\\ 0 & n\end{bmatrix}\begin{bmatrix}b_3^{(n)}\\ b_4^{(n)}\end{bmatrix}=\tilde{\zeta}\begin{bmatrix}b_1^{(n)}\\ b_2^{(n)}\end{bmatrix}+B\begin{bmatrix}b_1^{(n-1)}\\ b_2^{(n-1)}\end{bmatrix}$$
$$+A\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}b_3^{(n-1)}\\ b_4^{(n-1)}\end{bmatrix}, \tag{4.10b}$$

for $n=1,2,\ldots,$ and with the initial values

$$b^{(0)}=R^{-1}c^{(0)}=\begin{bmatrix}0\\ 0\\ 0\\ -1\end{bmatrix},\qquad b^{(-1)}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}. \tag{4.11}$$

The other solution that is bounded near the origin, $\tilde{f}$, is most easily obtained by noting that eqs. (2.10) are invariant under the interchanges

$$(h_1\leftrightarrow h_2)\,,\qquad (h_3\leftrightarrow h_4)\,,\qquad (\mu\leftrightarrow -\mu)\,. \tag{4.12}$$

In particular, this transforms

$$(c^{(0)}\leftrightarrow \tilde{c}^{(0)})\,,\qquad (\alpha\leftrightarrow\tilde{\alpha})\,. \tag{4.13}$$

Thus, $\tilde{f}$ is obtained from f by taking $\mu\to-\mu$, and interchanging the upper two and the lower two components among themselves.

It remains to consider the exceptional case where $\alpha-\tilde{\alpha}$ is a positive integer, i.e., where

$$2\mu=[(2j+1)^2-(2q)^2]^{1/2}=\text{positive integer}\,. \tag{4.14}$$

Since the integers $2|q|$ and $2j+1$ are either both odd or both even, (4.14) actually implies that μ itself is a positive integer. There are two types of solutions: they are

$$\mu = n_1 n_2 n_3 ,\quad 2|q| = (n_1^2 - n_2^2) n_3 ,\quad 2j+1 = (n_1^2 + n_2^2) n_3 ; \tag{4.15a}$$

$$\mu = (n_1^2 - n_2^2) n_3 ,\quad 2|q| = 4 n_1 n_2 n_3 ,\quad 2j+1 = 2(n_1^2 + n_2^2) n_3 , \tag{4.15b}$$

where n_1, n_2 and n_3 are three positive integers such that $n_1 > n_2$ and $n_1 + n_2$ is odd. It is seen from (4.15) that

(i) $2|q|$ can take all positive integer values except $2|q| = 1, 2, 4$;

(ii) μ cannot be equal to 1 but can take all other positive integer values.

In these exceptional cases, the ansatz (4.5) for $\check{f}$ has to be replaced by

$$\check{f} = \beta \tilde{f} \log \rho + \rho^{-\mu} \sum_{n=0} \check{c}^{(n)} \rho^n , \tag{4.16}$$

with $\check{c}^{(0)} = \tilde{c}^{(0)}$. The constant β is to be determined from the recursion relations.

With the rotation

$$\check{c}^{(n)} = R \check{b}^{(n)} , \tag{4.17}$$

we find that the components of the vectors $\check{b}^{(n)}$ satisfy

$$\begin{bmatrix} \check{b}_1^{(n)} \\ \check{b}_2^{(n)} \end{bmatrix} = -\tfrac{1}{2} \left\{ \begin{bmatrix} n-1-2\mu & 0 \\ 0 & n-1 \end{bmatrix} \begin{bmatrix} \check{b}_1^{(n-1)} \\ \check{b}_2^{(n-1)} \end{bmatrix} + \tilde{\zeta} \begin{bmatrix} \check{b}_3^{(n-1)} \\ \check{b}_4^{(n-1)} \end{bmatrix} + A \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \check{b}_1^{(n-2)} \\ \check{b}_2^{(n-2)} \end{bmatrix} + B \begin{bmatrix} \check{b}_3^{(n-2)} \\ \check{b}_4^{(n-2)} \end{bmatrix} + \beta \begin{bmatrix} b_1^{(n-1-2\mu)} \\ b_2^{(n-1-2\mu)} \end{bmatrix} \right\} , \tag{4.18a}$$

$$\begin{bmatrix} n & 0 \\ 0 & n-2\mu \end{bmatrix} \begin{bmatrix} \check{b}_3^{(n)} \\ \check{b}_4^{(n)} \end{bmatrix} = \tilde{\zeta} \begin{bmatrix} \check{b}_1^{(n)} \\ \check{b}_2^{(n)} \end{bmatrix} + B \begin{bmatrix} \check{b}_1^{(n-1)} \\ \check{b}_2^{(n-1)} \end{bmatrix} + A \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \check{b}_3^{(n-1)} \\ \check{b}_4^{(n-1)} \end{bmatrix} - \beta \begin{bmatrix} b_3^{(n-2\mu)} \\ b_4^{(n-2\mu)} \end{bmatrix} , \tag{4.18b}$$

for $n = 1, 2, \ldots, 2\mu - 1,\ 2\mu,\ 2\mu + 1, \ldots$, with the initial values

$$\check{b}^{(0)} = R^{-1} \check{c}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} , \qquad \check{b}^{(-1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{4.19}$$

Eqs. (4.18) determine $\check{b}^{(n)}$ recursively except $\check{b}_4^{(2\mu)}$. It is convenient to choose

$$\check{b}_4^{(2\mu)} = 0 . \tag{4.20}$$

With or without this choice, the lower component of (4.18b) with $n=2\mu$ gives

$$\beta=-\tilde{\zeta}\check{b}_2^{(2\mu)}-B\check{b}_2^{(2\mu-1)}-A\check{b}_3^{(2\mu-1)}, \tag{4.21}$$

where we have used $b_4^{(0)}=-1$ from (4.11).

As an example, the lowest values of q and j that lead to an exceptional case are $|q|=\frac{3}{2}$ and $j=2$ with $\mu=2$. In this case

$$b_4^{(0)}=-\tfrac{1}{3}AB\tilde{\zeta},$$

which is non-zero when $\zeta\neq 0$.

4.2. EXPANSIONS FOR LARGE ρ

Let us now turn to the solutions g and $\tilde{g}$ that are bounded as $\rho\to\infty$. These obey

$$Dg=0\,,\qquad D\tilde{g}=0\,. \tag{4.22}$$

The procedure to be followed here is a generalization of that of sect. 3 where $j=|q|-\frac{1}{2}$: we first introduce the variable x of eq. (3.17), then rescale and interchange the components of g by defining

$$\hat{g}=\begin{bmatrix} -|A+B|^{1/2}g_3 \\ -|A+B|^{1/2}g_4 \\ |A-B|^{1/2}\dfrac{\kappa}{|\kappa|}g_2 \\ |A-B|^{1/2}\dfrac{\kappa}{|\kappa|}g_1 \end{bmatrix}. \tag{4.23}$$

With

$$\hat{g}=e^{-1/x}\bar{g}, \tag{4.24}$$

the radial differential equations take the form

$$\bar{D}_\infty\bar{g}=0\,, \tag{4.25}$$

where

$$\bar{D}_\infty=\begin{bmatrix} \dfrac{d}{dx}+\dfrac{1}{x^2}-\dfrac{\mu}{x} & 0 & A+B & \dfrac{1}{x^2}-\left(\dfrac{A+B}{A-B}\right)^{1/2}\dfrac{\zeta}{x} \\ 0 & \dfrac{d}{dx}+\dfrac{1}{x^2}+\dfrac{\mu}{x} & \dfrac{1}{x^2}-\left(\dfrac{A+B}{A-B}\right)^{1/2}\dfrac{\zeta}{x} & A+B \\ A-B & \dfrac{1}{x^2}+\left(\dfrac{A-B}{A+B}\right)^{1/2}\dfrac{\zeta}{x} & \dfrac{d}{dx}+\dfrac{1}{x^2}-\dfrac{\mu}{x} & 0 \\ \dfrac{1}{x^2}+\left(\dfrac{A-B}{A+B}\right)^{1/2}\dfrac{\zeta}{x} & A-B & 0 & \dfrac{d}{dx}+\dfrac{1}{x^2}+\dfrac{\mu}{x} \end{bmatrix}. \tag{4.26}$$

Similar to eq. (3.21), the coefficients of the ζ-terms here are also different in the upper two and lower two equations. Further, we note that, in contrast to the $\bar{D}$ of eq. (4.4), the ζ-terms have become detached from the $A\pm B$ terms. This is due to the inversion and rescaling, but does not occur for $j=|q|-\frac{1}{2}$, since in that case the $A\pm B$ terms, the ζ/ρ and $1/\rho^2$ terms all multiply the same function.

With the ansätze

$$\bar{g}=x^{\alpha}\sum_{n=0} C^{(n)}x^{n}, \qquad \bar{\tilde{g}}=x^{\tilde{\alpha}}\sum_{n=0} \tilde{C}^{(n)}x^{n}, \tag{4.27}$$

we find the most singular terms in $\bar{D}_{\infty}$ annihilate

$$C^{(0)}=c^{(0)}, \qquad \tilde{C}^{(0)}=\tilde{c}^{(0)}, \tag{4.28}$$

where $c^{(0)}$ and $\tilde{c}^{(0)}$ are defined by eq. (4.6). In analogy with the $j=|q|-\frac{1}{2}$ case we find

$$\alpha=\tilde{\alpha}=\frac{-B}{\sqrt{A^2-B^2}}\zeta, \tag{4.29}$$

independent of μ.

The recursive determination of the coefficients is now straightforward. With the rotation (cf. eqs. (4.8), (4.9))

$$C^{(n)}=RB^{(n)}, \tag{4.30}$$

we find the vector coefficients $B^{(n)}$ to be given by

$$\begin{aligned}\begin{bmatrix} B_1^{(n)} \\ B_2^{(n)} \end{bmatrix} = -\tfrac{1}{2}\Bigg\{&\left(n-1-\frac{2B}{\sqrt{A^2-B^2}}\zeta\right)\begin{bmatrix} B_1^{(n-1)} \\ B_2^{(n-1)} \end{bmatrix}+\begin{bmatrix} 0 & \mu-\bar{\zeta} \\ -\mu-\bar{\zeta} & 0 \end{bmatrix}\begin{bmatrix} B_3^{(n-1)} \\ B_4^{(n-1)} \end{bmatrix} \\ &+A\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} B_1^{(n-2)} \\ B_2^{(n-2)} \end{bmatrix}+B\begin{bmatrix} B_3^{(n-2)} \\ B_4^{(n-2)} \end{bmatrix}\Bigg\},\end{aligned} \tag{4.31a}$$

$$\begin{aligned}\begin{bmatrix} B_3^{(n)} \\ B_4^{(n)} \end{bmatrix} = \frac{1}{n}\Bigg\{&\begin{bmatrix} 0 & \mu-\bar{\zeta} \\ -\mu-\bar{\zeta} & 0 \end{bmatrix}\begin{bmatrix} B_1^{(n)} \\ B_2^{(n)} \end{bmatrix}+B\begin{bmatrix} B_1^{(n-1)} \\ B_2^{(n-1)} \end{bmatrix} \\ &+A\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} B_3^{(n-1)} \\ B_4^{(n-1)} \end{bmatrix}\Bigg\}.\end{aligned} \tag{4.31b}$$

In order to construct g we first determine $\hat{g}$ from eqs. (4.24), (4.27), (4.30) and (4.31), and then identify the components of g in terms of those of $\hat{g}$ (eq. (4.23)). The other solution, $\tilde{g}$, that is bounded as $\rho\to\infty$ (or $x\to 0$), is obtained from g using the symmetry (4.12).

Similar to the $j=|q|-\frac{1}{2}$ case, the coefficients grow like factorials, and the series are asymptotic.

5. Matching and energy determination

The bound-state wave functions F, G and h_k are bounded as $\rho\to 0$ and as $\rho\to\infty$. Therefore,
for $j=|q|-\frac{1}{2}$:

$$\begin{aligned} F(\rho)&=\alpha_1 v_1(\rho)=\alpha_2 w_1(\rho), \\ G(\rho)&=\alpha_1 v_2(\rho)=\alpha_2 w_2(\rho), \end{aligned} \tag{5.1}$$

and for $j \geq |q| + \frac{1}{2}$:

$$h_k(\rho) = \beta_1 f_k(\rho) + \beta_2 \tilde{f}_k(\rho) = \beta_3 g_k(\rho) + \beta_4 \tilde{g}_k(\rho) , \tag{5.2}$$

for $k = 1, 2, 3, 4$, where the α_i and β_i are real numbers.

By imposing the condition (5.1) or (5.2) at some $\rho = \rho_0$, we get a set of homogeneous equations linear in the coefficients:

$$\alpha_1 v_k(\rho_0) - \alpha_2 w_k(\rho_0) = 0 , \qquad k = 1, 2 , \tag{5.3}$$

$$\beta_1 f_k(\rho_0) + \beta_2 \tilde{f}_k(\rho_0) - \beta_3 g_k(\rho_0) - \beta_4 \tilde{g}_k(\rho_0) = 0 , \qquad k = 1, 2, 3, 4 . \tag{5.4}$$

These equations determine the coefficients (up to an overall constant) and the eigenvalue parameter B. Clearly, the existence of a non-trivial solution requires in the two cases the following determinants to vanish:

$$\begin{vmatrix} v_1(\rho_0) & w_1(\rho_0) \\ v_2(\rho_0) & w_2(\rho_0) \end{vmatrix} = 0 , \tag{5.5}$$

$$\begin{vmatrix} f_1(\rho_0) & \tilde{f}_1(\rho_0) & g_1(\rho_0) & \tilde{g}_1(\rho_0) \\ f_2(\rho_0) & \tilde{f}_2(\rho_0) & g_2(\rho_0) & \tilde{g}_2(\rho_0) \\ f_3(\rho_0) & \tilde{f}_3(\rho_0) & g_3(\rho_0) & \tilde{g}_3(\rho_0) \\ f_4(\rho_0) & \tilde{f}_4(\rho_0) & g_4(\rho_0) & \tilde{g}_4(\rho_0) \end{vmatrix} = 0 . \tag{5.6}$$

From these equations, one may iteratively determine the eigenvalue B in the two cases.

When $\zeta = \mu = 0$, there is a natural choice of the matching point ρ_0 due to symmetry, namely at the geometric mean of the characteristic variables of the two regions, ρ and $[(A^2 - B^2)^{1/2}\rho]^{-1}$:

$$\rho_0 = (A^2 - B^2)^{-1/4} . \tag{5.7}$$

Even when ζ and μ are non-zero, this choice is often adequate.

Since the series expansions found in sects. 3 and 4 are only asymptotic, the small-ρ and large-ρ expansions will not have any overlapping region of validity. The solutions at ρ_0 therefore have to be obtained from a numerical integration of eqs. (2.9) and (2.10), using the series to determine initial values at small and large values of ρ.

6. The $|q| = \frac{1}{2}$ monopole-fermion system

In this and the following two sections we shall give some numerical results on the dyon-fermion bound states. These systems are characterized by three parameters, namely q, κ and ζ. As seen from (2.5) and (2.9), q and κ appear only in the combination $A = \frac{1}{2}\kappa|q|$ for the states of lowest angular momentum $j = |q| - \frac{1}{2}$. For other states with $j \geq |q| + \frac{1}{2}$, as seen from (2.10), all three parameters appear independently. However, as already noted in sect. 2, neither the sign of q nor that of κ enters in any essential way for these states.

In this section we concentrate on the simplest case where $|q| = \frac{1}{2}$ and $\zeta = 0$. In sect. 7, the results are generalized to the case $\zeta \neq 0$ but still with $|q| = \frac{1}{2}$. The cases with $|q| \geq 1$ are briefly considered in sect. 8.

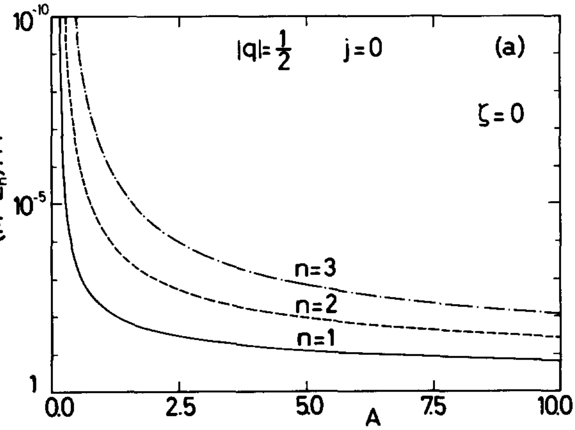
When $|q| = \frac{1}{2}$, A is simply

$$A = \tfrac{1}{4}\kappa. \tag{6.1}$$

In fig. 1 we show plots of the binding energies of the lowest states for $j = 0, 1, 2$, and 3. The notation here is that $n = 0$ refers to the zero-energy bound states given in sect. 2, while $n = 1, 2, \ldots$ denotes the higher bound states. The conditions for the existence of these higher bound states or excited states are for $|q| = \frac{1}{2}$ [1, 2]

$$\kappa > \tfrac{1}{2} \quad \text{for} \quad j = 0\,, \tag{6.2}$$

$$\kappa > \tfrac{1}{2}(2j+1)^2 - 1 \quad \text{for} \quad j \geq 1\,. \tag{6.3}$$



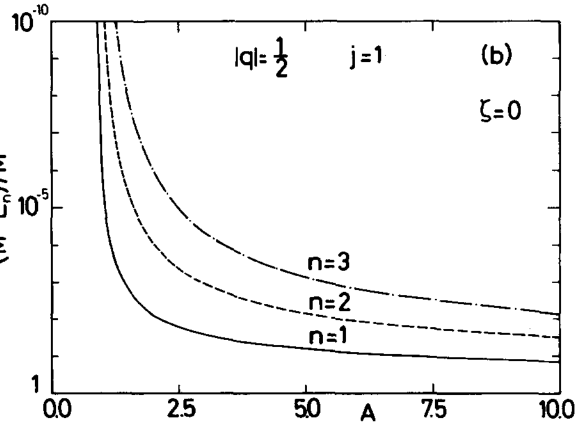


Fig. 1. Monopole-fermion binding energies $\varepsilon_n = (M - E_n)/M$ versus $A = \frac{1}{2}|q|\kappa$; (a) $j = 0$, (b) $j = 1$, (c) $j = 2$, and (d) $j = 3$. In each case three curves are given, for the levels $n = 1$, 2 and 3. The conditions for the presence of these levels are: $\kappa > \frac{1}{2}$ for $j = 0$; $|\kappa| > \frac{7}{2}$ for $j = 1$; $|\kappa| > \frac{23}{2}$ for $j = 2$; and $|\kappa| > \frac{47}{2}$ for $j = 3$. There is also a lower state ($n = 0$) at $\varepsilon_0 = (M - E_0)/M = 1$. For $j = 0$ these energies apply to any non-zero value of $|q|$, for $j \geq 1$ they are only valid for $|q| = \frac{1}{2}$. For $j \geq 1$ there are corresponding levels at $A \to -A$ [1, 2].

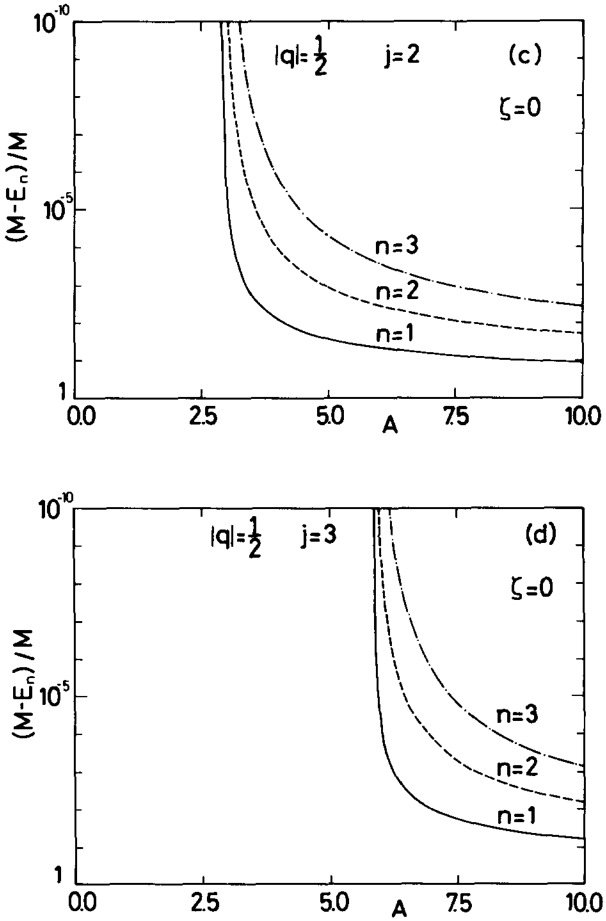


Fig. 1.—*cont.*

It is quite straightforward to determine the eigenvalues numerically by the iterative procedure outlined in sect. 5, provided we start at some good initial guess. Such guesses are conveniently provided by the explicit, analytic approximations obtained in refs. [4] and [11] for $j=0$ and for $j \geq 1$, respectively.

Some numerical values for the binding energies can be found in tables 1, 2 and 3 in sect. 7. For $j=0$ such values have been determined previously by the angle analysis [1, 4]. The values found here agree with those obtained in ref. [4].

In order to develop some intuition for these bound states, we also show the squared moduli of some radial wave functions corresponding to $A=2$ in fig. 2 ($j=0$) and fig. 3 ($j=1$). (For this value of A there are no states of higher angular momentum.) The $j=0$ wave functions are familiar from ref. [4]. The $j=1$ wave functions have in a sense two "large" and two "small" components. It should be

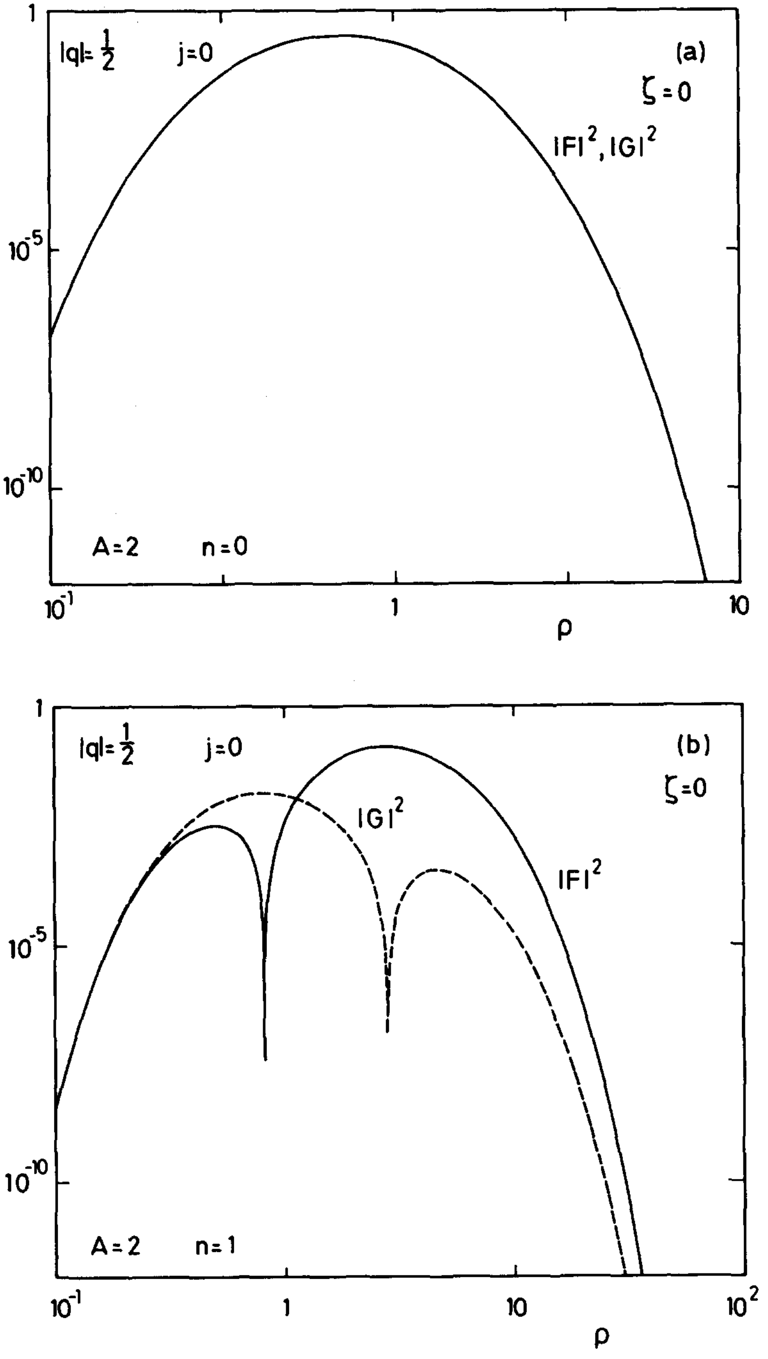
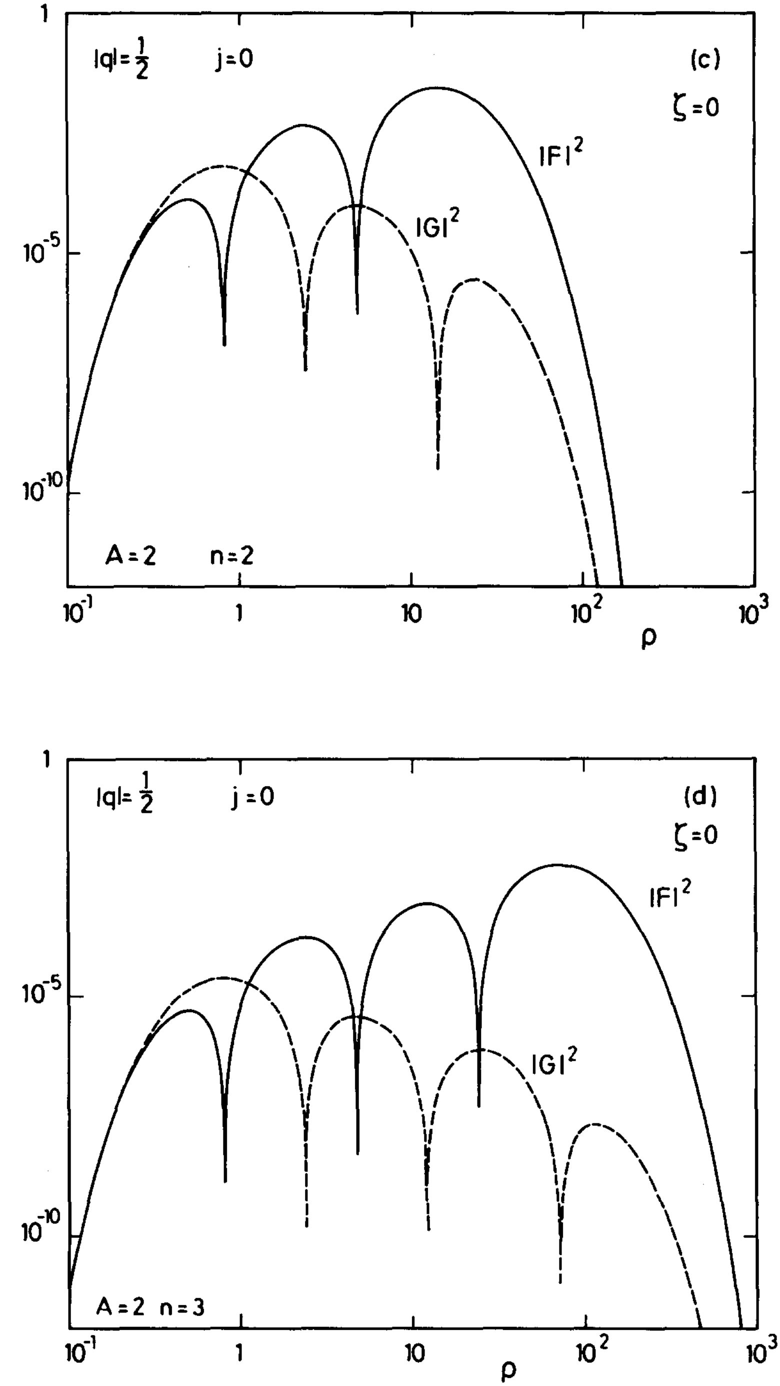
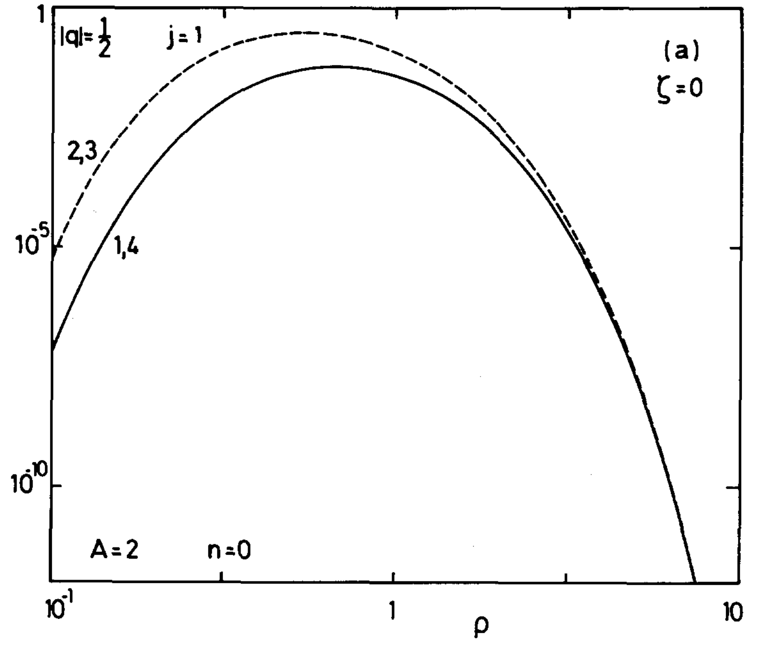


Fig. 2. Squares of the monopole-fermion radial wave functions F and G for $A=2$ and $j=0$. Four states are considered: (a) $n=0$, (b) $n=1$, (c) $n=2$, and (d) $n=3$. (All the minima in this figure, as well as those in figs. 3 and 6, are actually zeros of the wave functions.)



|q|= 1/2
j=0
(c)
ζ=0
|F|²
|G|²
A=2
n=2
1
10⁻⁵
10⁻¹⁰
10⁻¹
1
10
10²
10³
ρ
|q|= 1/2
j=0
(d)
ζ=0
|F|²
|G|²
A=2 n=3
1
10⁻⁵
10⁻¹⁰
10⁻¹
1
10
10²
10³
ρ

Fig. 2.—*cont.*



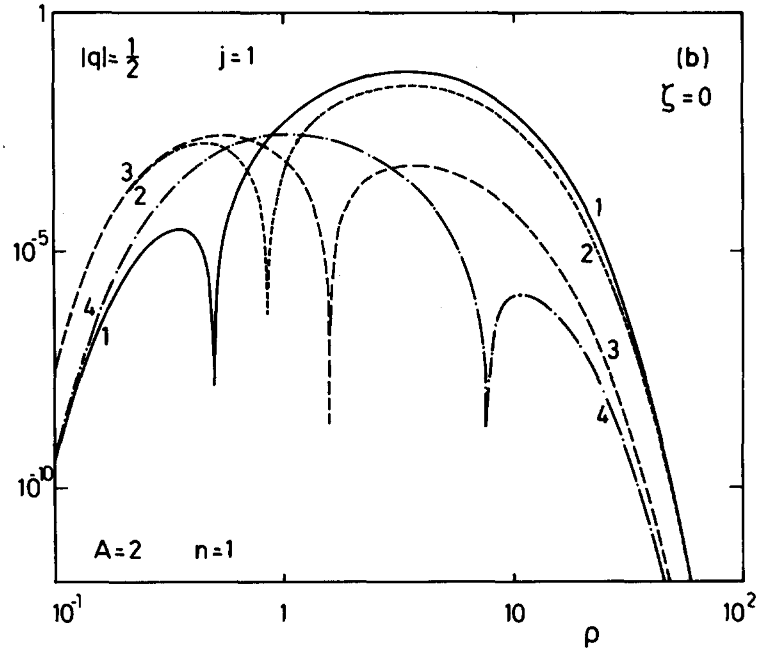
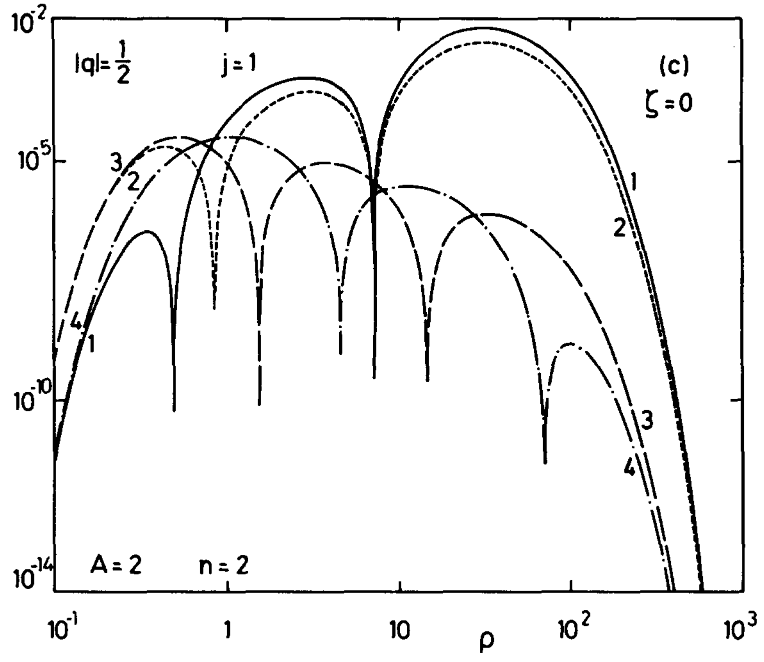


Fig. 3. Squares of the monopole-fermion radial wave functions h_1 (solid), h_2 (short-dashed), h_3 (long-dashed), and h_4 (dash-dotted), for $A=2$ and $j=1$. Four levels are considered: (a) $n=0$, (b) $n=1$, (c) $n=2$, and (d) $n=3$.


10^{-2}
10^{-5}
10^{-10}
10^{-14}
$|q| = \frac{1}{2}$
j = 1
(c)
ζ = 0
1
2
3
4
A = 2
n = 2
10^{-1}
1
10
ρ
10^{2}
10^{3}

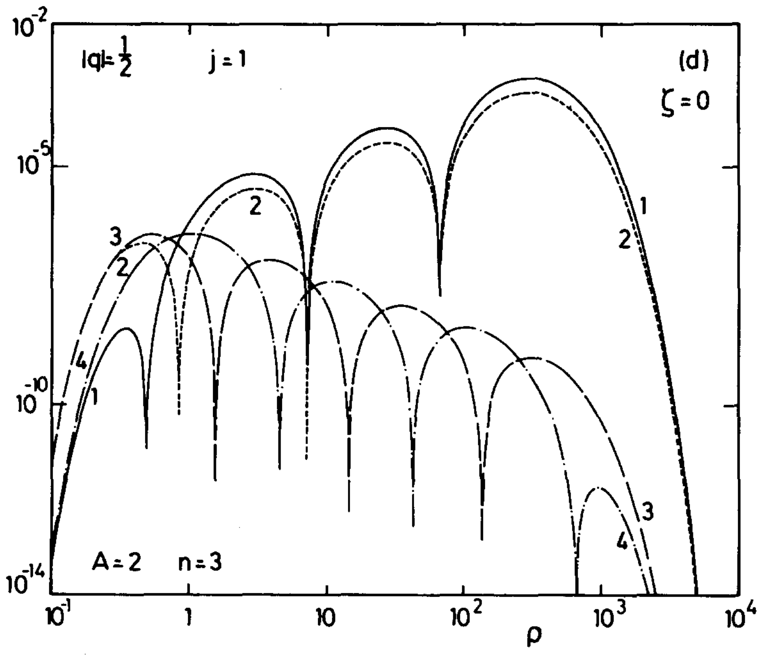

10^{-2}
10^{-5}
10^{-10}
10^{-14}
$|q| = \frac{1}{2}$
j = 1
(d)
ζ = 0
1
2
3
4
A = 2
n = 3
10^{-1}
1
10
10^{2}
10^{3}
10^{4}
ρ

Fig. 3.—*cont.*

noted, however, that there is a sort of transition at some intermediate value of ρ: whereas h_1 and h_2 dominate at large values of ρ, it is h_2 and h_3 which dominate at small ρ. Also, although for $j \geqslant 1$ there is no inversion symmetry of the kind found for $j=0$ [4], h_1 and h_4 are nevertheless qualitatively similar to each other's mirror image.

7. The $|q| = \frac{1}{2}$ dyon-fermion system

For each value of angular momentum j, the monopole-fermion system studied in the preceding section possesses an infinite number of bound states when and only

TABLE 1

Binding energies $(M-E)/M$ for the dyon-fermion system with $j=|q|-\frac{1}{2}$

ζ	n \ A	0.4482 (p)	1.0	1.5	2.0	2.5
-2α	3					
	2			$1.6319 \cdot 10^{-5}$	$4.4965 \cdot 10^{-4}$	$1.2734 \cdot 10^{-3}$
	1		$3.7532 \cdot 10^{-3}$	$1.1679 \cdot 10^{-2}$	$2.1122 \cdot 10^{-2}$	$3.0978 \cdot 10^{-2}$
	0	0.9814	0.9870	0.9891	0.9905	0.9914
$-\alpha$	3					$3.5825 \cdot 10^{-5}$
	2			$1.5383 \cdot 10^{-4}$	$6.8094 \cdot 10^{-4}$	$1.5788 \cdot 10^{-3}$
	1	$7.0407 \cdot 10^{-6}$	$4.6042 \cdot 10^{-3}$	$1.2785 \cdot 10^{-2}$	$2.2359 \cdot 10^{-2}$	$3.2287 \cdot 10^{-2}$
	0	0.9907	0.9935	0.9946	0.9952	0.9957
0	3	$4.6173 \cdot 10^{-11}$	$4.1337 \cdot 10^{-7}$	$7.1763 \cdot 10^{-6}$	$3.6214 \cdot 10^{-5}$	$1.0627 \cdot 10^{-4}$
	2	$1.1438 \cdot 10^{-7}$	$4.7757 \cdot 10^{-5}$	$3.1711 \cdot 10^{-4}$	$9.2814 \cdot 10^{-4}$	$1.8960 \cdot 10^{-3}$
	1	$2.8310 \cdot 10^{-4}$	$5.4882 \cdot 10^{-3}$	$1.3911 \cdot 10^{-2}$	$2.3610 \cdot 10^{-2}$	$3.3607 \cdot 10^{-2}$
	0	1.0000	1.0000	1.0000	1.0000	1.0000
α	3	$6.6592 \cdot 10^{-6}$	$1.7399 \cdot 10^{-5}$	$4.1372 \cdot 10^{-5}$	$9.3119 \cdot 10^{-5}$	$1.8830 \cdot 10^{-4}$
	2	$2.5608 \cdot 10^{-5}$	$1.4641 \cdot 10^{-4}$	$5.0058 \cdot 10^{-4}$	$1.1898 \cdot 10^{-3}$	$2.2246 \cdot 10^{-3}$
	1	$6.6071 \cdot 10^{-4}$	$6.4034 \cdot 10^{-3}$	$1.5055 \cdot 10^{-2}$	$2.4874 \cdot 10^{-2}$	$3.4936 \cdot 10^{-2}$
	0	1.0093	1.0065	1.0054	1.0048	1.0043
2α	3	$2.2461 \cdot 10^{-5}$	$4.6676 \cdot 10^{-5}$	$8.7998 \cdot 10^{-5}$	$1.6122 \cdot 10^{-4}$	$2.8007 \cdot 10^{-4}$
	2	$7.4928 \cdot 10^{-5}$	$2.6826 \cdot 10^{-4}$	$7.0154 \cdot 10^{-4}$	$1.4650 \cdot 10^{-3}$	$2.5640 \cdot 10^{-3}$
	1	$1.1115 \cdot 10^{-3}$	$7.3482 \cdot 10^{-3}$	$1.6217 \cdot 10^{-2}$	$2.6151 \cdot 10^{-2}$	$3.6276 \cdot 10^{-2}$
	0	1.0186	1.0131	1.0109	1.0095	1.0086
0.1	3	$7.0495 \cdot 10^{-4}$	$9.4958 \cdot 10^{-4}$	$1.2106 \cdot 10^{-3}$	$1.5191 \cdot 10^{-3}$	$1.8812 \cdot 10^{-3}$
	2	$1.7905 \cdot 10^{-3}$	$2.8186 \cdot 10^{-3}$	$4.0277 \cdot 10^{-3}$	$5.5066 \cdot 10^{-3}$	$7.2326 \cdot 10^{-3}$
	1	$9.8771 \cdot 10^{-3}$	$2.0286 \cdot 10^{-2}$	$3.1077 \cdot 10^{-2}$	$4.2023 \cdot 10^{-2}$	$5.2682 \cdot 10^{-2}$
	0	1.1272	1.0894	1.0744	1.0652	1.0588
0.5	3	$1.3619 \cdot 10^{-2}$	$1.4560 \cdot 10^{-2}$	$1.5586 \cdot 10^{-2}$	$1.6672 \cdot 10^{-2}$	$1.7798 \cdot 10^{-2}$
	2	$2.9858 \cdot 10^{-2}$	$3.2581 \cdot 10^{-2}$	$3.5533 \cdot 10^{-2}$	$3.8615 \cdot 10^{-2}$	$4.1748 \cdot 10^{-2}$
	1	$1.0480 \cdot 10^{-1}$	$1.1505 \cdot 10^{-1}$	$1.2560 \cdot 10^{-1}$	$1.3571 \cdot 10^{-1}$	$1.4520 \cdot 10^{-1}$
	0	1.6312	1.4458	1.3713	1.3255	1.2936
1.0	3	$5.3116 \cdot 10^{-1}$	$5.1147 \cdot 10^{-2}$	$5.1379 \cdot 10^{-2}$	$5.2177 \cdot 10^{-2}$	$5.3236 \cdot 10^{-2}$
	2	$1.1192 \cdot 10^{-1}$	$1.0547 \cdot 10^{-1}$	$1.0537 \cdot 10^{-1}$	$1.0678 \cdot 10^{-1}$	$1.0882 \cdot 10^{-1}$
	1	$3.4010 \cdot 10^{-1}$	$3.0179 \cdot 10^{-1}$	$2.9380 \cdot 10^{-1}$	$2.9220 \cdot 10^{-1}$	$2.9317 \cdot 10^{-1}$
	0		1.8825	1.7387	1.6487	1.5857

TABLE 2

Binding energies $(M-E)/M$ for the dyon-fermion system with $j=|q|-\frac{1}{2}$

ζ	n \ A	1.5	2.0	2.5	4.0	6.0
-2α	3				$6.0192 \cdot 10^{-5}$	$1.0760 \cdot 10^{-3}$
	2			$6.8333 \cdot 10^{-5}$	$2.6096 \cdot 10^{-3}$	$9.7366 \cdot 10^{-3}$
	1	$3.9681 \cdot 10^{-4}$	$5.3724 \cdot 10^{-3}$	$1.3395 \cdot 10^{-2}$	$4.1931 \cdot 10^{-2}$	$7.7223 \cdot 10^{-2}$
	0	0.9859	0.9883	0.9899	0.9923	0.9939
$-\alpha$	3				$1.5080 \cdot 10^{-4}$	$1.2646 \cdot 10^{-3}$
	2		$<10^{-8}$?	$2.2601 \cdot 10^{-4}$	$2.9815 \cdot 10^{-3}$	$1.0259 \cdot 10^{-2}$
	1	$8.9792 \cdot 10^{-4}$	$6.2916 \cdot 10^{-3}$	$1.4523 \cdot 10^{-2}$	$4.3276 \cdot 10^{-2}$	$7.8597 \cdot 10^{-2}$
	0	0.9930	0.9942	0.9949	0.9962	0.9969
0	3	$7.6228 \cdot 10^{-9}$	$9.9149 \cdot 10^{-7}$	$1.0452 \cdot 10^{-5}$	$2.5213 \cdot 10^{-4}$	$1.4589 \cdot 10^{-3}$
	2	$3.3440 \cdot 10^{-6}$	$8.4976 \cdot 10^{-5}$	$4.0622 \cdot 10^{-4}$	$3.3627 \cdot 10^{-3}$	$1.0788 \cdot 10^{-2}$
	1	$1.4629 \cdot 10^{-3}$	$7.2381 \cdot 10^{-3}$	$1.5669 \cdot 10^{-2}$	$4.4630 \cdot 10^{-2}$	$7.9975 \cdot 10^{-2}$
	0	1.0000	1.0000	1.0000	1.0000	1.0000
α	3	$9.5724 \cdot 10^{-6}$	$2.1391 \cdot 10^{-5}$	$4.8675 \cdot 10^{-5}$	$3.6267 \cdot 10^{-4}$	$1.6586 \cdot 10^{-3}$
	2	$5.0242 \cdot 10^{-5}$	$2.0330 \cdot 10^{-4}$	$6.0506 \cdot 10^{-4}$	$3.7529 \cdot 10^{-3}$	$1.1321 \cdot 10^{-2}$
	1	$2.0811 \cdot 10^{-3}$	$8.2105 \cdot 10^{-3}$	$1.6831 \cdot 10^{-2}$	$4.5991 \cdot 10^{-2}$	$8.1358 \cdot 10^{-2}$
	0	1.0070	1.0058	1.0051	1.0038	1.0031
2α	3	$2.9762 \cdot 10^{-5}$	$5.4291 \cdot 10^{-5}$	$9.9121 \cdot 10^{-5}$	$4.8150 \cdot 10^{-4}$	$1.8635 \cdot 10^{-3}$
	2	$1.2263 \cdot 10^{-4}$	$3.4322 \cdot 10^{-4}$	$8.2036 \cdot 10^{-4}$	$4.1518 \cdot 10^{-3}$	$1.1860 \cdot 10^{-2}$
	1	$2.7463 \cdot 10^{-3}$	$9.2079 \cdot 10^{-3}$	$1.8010 \cdot 10^{-2}$	$4.7360 \cdot 10^{-2}$	$8.2746 \cdot 10^{-2}$
	0	1.0141	1.0117	1.0102	1.0077	1.0061
0.1	3	$7.8690 \cdot 10^{-4}$	$1.0039 \cdot 10^{-3}$	$1.2629 \cdot 10^{-3}$	$2.3532 \cdot 10^{-3}$	$4.6073 \cdot 10^{-3}$
	2	$2.1168 \cdot 10^{-3}$	$3.0613 \cdot 10^{-3}$	$4.2736 \cdot 10^{-3}$	$9.4049 \cdot 10^{-3}$	$1.8530 \cdot 10^{-2}$
	1	$1.3115 \cdot 10^{-2}$	$2.2489 \cdot 10^{-2}$	$3.2932 \cdot 10^{-2}$	$6.3951 \cdot 10^{-2}$	$9.9311 \cdot 10^{-2}$
	0	1.0964	1.0799	1.0695	1.0527	1.0420
0.5	3	$1.3582 \cdot 10^{-2}$	$1.4597 \cdot 10^{-2}$	$1.5656 \cdot 10^{-2}$	$1.9033 \cdot 10^{-2}$	$2.3861 \cdot 10^{-2}$
	2	$2.9638 \cdot 10^{-2}$	$3.2609 \cdot 10^{-2}$	$3.5671 \cdot 10^{-2}$	$4.5068 \cdot 10^{-2}$	$5.7457 \cdot 10^{-2}$
	1	$1.0146 \cdot 10^{-1}$	$1.1378 \cdot 10^{-1}$	$1.2512 \cdot 10^{-1}$	$1.5411 \cdot 10^{-1}$	$1.8439 \cdot 10^{-1}$
	0	1.4806	1.3989	1.3472	1.2631	1.2097
1.0	3	$4.9095 \cdot 10^{-2}$	$4.9676 \cdot 10^{-2}$	$5.0630 \cdot 10^{-2}$	$5.4239 \cdot 10^{-2}$	$5.9745 \cdot 10^{-2}$
	2	$1.0010 \cdot 10^{-1}$	$1.0120 \cdot 10^{-1}$	$1.0309 \cdot 10^{-1}$	$1.1053 \cdot 10^{-1}$	$1.2130 \cdot 10^{-1}$
	1	$2.8390 \cdot 10^{-1}$	$2.8254 \cdot 10^{-1}$	$2.8367 \cdot 10^{-1}$	$2.9250 \cdot 10^{-1}$	$3.0676 \cdot 10^{-1}$
	0	1.9493	1.7924	1.6915	1.5253	1.4191

when κ satisfies the conditions (6.2) or (6.3). This is to be contrasted with the corresponding situation for the dyon. When the Coulomb interaction is attractive ($\zeta>0$), there is an infinite number of bound states for each j as long as $\kappa\neq 0$ [13]. When the Coulomb interaction is repulsive ($\zeta<0$), the number of bound states is finite, or maybe even zero. A survey of binding energies is given in tables 1, 2 and 3 for $j=0$, 1 and 2, respectively, and for a selection of A and ζ values.

In figs. 4 and 5 we show how the binding energy can change dramatically when the Coulomb interaction is turned on. Fig. 4 gives the "zero-energy" level, $n=0$.

TABLE 3

Binding energies $(M-E)/M$ for the dyon-fermion system with $|q|=\frac{1}{2}$ and $j=2$

ζ	n \ A	4.0	6.0	8.0	10.0	12.0
-2α	3		$6.6520 \cdot 10^{-5}$	$1.0813 \cdot 10^{-3}$	$3.1836 \cdot 10^{-3}$	$6.1262 \cdot 10^{-3}$
	2		$2.6322 \cdot 10^{-3}$	$9.6757 \cdot 10^{-3}$	$1.8737 \cdot 10^{-2}$	$2.8487 \cdot 10^{-2}$
	1	$5.7739 \cdot 10^{-3}$	$4.1429 \cdot 10^{-2}$	$7.6402 \cdot 10^{-2}$	$1.0603 \cdot 10^{-1}$	$1.3109 \cdot 10^{-1}$
	0	0.9906	0.9929	0.9941	0.9949	0.9954
$-\alpha$	3		$1.5784 \cdot 10^{-4}$	$1.2687 \cdot 10^{-3}$	$3.4405 \cdot 10^{-3}$	$6.4337 \cdot 10^{-3}$
	2	$4.6832 \cdot 10^{-6}$	$2.9985 \cdot 10^{-3}$	$1.0191 \cdot 10^{-2}$	$1.9331 \cdot 10^{-2}$	$2.9125 \cdot 10^{-2}$
	1	$6.6644 \cdot 10^{-3}$	$4.2723 \cdot 10^{-2}$	$7.7741 \cdot 10^{-2}$	$1.0734 \cdot 10^{-1}$	$1.3237 \cdot 10^{-1}$
	0	0.9953	0.9965	0.9971	0.9975	0.9977
0	3	$1.3351 \cdot 10^{-6}$	$2.5949 \cdot 10^{-4}$	$1.4617 \cdot 10^{-3}$	$3.7013 \cdot 10^{-3}$	$6.7442 \cdot 10^{-3}$
	2	$1.0028 \cdot 10^{-4}$	$3.3736 \cdot 10^{-3}$	$1.0711 \cdot 10^{-2}$	$1.9929 \cdot 10^{-2}$	$2.9767 \cdot 10^{-2}$
	1	$7.5774 \cdot 10^{-3}$	$4.4023 \cdot 10^{-2}$	$7.9083 \cdot 10^{-2}$	$1.0866 \cdot 10^{-1}$	$1.3365 \cdot 10^{-1}$
	0	1.0000	1.0000	1.0000	1.0000	1.0000
α	3	$2.2929 \cdot 10^{-5}$	$3.7007 \cdot 10^{-4}$	$1.6599 \cdot 10^{-3}$	$3.9658 \cdot 10^{-3}$	$7.0575 \cdot 10^{-3}$
	2	$2.2290 \cdot 10^{-4}$	$3.7572 \cdot 10^{-3}$	$1.1236 \cdot 10^{-2}$	$2.0530 \cdot 10^{-2}$	$3.0411 \cdot 10^{-2}$
	1	$8.5119 \cdot 10^{-3}$	$4.5330 \cdot 10^{-2}$	$8.0430 \cdot 10^{-2}$	$1.0998 \cdot 10^{-1}$	$1.3493 \cdot 10^{-1}$
	0	1.0047	1.0035	1.0029	1.0026	1.0023
2α	3	$5.6829 \cdot 10^{-5}$	$4.8868 \cdot 10^{-4}$	$1.8632 \cdot 10^{-3}$	$4.2339 \cdot 10^{-3}$	$7.3737 \cdot 10^{-3}$
	2	$3.6570 \cdot 10^{-4}$	$4.1491 \cdot 10^{-3}$	$1.1765 \cdot 10^{-2}$	$2.1135 \cdot 10^{-2}$	$3.1057 \cdot 10^{-2}$
	1	$9.4668 \cdot 10^{-3}$	$4.6644 \cdot 10^{-2}$	$8.1781 \cdot 10^{-2}$	$1.1131 \cdot 10^{-1}$	$1.3621 \cdot 10^{-1}$
	0	1.0094	1.0071	1.0059	1.0051	1.0046
0.1	3	$1.0026 \cdot 10^{-3}$	$2.3425 \cdot 10^{-3}$	$4.5796 \cdot 10^{-3}$	$7.6276 \cdot 10^{-3}$	$1.1277 \cdot 10^{-2}$
	2	$3.0436 \cdot 10^{-3}$	$9.2845 \cdot 10^{-3}$	$1.8318 \cdot 10^{-2}$	$2.8465 \cdot 10^{-2}$	$3.8823 \cdot 10^{-2}$
	1	$2.1958 \cdot 10^{-2}$	$6.2520 \cdot 10^{-2}$	$9.7891 \cdot 10^{-2}$	$1.2703 \cdot 10^{-1}$	$1.5141 \cdot 10^{-1}$
	0	1.0642	1.0484	1.0401	1.0349	1.0313
0.5	3	$1.3929 \cdot 10^{-2}$	$1.8578 \cdot 10^{-2}$	$2.3496 \cdot 10^{-2}$	$2.8605 \cdot 10^{-2}$	$3.3807 \cdot 10^{-2}$
	2	$3.0514 \cdot 10^{-2}$	$4.3577 \cdot 10^{-2}$	$5.6295 \cdot 10^{-2}$	$6.8333 \cdot 10^{-2}$	$7.9602 \cdot 10^{-2}$
	1	$1.0319 \cdot 10^{-1}$	$1.4765 \cdot 10^{-1}$	$1.8017 \cdot 10^{-1}$	$2.0555 \cdot 10^{-1}$	$2.2634 \cdot 10^{-1}$
	0	1.3209	1.2419	1.2006	1.1747	1.1565
1.0	3	$4.5688 \cdot 10^{-2}$	$5.2170 \cdot 10^{-2}$	$5.8395 \cdot 10^{-2}$	$6.4416 \cdot 10^{-2}$	$7.0238 \cdot 10^{-2}$
	2	$9.0543 \cdot 10^{-2}$	$1.0517 \cdot 10^{-1}$	$1.1793 \cdot 10^{-1}$	$1.2935 \cdot 10^{-1}$	$1.3972 \cdot 10^{-1}$
	1	$2.4390 \cdot 10^{-1}$	$2.7568 \cdot 10^{-1}$	$2.9741 \cdot 10^{-1}$	$3.1429 \cdot 10^{-1}$	$3.2825 \cdot 10^{-1}$
	0	1.6399	1.4831	1.4009	1.3491	1.3130

Two values of A ($A=1.0$ and 2.0) and two values of the angular momentum ($j=0$ and 1) are considered.

We note that for the smaller value of A, and for the larger value of angular momentum, the state is more susceptible to the influence of the Coulomb field. Thus, for $A=1.0$ and $j=1$, around $\zeta \simeq 0.80$ the binding reaches its maximum value of $\varepsilon_0=(M-E_0)/M=2$, beyond which the state no longer exists. For negative values of ζ, the binding gets increasingly weaker as $|\zeta|$ increases, until at some point the state is "pushed up" into the continuum.

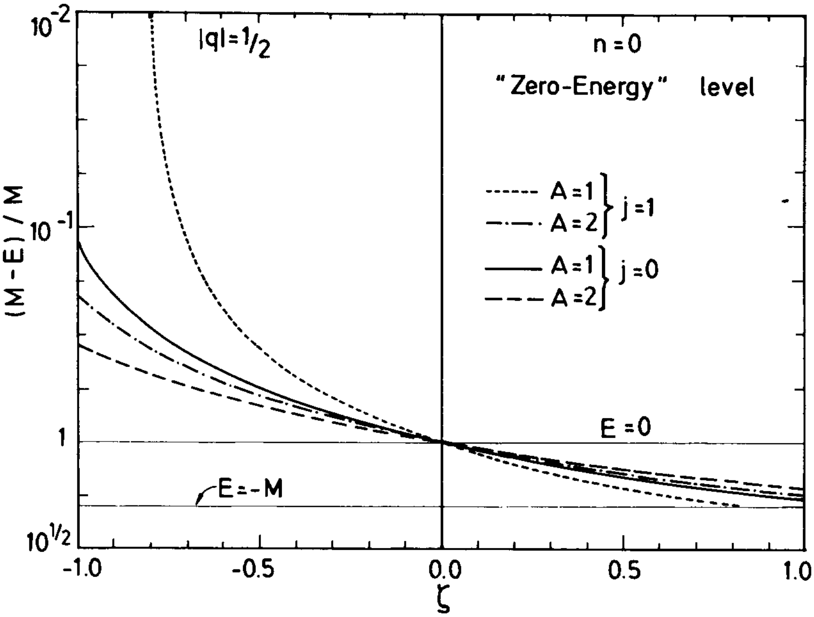


Fig. 4. Dyon-fermion binding energies for the "zero-energy" state versus ζ, the Coulomb-interaction strength parameter. A "zero-energy" state is defined as a state with the property of $E \to 0$ as $\zeta \to 0$. Two values of the parameter A are considered, $A = 1.0$ and 2.0, and two angular momenta, $j = 0$ and 1. The $A = 1, j = 1$ state (short-dashed) is seen to reach maximum binding $\varepsilon_0 = (M - E_0)/M = 2$ around $\zeta = 0.80$.

In fig. 5 we show the ζ-dependence of the levels labelled by $n = 1$, again for $A = 1.0$ and 2.0, and for $j = 0$ and 1. The energy levels of these more weakly bound states are seen to be much more susceptible to the Coulomb force than was the case for the $n = 0$ states.

The wave functions for the $j = 0$, $\zeta = \alpha$, $\kappa < 0$ states are rather different from those describing $\kappa > 0$. The moduli of these wave functions are shown in fig. 6 for $A = -2$. The corresponding wave functions for $A = +2$ and $\zeta = \alpha$ are very similar to those for $A = +2$ and $\zeta = 0$ (they are just "pulled in" a little because of the extra attraction) which were shown in fig. 2.

Comparing now the $n = 1$ wave functions (figs. 2b and 6a) we note that whereas for $A > 0$ F and G each has one node, for $A < 0$ only G has a node. This qualitative difference is closely related to the non-existence of the "zero-energy" state ($n = 0$) for $j = 0$ and $\kappa < 0$, and can be understood by considering the boundary conditions.

It is seen from (3.9) that, in the limit $\rho \to 0$, F and G have opposite signs. Similarly, it follows from (3.18) that

$$\lim_{\rho \to \infty} F/G = -\frac{\kappa}{|\kappa|}\left(\frac{A+B}{A-B}\right)^{1/2}, \tag{7.1}$$

and hence, in this limit $\rho \to \infty$, the relative sign of F and G is opposite to that of

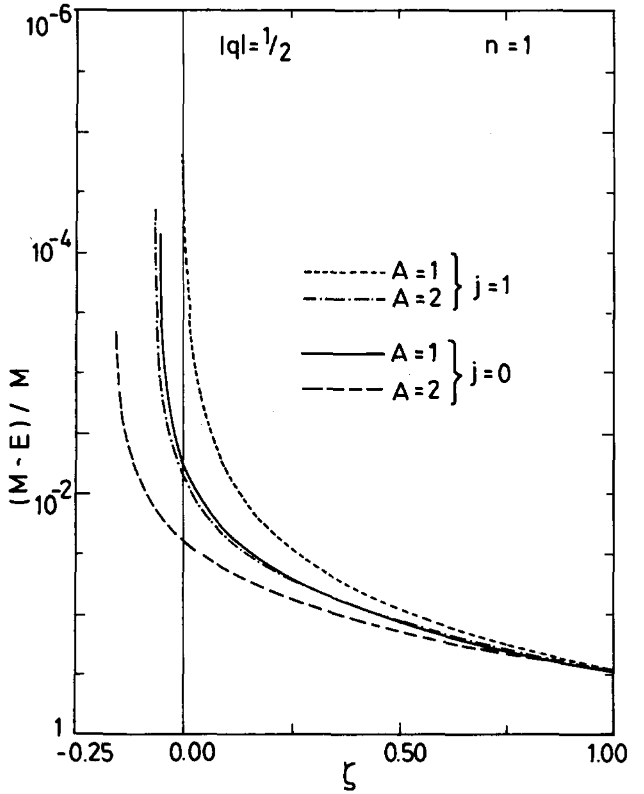


Fig. 5. Same as fig. 4 for the energy level labelled by $n = 1$.

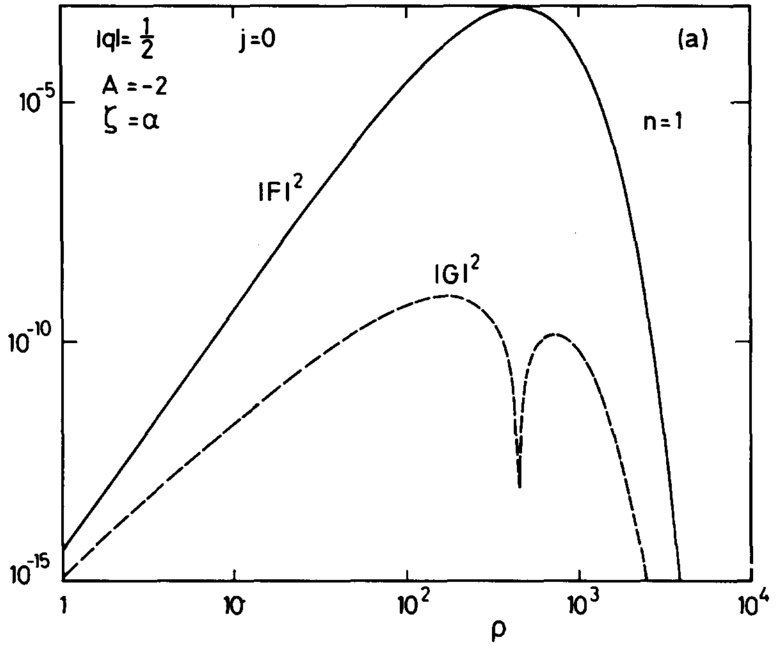
κ. A comparison of these two statements shows that for $\kappa > 0$ F and G will have the same number of nodes (modulo 2). For $\kappa < 0$, on the other hand, the number of nodes will differ by one (modulo 2). Since for $n = 1$ and $\kappa < 0$ F has no nodes (cf. fig. 6a), this must be the lowest state (among the angular momentum 0 states).

Negative values of κ lead to a peculiar spectrum, in the sense that the lowest state among those of the lowest possible angular momentum, $j = 0$, is not the lowest of all possible states. There are several states of non-zero angular momentum which are lower than the lowest $j = 0$ state.

We conclude this section with a comparison of the lowest levels for $\kappa > 0$, for $j = 0$, 1 and 2 (fig. 7). The levels are seen to cross, in the sense that for certain values of A (or κ) there exist two states of different angular momentum having the same binding energy.

8. Higher charges, $|q| \geq 1$

We now consider briefly the case where $|q| > \frac{1}{2}$. In this general case the binding energy is a function of five parameters: $|q|$, κ, ζ, j and n. We here limit ourselves to presenting two sets of curves.



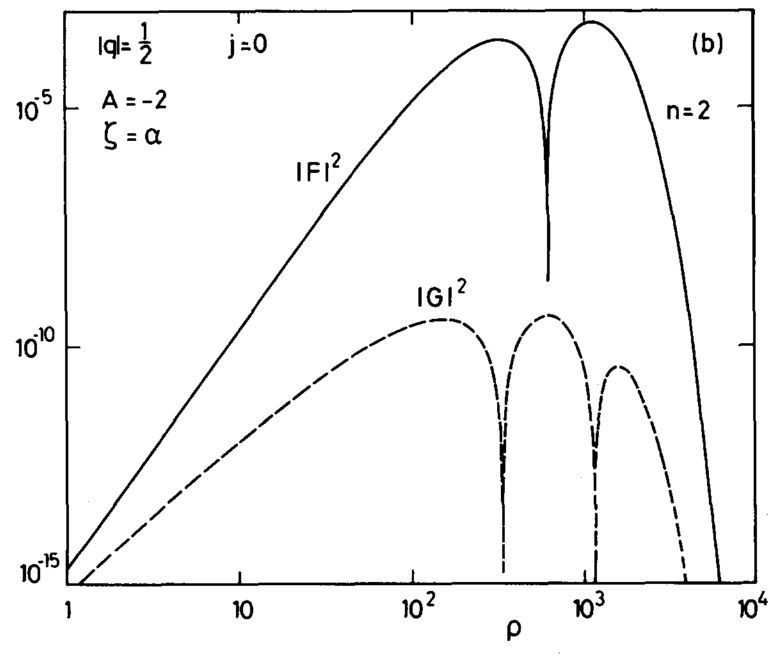


Fig. 6. Squares of the dyon-fermion radial wave functions F and G for $A=-2$ and $j=0$. The binding energies of the three states with (a) $n=1$, (b) $n=2$ and (c) $n=3$ are given respectively by $10^6(M-E)/M=$ 4.058, 2.099 and 1.280.

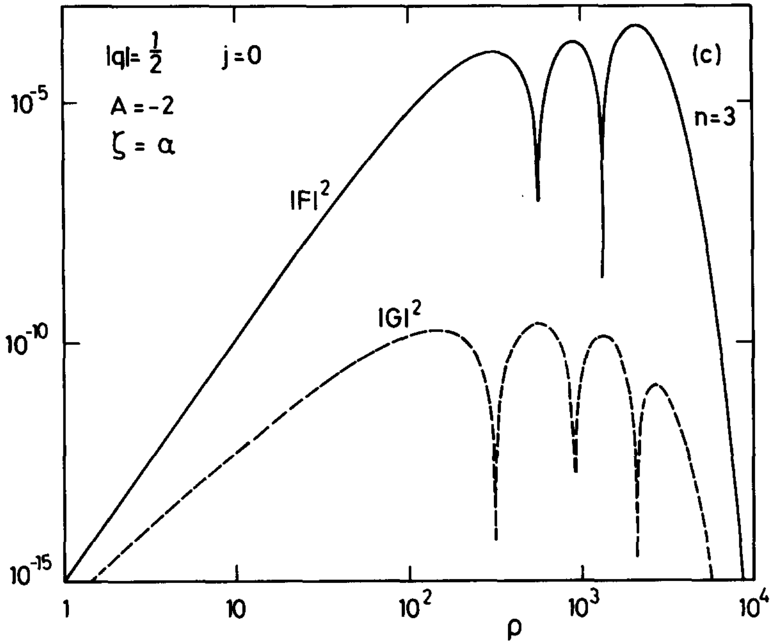


Fig. 6.—*cont.*

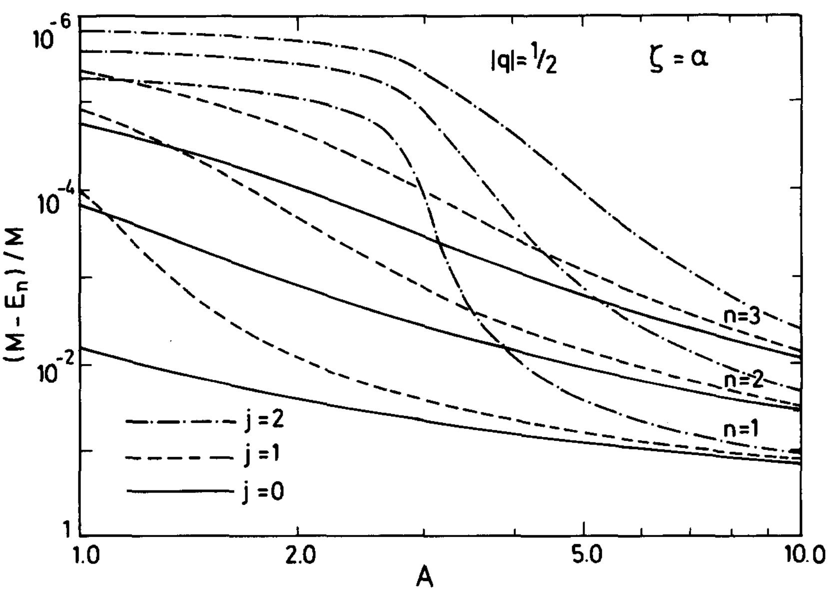
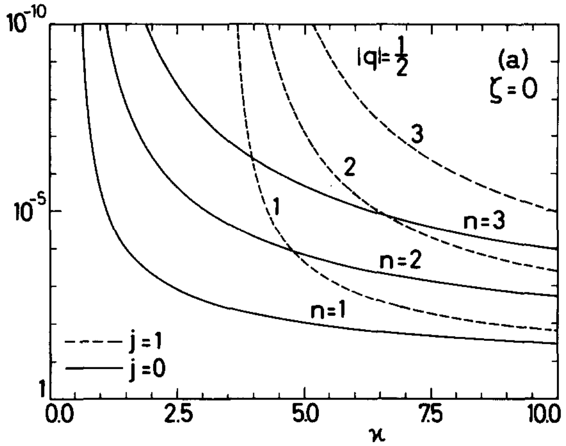
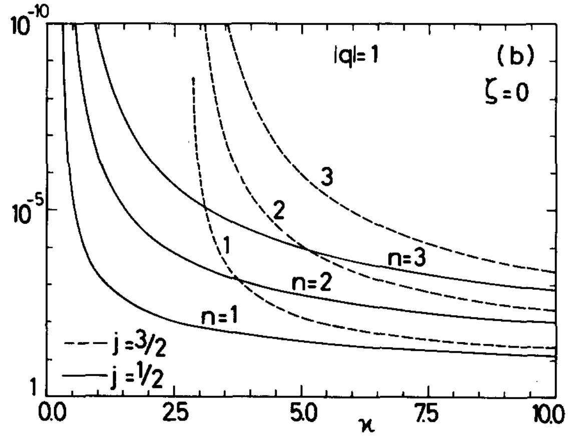


Fig. 7. Dyon-fermion binding energies for fixed $\zeta = \alpha \simeq \frac{1}{137}$ versus A. Three states ($n = 1, 2$ and 3) are shown for each of the three angular momenta, $j = 0$, 1 and 2. Levels of different angular momentum are seen to cross at certain values of A.





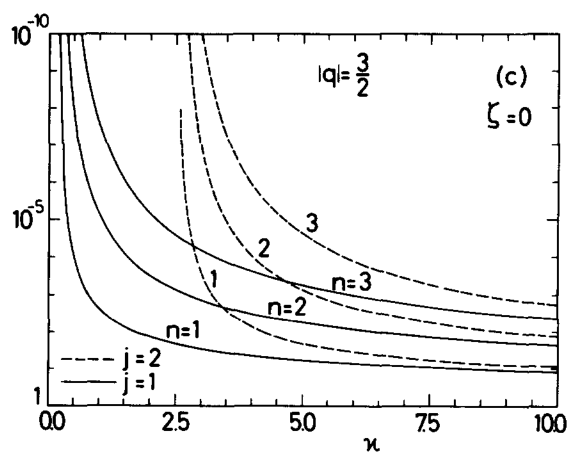


Fig. 8. Monopole-fermion binding energies versus κ, for (a) $|q|=\frac{1}{2}$, (b) $|q|=1$, and (c) $|q|=\frac{3}{2}$. In each case, three states ($n=1$, 2 and 3) are shown for each of the two angular momenta $j=|q|-\frac{1}{2}$ and $j=|q|+\frac{1}{2}$.

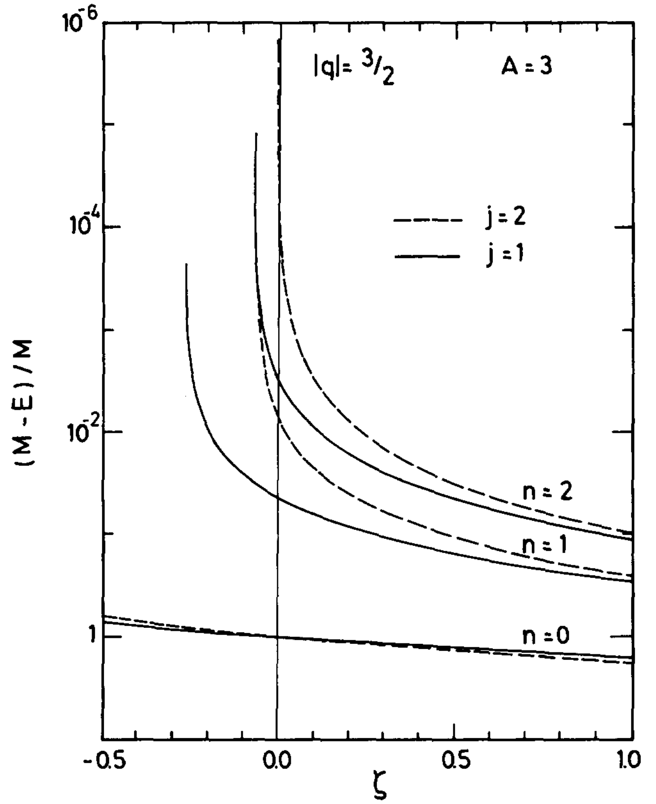


Fig. 9. Dyon-fermion binding energies versus ζ, for $|q|=\frac{3}{2}$ and $\kappa=4$ (or $A=3$). Three states ($n=0$, 1 and 2) are shown for each of the two angular momenta $j=1$ and 2. Note that $j=2$ is a logarithmic case, while $j=1$ is not.

For the three lowest values of $|q|$, $|q|=\frac{1}{2}$, 1 and $\frac{3}{2}$, and for $\zeta=0$, we show in fig. 8 the binding energies for some of the lowest states, for a range of κ-values. As one would expect, for increasing values of $|q|$ (and κ) the binding gets stronger.

In sect. 4, logarithms are found to appear in a number of special cases specified by eq. (4.15). Among these cases, the lowest one is $|q|=\frac{3}{2}$ and $j=2$ with $\zeta\neq 0$. In fig. 9 we show the binding energies for this case of $|q|=\frac{3}{2}$ with the choice of $\kappa=4$ (and thus $A=3$), as functions of ζ. For comparison, the case $j=1$ is included in addition to $j=2$.

9. Concluding remarks

In this paper we have studied the bound states for a monopole-fermion system or more generally a dyon-fermion system. The formalism has been used to give numerical results, both for the binding energy and the bound-state wave function. However, because of the complexity of the systems, it is not possible to obtain closed, exact formulas for either, with the exception of the zero-energy state of (2.13) and (2.14). In order to gain further physical insight it is desirable to study various special and limiting cases by analytic methods.

Some examples of the limiting cases of interest are

(A) $\kappa \to 0$,

(B) $\kappa \to \infty$,

(C) $\zeta \to \infty$,

(D) the case of weak binding $\frac{M-E}{M} \ll 1$.

For the states of the lowest angular momentum, case A has been studied before [5]. The generalization to higher angular momenta is straightforward because, in the limit $\kappa \to 0$ with finite ρ, eq. (2.10) decouples and thus the solution can be expressed in terms of a confluent hypergeometric function in the same way as eq. (2.9). Both case B and C can be treated to a large extent by the WKB method. Case B is studied for the monopole and dyon cases with the lowest angular momentum in papers III and IV, respectively. Case D is especially interesting for several reasons. First, as seen from the numerical results here, many if not most of the bound states are indeed weakly bound. Secondly, the analytic results are expected to be useful for various applications, such as the study of radiative capture [14]. For these reasons papers II, V, and VI in this series will deal with various aspects of this case of weak binding.

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