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MONOPOLE-FERMION AND DYON-FERMION BOUND STATES (II). Weakly bound states for the lowest angular momentum

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We investigate weakly bound dyon-fermion states of the lowest angular momentum. Both Coulomb attraction and Coulomb repulsion are studied. Binding energies are given by a transcendental equation which is solved explicitly in a number of limiting cases. Normalized wave functions are given in terms of Bessel functions and confluent hypergeometric functions.

1. Introduction

The dyon-fermion system is interesting because of its intermediate position between the monopole-fermion system and the Coulomb system (e.g. the hydrogen atom). It is described by the hamiltonian [1]

$$H = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - ZeA) + \beta M - \frac{\zeta}{r} - \kappa q \beta \boldsymbol{\sigma} \cdot \boldsymbol{r}/(2Mr^3)\,, \tag{1.1}$$

where

$$q = Zeg = \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots,$$
$$\zeta = -ZZ_d e^2 = -ZZ_d \alpha\,. \tag{1.2}$$

Here the dyon has a magnetic charge g and an electric charge $Z_d e$, and the fermion has an electric charge Ze, mass M, and an anomalous magnetic moment κ.

The present paper is the second in a series which deals with dyon-fermion bound states. We have described in the first paper [2] the general properties of these states with detailed, accurate numerical results for the binding energies and wave functions. In this paper we concentrate on the weakly bound dyon-fermion bound states of the lowest angular momentum. The corresponding wave functions can be expressed in the form [3, 4]

$$\psi(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} \frac{\kappa q}{|\kappa q|} F(r)\eta_{jj_z}(\hat{\boldsymbol{r}}) \\ -iG(r)\eta_{jj_z}(\hat{\boldsymbol{r}}) \end{pmatrix}, \tag{1.3}$$

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involving two scalar functions $F(r)$ and $G(r)$, where η_{jj_z} is an eigensector of angular momentum.

With this expression (1.3), the eigenvalue equation

$$H\psi = E\psi \tag{1.4}$$

leads to the two coupled radial equations

$$\frac{\mathrm{d}G}{\mathrm{d}r}=\left[\frac{\kappa}{|\kappa|}\left(M-E-\frac{\zeta}{r}\right)-\frac{|\kappa q|}{2Mr^2}\right]F,$$

$$\frac{\mathrm{d}F}{\mathrm{d}r}=\left[\frac{\kappa}{|\kappa|}\left(M+E+\frac{\zeta}{r}\right)-\frac{|\kappa q|}{2Mr^2}\right]G. \tag{1.5}$$

With the notation

$$A=\tfrac{1}{2}\kappa|q|\,,\qquad B=\tfrac{1}{2}\kappa|q|E/M\,,$$

$$\rho=\frac{2M}{|\kappa q|}r, \tag{1.6}$$

$$\tilde{\zeta}=\frac{\kappa}{|\kappa|}\zeta, \tag{1.7}$$

eqs. (1.5) then take the form

$$\frac{\mathrm{d}G}{\mathrm{d}\rho}=\left(A-B-\frac{\tilde{\zeta}}{\rho}-\frac{1}{\rho^2}\right)F,$$

$$\frac{\mathrm{d}F}{\mathrm{d}\rho}=\left(A+B+\frac{\tilde{\zeta}}{\rho}-\frac{1}{\rho^2}\right)G. \tag{1.8}$$

It is the purpose of this paper to give analytic solutions to these eigenvalue equations in the limit of weak binding

$$\varepsilon=(M-E)/M\ll 1\,, \tag{1.9}$$

and with

$$|\zeta|\ll 1\,,$$

$$|A+A^{-1}|=\mathrm{O}(1) \tag{1.10}$$

(which merely means that $|A|$ is neither small nor large). The special case of $\zeta=0$, corresponding to the monopole-fermion system, has been discussed previously [5]. The limit of large A, but without the constraint of weak binding, is studied in papers III [6] and IV [7].

Under these conditions (1.9) and (1.10), the bound-state wave functions are given by (2.6), (2.7), (3.4) and (3.6). The binding energy is obtained in sect. 4, especially (4.3). The normalization of the wave function is given by (6.13) and (6.14).

2. Radial wave function in the interior region

Consider first the region where

$$\rho \ll \min[|\zeta|^{-1}, |A-B|^{-1/2}], \tag{2.1}$$

which will be referred to as the interior region. In this region

$$|A-B| \ll \rho^{-2},$$
$$|\tilde{\xi}|/\rho \ll \rho^{-2}, \tag{2.2}$$

and therefore (1.8) can be approximated by

$$\frac{dG}{d\rho} = -\frac{1}{\rho^2}F,$$
$$\frac{dF}{d\rho} = \left(A+B-\frac{1}{\rho^2}\right)G. \tag{2.3}$$

Elimination of F gives

$$\frac{d^2G}{dx^2} - \left(1 - \frac{A+B}{x^2}\right)G = 0, \tag{2.4}$$

where

$$x = 1/\rho. \tag{2.5}$$

Since there is no dependence on the electric charge parameter ζ, the previous treatment [5] of the monopole-fermion case can be used to yield

$$G = N_1\sqrt{x}K_\nu(x), \tag{2.6}$$

$$F = N_1\frac{d}{dx}[\sqrt{x}K_\nu(x)], \tag{2.7}$$

with

$$\nu = \tfrac{1}{2}[1-4(A+B)]^{1/2}. \tag{2.8}$$

This is the required answer for the interior region (2.1).

In the monopole-fermion case, this quantity ν has to be imaginary, i.e., for the existence of excited bound states [4]

$$A > \tfrac{1}{8} \quad (\zeta = 0). \tag{2.9}$$

We shall see that the critical value of A now will depend on ζ, and that ν no longer has to be imaginary.

Since

$$K_\nu(x) \simeq \frac{\pi}{2\sin(\pi\nu)}\left[\frac{(\frac{1}{2}x)^{-\nu}}{\Gamma(1-\nu)} - \frac{(\frac{1}{2}x)^{\nu}}{\Gamma(1+\nu)}\right] \tag{2.10}$$

for small x, it follows from (2.1), (2.6) and (2.7) that

$$F \simeq \frac{N_1\pi}{2\sin(\pi\nu)}\rho^{1/2}\left[\frac{(\frac{1}{2}-\nu)2^{\nu}\rho^{\nu}}{\Gamma(1-\nu)}-\frac{(\frac{1}{2}+\nu)2^{-\nu}\rho^{-\nu}}{\Gamma(1+\nu)}\right], \tag{2.11}$$

$$G \simeq \frac{N_1\pi}{2\sin(\pi\nu)}\rho^{-1/2}\left[\frac{2^{\nu}\rho^{\nu}}{\Gamma(1-\nu)}-\frac{2^{-\nu}\rho^{-\nu}}{\Gamma(1+\nu)}\right], \tag{2.12}$$

provided that

$$1 \ll \rho \ll \min\left[|\zeta|^{-1}, |A-B|^{-1/2}\right]. \tag{2.13}$$

The conditions (1.9) and (1.10) guarantee that this range (2.13) for ρ is not empty.

These expansions (2.11) and (2.12) require that $\nu < 1$ or ν imaginary, i.e., $A+B > -\frac{3}{4}$. Beyond that, there is no justification in keeping the $\rho^{-\nu}$ term; for example, at $\nu = 1$ the right-hand sides are not finite.

3. Radial wave function in the exterior region

The radial wave function in the exterior region

$$\rho \gg 1 \tag{3.1}$$

is only slightly more complicated. In this region, (1.8) can be approximated by

$$\frac{dG}{d\rho} = \left(A - B - \frac{\tilde{\zeta}}{\rho} - \frac{1}{\rho^2}\right)F,$$

$$\frac{dF}{d\rho} = (A+B)G. \tag{3.2}$$

With

$$z = 2(A^2 - B^2)^{1/2}\rho,$$

elimination of G gives

$$\frac{d^2F}{dz^2} + \left[-\frac{1}{4} + \frac{1}{2}\left(\frac{A+B}{A-B}\right)^{1/2}\frac{\zeta}{z} + \frac{A+B}{z^2}\right]F = 0. \tag{3.3}$$

This is the Whittaker equation [8], whose exponentially decreasing solution is

$$\begin{aligned} F &= N_2 W_{\lambda\nu}(z) \\ &= N_2 z^{1/2+\nu} e^{-z/2}\Psi(\tfrac{1}{2} - \lambda + \nu, 1 + 2\nu; z), \end{aligned} \tag{3.4}$$

where N_2 is a normalization constant to be determined later, W is the Whittaker function, Ψ is the confluent hypergeometric function, ν is given by eq. (2.8) and

$$\lambda = \frac{1}{2}\left(\frac{A+B}{A-B}\right)^{1/2}\zeta. \tag{3.5}$$

From eqs. (3.2) and (3.4), we find

$$G=2N_2\left(\frac{A-B}{A+B}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}z}W_{\lambda\nu}(z)$$

$$=2N_2\left(\frac{A-B}{A+B}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}z}[z^{1/2+\nu}\,\mathrm{e}^{-z/2}\Psi(\tfrac{1}{2}-\lambda+\nu,1+2\nu;z)]\,. \qquad (3.6)$$

The solutions (3.4) and (3.6) reduce to those of the monopole-fermion case when $\zeta\to 0$. If we observe the identity [8]

$$W_{0\nu}(z)=\left(\frac{z}{\pi}\right)^{1/2}K_\nu(\tfrac{1}{2}z)\,, \qquad (3.7)$$

the exterior solutions of ref. [5] follow.

It is instructive to compare the exterior region with the case of the hydrogen atom. Formally, eq. (3.3) is the same as the radial equation in the non-relativistic limit of the hydrogen Dirac equation. Apart from a change of scale in the radial variable, the orbital angular momentum term in the equation for the hydrogen atom, $l(l+1)/r^2$, has been replaced by the anomalous magnetic moment interaction, $-(A+B)/r^2$. The dyon-monopole system is thus (in the state of lowest angular momentum) at large separations similar to a "hydrogen atom" with an "angular-momentum" term which may be negative or positive depending on the sign of κ. Of course, unlike the case of the hydrogen atom, (3.3) here holds only in the exterior region (3.1).

Finally, using [8]

$$W_{\lambda\nu}(z)\simeq z^{1/2}\left[\frac{\Gamma(-2\nu)z^\nu}{\Gamma(\frac{1}{2}-\lambda-\nu)}+\frac{\Gamma(2\nu)z^{-\nu}}{\Gamma(\frac{1}{2}-\lambda+\nu)}\right], \qquad (3.8)$$

valid for small z, we get from (3.4) and (3.6)

$$F\simeq 2^{1/2}(A^2-B^2)^{1/4}N_2\rho^{1/2}$$
$$\times\left[\frac{\Gamma(-2\nu)2^\nu(A^2-B^2)^{\nu/2}\rho^\nu}{\Gamma(\frac{1}{2}-\lambda-\nu)}+\frac{\Gamma(2\nu)2^{-\nu}(A^2-B^2)^{-\nu/2}\rho^{-\nu}}{\Gamma(\frac{1}{2}-\lambda+\nu)}\right], \qquad (3.9)$$

$$G\simeq 2^{1/2}(A^2-B^2)^{1/4}(A+B)^{-1}N_2\rho^{-1/2}$$
$$\times\left[\frac{(\frac{1}{2}+\nu)\Gamma(-2\nu)2^\nu(A^2-B^2)^{\nu/2}\rho^\nu}{\Gamma(\frac{1}{2}-\lambda-\nu)}+\frac{(\frac{1}{2}-\nu)\Gamma(2\nu)2^{-\nu}(A^2-B^2)^{-\nu/2}\rho^{-\nu}}{\Gamma(\frac{1}{2}-\lambda+\nu)}\right], \qquad (3.10)$$

provided that

$$1\ll\rho\ll(A^2-B^2)^{-1/2}\,. \qquad (3.11)$$

4. Binding energy

The binding energy ε, defined by (1.9), is determined by matching the solutions in the interior and exterior regions as given in sects. 2 and 3 respectively. Under the assumptions (1.9) and (1.10), this matching is to be carried out in the range (2.13), which is contained in (3.11). By equating (2.11) and (2.12) to (3.9) and (3.10) respectively, we get

$$\frac{N_1\pi}{2\sin(\pi\nu)}\frac{\frac{1}{2}-\nu}{\Gamma(1-\nu)}=N_2[2(A^2-B^2)^{1/2+\nu}]^{1/2}\frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\lambda-\nu)},$$

$$\frac{-N_1\pi}{2\sin(\pi\nu)}\frac{\frac{1}{2}+\nu}{\Gamma(1+\nu)}=N_2[2(A^2-B^2)^{1/2-\nu}]^{1/2}\frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}-\lambda+\nu)}. \tag{4.1}$$

Therefore, the transcendental equation for the determination of ε is

$$(A^2-B^2)^\nu=-\frac{(\frac{1}{2}-\nu)\Gamma(1+\nu)\Gamma(2\nu)\Gamma(\frac{1}{2}-\lambda-\nu)}{(\frac{1}{2}+\nu)\Gamma(1-\nu)\Gamma(-2\nu)\Gamma(\frac{1}{2}-\lambda+\nu)}, \tag{4.2}$$

or alternatively by the Legendre duplication formula

$$\left(\frac{A^2-B^2}{16}\right)^\nu=\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}\left[\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right]^2\frac{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\lambda-\nu)}{\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}-\lambda+\nu)}. \tag{4.3}$$

As discussed at the end of sect. 2, (4.3) is valid only for $A+B>-\frac{3}{4}$.

Eq. (4.3) simplifies when A is negative. In this case, $\nu>\frac{1}{2}$, and hence the left-hand side is small. When this left-hand side is replaced by zero, the solution is simply

$$-\tfrac{1}{2}-\lambda+\nu=-n, \tag{4.4}$$

where $n=1,2,\dots$ is a positive integer. By (3.5), the binding energy is in this case

$$\varepsilon=\frac{\zeta^2}{2(n-\frac{1}{2}+\nu)^2}, \tag{4.5}$$

provided that $\zeta>0$, i.e., the Coulomb force being attractive. Physically, this merely confirms that attractive Coulomb interaction is needed to overcome the magnetic repulsion due to negative κ.

So far, (4.4) and equivalently (4.5) have been obtained under the assumption $A>-\frac{3}{8}$. Otherwise (4.3) cannot be used. This additional assumption is however not needed. The argument is as follows. For $A<0$, ν is real and larger than $\frac{1}{2}$. Thus, by (2.6) and (2.7), the wave function G in the interior region is an increasing function of ρ. In order to connect to the wave functions in the exterior region as given by (3.4) and (3.6), the $W_{\lambda\nu}(z)$ must be approximately a multiple of $M_{\lambda\nu}(z)$. This gives (4.4) immediately. Thus (4.4) and (4.5) hold approximately for all negative values of A, including those where ν is a positive integer.

For this case of values of A that are negative and not too small, the wave functions in the exterior region simplify: when eq. (4.4) is satisfied, we have [8]

$$W_{\lambda\nu}(z)=(-1)^{n-1}(n-1)!z^{1/2+\nu}\,e^{-z/2}L_{n-1}^{2\nu}(z), \tag{4.6}$$

where $L_{n-1}^{2\nu}(z)$ is a Laguerre polynomial.

5. Limiting cases of small and large ζ^2/ε

Eq. (4.2), or equivalently eq. (4.3) is transcendental and hence can only be solved numerically. There are however a number of limiting cases where an explicit solution is possible. In this section we consider the two cases where $|\lambda| \ll 1$ and $|\lambda| \gg 1$.

5.1. CASE OF SMALL ζ^2/ε

With $\zeta = 0$ and $A > \frac{1}{8}$, eq. (4.3) gives the spectrum obtained in ref. [5],

$$\varepsilon_0 = \frac{M-E}{M}\bigg|_{\zeta=0} = \frac{8}{A^2} \exp\left\{-\frac{2}{\beta}[\pi n + \psi - 2\varphi]\right\}, \tag{5.1}$$

where the different levels are labelled by $n = 1, 2, 3, \ldots$, and with

$$\begin{aligned} \beta &= \tfrac{1}{2}(8A-1)^{1/2}, \\ \varphi &= \arg \Gamma(1 + i\beta), \\ \psi &= \arg(\tfrac{1}{2} + i\beta) \qquad \text{or} \qquad \cos\psi = 1/\sqrt{8A}. \end{aligned} \tag{5.2}$$

In the limit

$$|\zeta/\sqrt{\varepsilon}| \ll 1, \tag{5.3}$$

or equivalently $|\lambda| \ll 1$, corrections to the spectrum (5.1) can be determined perturbatively. For this purpose, we expand

$$\Gamma(\tfrac{1}{2} - \lambda \pm i\beta) \simeq \Gamma(\tfrac{1}{2} \pm i\beta)[1 - \psi(\tfrac{1}{2} \pm i\beta)\lambda], \tag{5.4}$$

where

$$\psi(z) = \frac{d}{dz} \log \Gamma(z). \tag{5.5}$$

Using further the identity

$$\psi(\tfrac{1}{2} + i\beta) - \psi(\tfrac{1}{2} - i\beta) = i\pi \tanh(\pi\beta), \tag{5.6}$$

we obtain to lowest order in $\zeta/\sqrt{\varepsilon_0}$,

$$\varepsilon = \varepsilon_0\left[1 + \frac{\zeta}{\sqrt{2\varepsilon_0}} \frac{\pi}{\beta} \tanh(\pi\beta)\right]. \tag{5.7}$$

5.2. CASE OF LARGE ζ^2/ε WITH ATTRACTIVE COULOMB FORCE

Physically, the two cases of Coulomb attraction and Coulomb repulsion are qualitatively different: with Coulomb attraction, there is an infinite number of bound states just like for the hydrogen atom, whereas with Coulomb repulsion there is at most a finite number of bound states. Since large ζ^2/ε holds for the very weakly bound states, these two cases, corresponding to $\lambda \gg 1$ and $-\lambda \gg 1$, respectively, must be treated separately.

For $\lambda \gg 1$, we use the reflection formula and Stirling's formula to expand

$$\frac{\Gamma(\frac{1}{2}-\lambda-\nu)}{\Gamma(\frac{1}{2}-\lambda+\nu)}=\frac{\cos[\pi(\lambda-\nu)]}{\cos[\pi(\lambda+\nu)]}\frac{\Gamma(\frac{1}{2}+\lambda-\nu)}{\Gamma(\frac{1}{2}+\lambda+\nu)}$$

$$\simeq\frac{\cos[\pi(\lambda-\nu)]}{\cos[\pi(\lambda+\nu)]}\lambda^{-2\nu}. \tag{5.8}$$

With the abbreviation

$$G=(\zeta A)^{2\nu_0}\frac{\frac{1}{2}+\nu_0}{\frac{1}{2}-\nu_0}\frac{\Gamma(1-\nu_0)}{\Gamma(1+\nu_0)}\frac{\Gamma(-2\nu_0)}{\Gamma(2\nu_0)}, \tag{5.9}$$

where

$$\nu_0=(\tfrac{1}{4}-2A)^{1/2}, \tag{5.10}$$

the eigenvalue equation (4.2) takes the form

$$G\cos[\pi(\lambda+\nu_0)]+\cos[\pi(\lambda-\nu_0)]=0. \tag{5.11}$$

This equation can be solved explicitly for λ:

$$\lambda=\frac{1}{\pi}\arctan\left[\frac{G+1}{G-1}\cot(\pi\nu_0)\right]+N, \tag{5.12}$$

where N is some large positive integer. It should be noted that ν_0 may be imaginary ($A>\frac{1}{8}$) in which case G is just a phase factor, or real ($A<\frac{1}{8}$). In this case of Coulomb attraction and for highly excited states, the binding energy is then given explicitly as [cf. eq. (3.5)]

$$\varepsilon=\tfrac{1}{2}\zeta^2\left\{N+\frac{1}{\pi}\arctan\left[\frac{G+1}{G-1}\cot(\pi\nu_0)\right]\right\}^{-2}. \tag{5.13}$$

These levels are thus shifted with respect to those of the hydrogen atom by an amount which depends on A and ζ.

5.3. CASE OF LARGE ζ^2/ε WITH REPULSIVE COULOMB FORCE

When the Coulomb force is repulsive ($\lambda<0$), bound states can exist only if the magnetic interaction is sufficiently attractive, i.e., if $A>\frac{1}{8}$ and hence ν_0 is purely imaginary. The most loosely bound ones among these states can be studied by an expansion similar to that of subsect. 5.2.

For $-\lambda \gg 1$, we thus use the Stirling formula to obtain

$$\frac{\Gamma(\frac{1}{2}-\lambda-\nu_0)}{\Gamma(\frac{1}{2}-\lambda+\nu_0)}=\frac{\Gamma(\frac{1}{2}+|\lambda|-\nu_0)}{\Gamma(\frac{1}{2}+|\lambda|+\nu_0)}\simeq|\lambda|^{-2\nu_0}\exp\left(-\frac{2A\nu_0}{3\lambda^2}\right). \tag{5.14}$$

States with large $-\lambda$ are present only if

$$-\psi+2\varphi+\arg\Gamma(\tfrac{1}{2}+i\beta)-\beta\log(\tfrac{1}{4}|\zeta|A)=\pi N+\delta\,, \tag{5.15}$$

where δ is small and positive, and the definitions of (5.2) have been used. In terms of this δ, λ is given by

$$\lambda=-\left(\frac{A\beta}{3\delta}\right)^{1/2}, \tag{5.16}$$

and hence the binding energy is

$$\varepsilon=\frac{3\zeta^2\delta}{2A\beta}. \tag{5.17}$$

Eq. (5.15) also shows that the number of bound states with the lowest angular momentum $j=|q|-\frac{1}{2}$ is

$$\frac{\beta}{\pi}\ln\frac{1}{|\zeta|}+\mathrm{O}(1) \tag{5.18}$$

for small negative ζ.

6. Normalization of eigensections

The normalization condition is

$$\int_0^\infty r^2\,\mathrm{d}r\{|f(r)|^2+|g(r)|^2\}=1\,, \tag{5.1}$$

which in terms of $F(\rho)$ and $G(\rho)$ becomes

$$\frac{A}{M}\int_0^\infty \mathrm{d}\rho\,\{F^2(\rho)+G^2(\rho)\}=1\,. \tag{6.2}$$

Changing to "natural" variables for the two regions, and using the results of sects. 2 and 3, we get

$$N_1^2\int_{x_1}^\infty\frac{\mathrm{d}x}{x^2}\left\{xK_\nu^2(x)+\left(\frac{\mathrm{d}}{\mathrm{d}x}[\sqrt{x}K_\nu(x)]\right)^2\right\}$$
$$+\frac{N_2^2}{2(A^2-B^2)^{1/2}}\int_{x_2}^\infty\mathrm{d}x\left\{W_{\lambda\nu}^2(x)+4\left(\frac{A-B}{A+B}\right)\left[\frac{\mathrm{d}}{\mathrm{d}x}W_{\lambda\nu}(x)\right]^2\right\}=\frac{M}{A}, \tag{6.3}$$

where

$$x_1=\frac{1}{\rho_0},$$
$$x_2=2(A^2-B^2)^{1/2}\rho_0\,, \tag{6.4}$$

and ρ_0 satisfies (2.13).

Of the four integrals on the left-hand side of (6.3), only one is of importance, and hence

$$N_2^2 \simeq \frac{2M(A^2-B^2)^{1/2}}{\cdot A I_{\lambda\nu}}, \tag{6.5}$$

where

$$I_{\lambda\nu} = \int_{x_2}^{\infty} \mathrm{d}x\, W_{\lambda\nu}^2(x) . \tag{6.6}$$

The argument for this approximation is as follows. First, by (4.1), N_1^2 and N_2^2 are related by

$$N_1^2 = \frac{2}{\cos(\pi\nu)} \left(\frac{A-B}{A+B}\right)^{1/2} [\Gamma(\tfrac{1}{2}-\lambda+\nu)\Gamma(\tfrac{1}{2}-\lambda-\nu)]^{-1} N_2^2 . \tag{6.7}$$

Because of the assumption (1.9) of weak binding, it follows from (6.7) that N_1^2 is much smaller than N_2^2. Next, consider the case where ν is purely imaginary. In this case, the integration to the small arguments x_1 or x_2 can give at most a logarithmic factor. Therefore (6.5) follows. Finally, when ν is real, the interior wave functions in terms of Bessel functions are non-oscillatory, while the exterior ones in terms of Whittaker functions are oscillatory. Therefore, in this case, the major contributions must come from the exterior region, and the approximation (6.5) is again obtained. Note that the integral involving $[\mathrm{d}W_{\lambda\nu}(x)/\mathrm{d}x]^2$ is smaller because

$$\frac{A-B}{A+B} x_2^{-2} \ll 1 . \tag{6.8}$$

We proceed to evaluate the $I_{\lambda\nu}$ of (6.6). First, we replace the lower limit of integration by 0:

$$I_{\lambda\nu} = \int_0^{\infty} \mathrm{d}x\, W_{\lambda\nu}^2(x) . \tag{6.9}$$

This is clearly justified for $A+B > -\frac{3}{4}$; it is also justified for negative A because of (4.4). In order to evaluate this integral (6.9), we note that

$$\begin{aligned} &\tfrac{1}{2} W_{\lambda\nu}^2(x) - \frac{\lambda}{x} W_{\lambda\nu}^2(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\Bigg[\tfrac{1}{4} x W_{\lambda\nu}^2(x) - \frac{\frac{1}{4}-\nu^2}{x} W_{\lambda\nu}^2(x) - \lambda W_{\lambda\nu}^2(x) - x\left(\frac{\mathrm{d}}{\mathrm{d}x} W_{\lambda\nu}(x)\right)^2 \\ &\quad + W_{\lambda\nu}(x) \frac{\mathrm{d}}{\mathrm{d}x} W_{\lambda\nu}(x)\Bigg] , \end{aligned} \tag{6.10}$$

since $W_{\lambda\nu}(x)$ satisfies the differential equation (3.3). We then use [9]

$$\int_0^{\infty} \frac{\mathrm{d}x}{x} W_{\lambda\nu}^2(x) = \frac{\pi}{\sin(2\pi\nu)} \frac{\psi(\frac{1}{2}-\lambda+\nu) - \psi(\frac{1}{2}-\lambda-\nu)}{\Gamma(\frac{1}{2}-\lambda+\nu)\Gamma(\frac{1}{2}-\lambda-\nu)} , \tag{6.11}$$

take the limit of the right-hand side of eq. (6.10) as $x \to 0$, and find

$$I_{\lambda\nu} = \frac{2\pi}{\sin(2\pi\nu)} [\Gamma(\tfrac{1}{2}-\lambda+\nu)\Gamma(\tfrac{1}{2}-\lambda-\nu)]^{-1}\{2\nu+\lambda[\psi(\tfrac{1}{2}-\lambda+\nu)-\psi(\tfrac{1}{2}-\lambda-\nu)]\}\,. \tag{6.12}$$

A more direct, but somewhat subtle way of obtaining this result is given in the appendix.

The normalization constants are thus given by

$$N_2^2 = \frac{\sin(2\pi\nu)}{\pi}[2M(M-E)]^{1/2}\Gamma(\tfrac{1}{2}-\lambda+\nu)\Gamma(\tfrac{1}{2}-\lambda-\nu) \times\{2\nu+\lambda[\psi(\tfrac{1}{2}-\lambda+\nu)-\psi(\tfrac{1}{2}-\lambda-\nu)]\}^{-1}\,, \tag{6.13}$$

together with

$$\frac{N_1}{N_2} = \frac{4}{1-2\nu}[2(A^2-B^2)^{1/2+\nu}]^{1/2}\frac{\Gamma(-2\nu)}{\Gamma(\nu)\Gamma(\tfrac{1}{2}-\lambda-\nu)}\,. \tag{6.14}$$

Note that the first equation of (4.1) is used to get (6.14). The second equation is not suitable for this purpose because it fails when ν is a positive integer.

Eqs. (6.13) and (6.14) are the general formulae for the normalization constants; they are valid for both real and purely imaginary values of ν. For negative values of A, they simplify in much the same way as in sect. 4. By (4.4)

$$\frac{\Gamma(\tfrac{1}{2}-\lambda+\nu)}{\psi(\tfrac{1}{2}-\lambda+\nu)} = \frac{(-1)^n}{(n-1)!}\,, \tag{6.15}$$

and hence

$$N_2^2 = \frac{[2M(M-E)]^{1/2}}{(n-\tfrac{1}{2}+\nu)(n-1)!\,\Gamma(n+2\nu)}\,, \tag{6.16}$$

together with

$$\frac{N_1}{N_2} = (-1)^n\frac{4}{1-2\nu}[2(A^2-B^2)^{1/2+\nu}]^{1/2}\frac{\Gamma(n+2\nu)}{\Gamma(\nu)\Gamma(1+2\nu)}\,. \tag{6.17}$$

7. Numerical results

In this section we shall give a few numerical results. These serve to supplement those given in paper I[2], and also to determine how accurate the weak-binding approximation is.

In table 1 we present binding energies for some of the lowest states, considering positive as well as negative values of A (or κ), and for a few values of ζ, corresponding to Coulomb attraction as well as repulsion. The agreement between the values obtained in the present limit of weak binding and those obtained from the accurate

TABLE 1

Dyon-fermion binding energies $(M-E)/M$ for $j=|q|-\frac{1}{2}$ and for a few values of A and ζ

A	n	$\zeta=-\alpha$		$\zeta=\alpha$		$\zeta=0.1$		$\zeta=0.5$	
		Numerical	WBA	Numerical	WBA	Numerical	WBA	Numerical	WBA
4.0	3	$5.371\cdot10^{-4}$	$5.365\cdot10^{-4}$	$8.418\cdot10^{-4}$	$8.408\cdot10^{-4}$	$3.314\cdot10^{-3}$	$3.305\cdot10^{-3}$	$2.134\cdot10^{-2}$	$2.088\cdot10^{-2}$
	2	$6.072\cdot10^{-3}$	$6.026\cdot10^{-3}$	$7.007\cdot10^{-3}$	$6.954\cdot10^{-3}$	$1.353\cdot10^{-2}$	$1.341\cdot10^{-2}$	$5.116\cdot10^{-2}$	$4.955\cdot10^{-2}$
	1	$6.081\cdot10^{-2}$	$5.743\cdot10^{-2}$	$6.359\cdot10^{-2}$	$6.008\cdot10^{-2}$	$8.177\cdot10^{-2}$	$7.733\cdot10^{-2}$	$1.701\cdot10^{-1}$	$1.583\cdot10^{-1}$
2.0	3			$9.312\cdot10^{-5}$	$9.311\cdot10^{-5}$	$1.519\cdot10^{-3}$	$1.516\cdot10^{-3}$	$1.624\cdot10^{-2}$	$1.667\cdot10^{-2}$
	2	$6.809\cdot10^{-4}$	$6.802\cdot10^{-4}$	$1.190\cdot10^{-3}$	$1.189\cdot10^{-3}$	$5.507\cdot10^{-2}$	$5.485\cdot10^{-2}$	$3.861\cdot10^{-2}$	$3.731\cdot10^{-2}$
	1	$2.236\cdot10^{-2}$	$2.187\cdot10^{-2}$	$2.487\cdot10^{-2}$	$2.434\cdot10^{-2}$	$4.202\cdot10^{-2}$	$4.105\cdot10^{-2}$	$1.357\cdot10^{-1}$	$1.281\cdot10^{-1}$
1.0	3			$1.740\cdot10^{-5}$	$1.740\cdot10^{-5}$	$9.496\cdot10^{-4}$	$9.479\cdot10^{-4}$	$1.456\cdot10^{-2}$	$1.411\cdot10^{-2}$
	2			$1.464\cdot10^{-4}$	$1.464\cdot10^{-4}$	$2.819\cdot10^{-3}$	$2.811\cdot10^{-3}$	$3.258\cdot10^{-2}$	$3.127\cdot10^{-2}$
	1	$4.604\cdot10^{-3}$	$4.583\cdot10^{-3}$	$6.403\cdot10^{-3}$	$6.376\cdot10^{-3}$	$2.029\cdot10^{-2}$	$2.013\cdot10^{-2}$	$1.151\cdot10^{-2}$	$1.086\cdot10^{-1}$
0.5	3			$7.311\cdot10^{-6}$	$7.311\cdot10^{-6}$	$7.264\cdot10^{-4}$	$7.255\cdot10^{-4}$	$1.369\cdot10^{-2}$	$1.319\cdot10^{-2}$
	2			$3.040\cdot10^{-5}$	$3.040\cdot10^{-5}$	$1.874\cdot10^{-3}$	$1.872\cdot10^{-3}$	$3.005\cdot10^{-2}$	$2.863\cdot10^{-2}$
	1	$1.239\cdot10^{-4}$	$1.238\cdot10^{-4}$	$9.333\cdot10^{-4}$	$9.332\cdot10^{-4}$	$1.072\cdot10^{-2}$	$1.070\cdot10^{-2}$	$1.055\cdot10^{-1}$	$9.884\cdot10^{-2}$
−0.5	3			$2.034\cdot10^{-6}$	$2.034\cdot10^{-6}$	$3.839\cdot10^{-4}$	$3.798\cdot10^{-4}$	$1.024\cdot10^{-2}$	$1.220\cdot10^{-2}$
	2			$3.885\cdot10^{-6}$	$3.884\cdot10^{-6}$	$7.344\cdot10^{-4}$	$7.239\cdot10^{-4}$	$1.997\cdot10^{-2}$	$2.552\cdot10^{-2}$
	1			$1.017\cdot10^{-5}$	$1.017\cdot10^{-5}$	$1.928\cdot10^{-3}$	$1.888\cdot10^{-3}$	$5.411\cdot10^{-2}$	$7.940\cdot10^{-2}$
−1.0	3			$1.664\cdot10^{-6}$	$1.664\cdot10^{-6}$	$3.131\cdot10^{-4}$	$3.125\cdot10^{-4}$	$8.089\cdot10^{-3}$	$7.774\cdot10^{-3}$
	2			$2.958\cdot10^{-6}$	$2.958\cdot10^{-6}$	$5.568\cdot10^{-4}$	$5.557\cdot10^{-4}$	$1.448\cdot10^{-2}$	$1.376\cdot10^{-2}$
	1			$6.656\cdot10^{-6}$	$6.656\cdot10^{-6}$	$1.254\cdot10^{-3}$	$1.251\cdot10^{-3}$	$3.283\cdot10^{-2}$	$3.066\cdot10^{-2}$
−2.0	3			$1.280\cdot10^{-6}$	$1.280\cdot10^{-6}$	$2.405\cdot10^{-4}$	$2.403\cdot10^{-4}$	$6.102\cdot10^{-3}$	$6.099\cdot10^{-3}$
	2			$2.099\cdot10^{-6}$	$2.099\cdot10^{-6}$	$3.945\cdot10^{-4}$	$3.942\cdot10^{-4}$	$1.003\cdot10^{-2}$	$1.004\cdot10^{-2}$
	1			$4.058\cdot10^{-6}$	$4.058\cdot10^{-6}$	$7.626\cdot10^{-4}$	$7.620\cdot10^{-4}$	$1.938\cdot10^{-2}$	$1.947\cdot10^{-2}$
−4.0	3			$9.225\cdot10^{-7}$	$9.225\cdot10^{-7}$	$1.733\cdot10^{-4}$	$1.732\cdot10^{-4}$	$4.360\cdot10^{-3}$	$4.329\cdot10^{-3}$
	2			$1.393\cdot10^{-6}$	$1.393\cdot10^{-6}$	$2.616\cdot10^{-4}$	$2.615\cdot10^{-4}$	$6.581\cdot10^{-3}$	$6.535\cdot10^{-3}$
	1			$2.341\cdot10^{-6}$	$2.341\cdot10^{-6}$	$4.398\cdot10^{-4}$	$4.397\cdot10^{-4}$	$1.105\cdot10^{-2}$	$1.099\cdot10^{-2}$

"Numerical" refers to the method of paper I [2], whereas "WBA" refers to the weak binding approximation of the present paper. (The WBA results improve slightly if one replaces $A+B$ in the eigenvalue equation by $2A$.)

approach of paper I is in general excellent. Exceptions consist of the cases where our assumptions are not satisfied, e.g., where the binding gets fairly strong (large A, $n=1$) or where ζ is comparable to A.

Some wave functions are shown in figs. 1–4 for the case of Coulomb attraction. For fixed $\zeta=\alpha$, two positive values of A are considered in figs. 1 and 2. It is seen that for the larger value of A the wave functions do not extend as far out, due to the stronger magnetic attraction. Also, the higher states ($n=3$) have a marked Coulomb distortion; their outer oscillations have much shorter "period" (in $\log \rho$) than the inner ones. (Compare the inversion symmetry valid for the monopole case [5].)

Similarly, figs. 3 and 4 show the corresponding wave functions for A negative: $A=-\frac{1}{2}$ in fig. 3 and $A=-1$ in fig. 4. In these cases the Coulomb interaction provides the only attraction, the magnetic interaction being repulsive. It should be noted that, in contrast to the cases of positive A (or κ) considered in figs. 1 and 2, the wave function F now has one node less than G. (Also, it is seen that no problem arises for $A<-\frac{3}{4}$; compare the discussion at the end of sect. 4.)

In figs. 1–4, exact wave functions are also given, as obtained numerically by the method described in paper I. The agreement is excellent.

For the case of Coulomb repulsion, we show in fig. 5 the relationship between values of A and ζ that correspond to the onset of various levels. These curves are given by eq. (5.15). From these curves one can also read off how many bound states exist for some specified values of A and ζ as the number of curves that lie "outside" the point (A, ζ).

The curves given in fig. 5 were obtained under the assumption that $|\zeta| \ll 1$. The corresponding exact curves, valid also for larger values of $-\zeta$, have been shown, too. They are obtained as follows. Consider some value of ζ and $B=A$ (for zero binding energy). Then the exact differential equations (1.8) can be integrated out from some small value of ρ[2] and the asymptotic behaviour determined. By adjusting the value of A, one can find the values A_n that correspond to the bound states labelled n (where F and G have n nodes and are bounded as $\rho \to \infty$). Such points (A_n, ζ) define the exact curves given in fig. 5.

8. Range of validity

Even for the states of minimum angular momentum $j=|q|-\frac{1}{2}$, the limit of weak binding treated here is only one of the interesting limits. Another interesting limit, $A \gg 1$, is dealt with in papers III and IV for the monopole and dyon cases respectively.

For clarity of presentation we have here assumed that $|A|$ is neither small nor large [see eq. (1.10)]. Actually, the method employed does not require this assumption. In this section, we discuss the conditions for the validity of the present procedure, including that of the basic formula (4.3), when $A>-\frac{3}{4}$. Without loss of generality, A and B are taken to have the same sign.

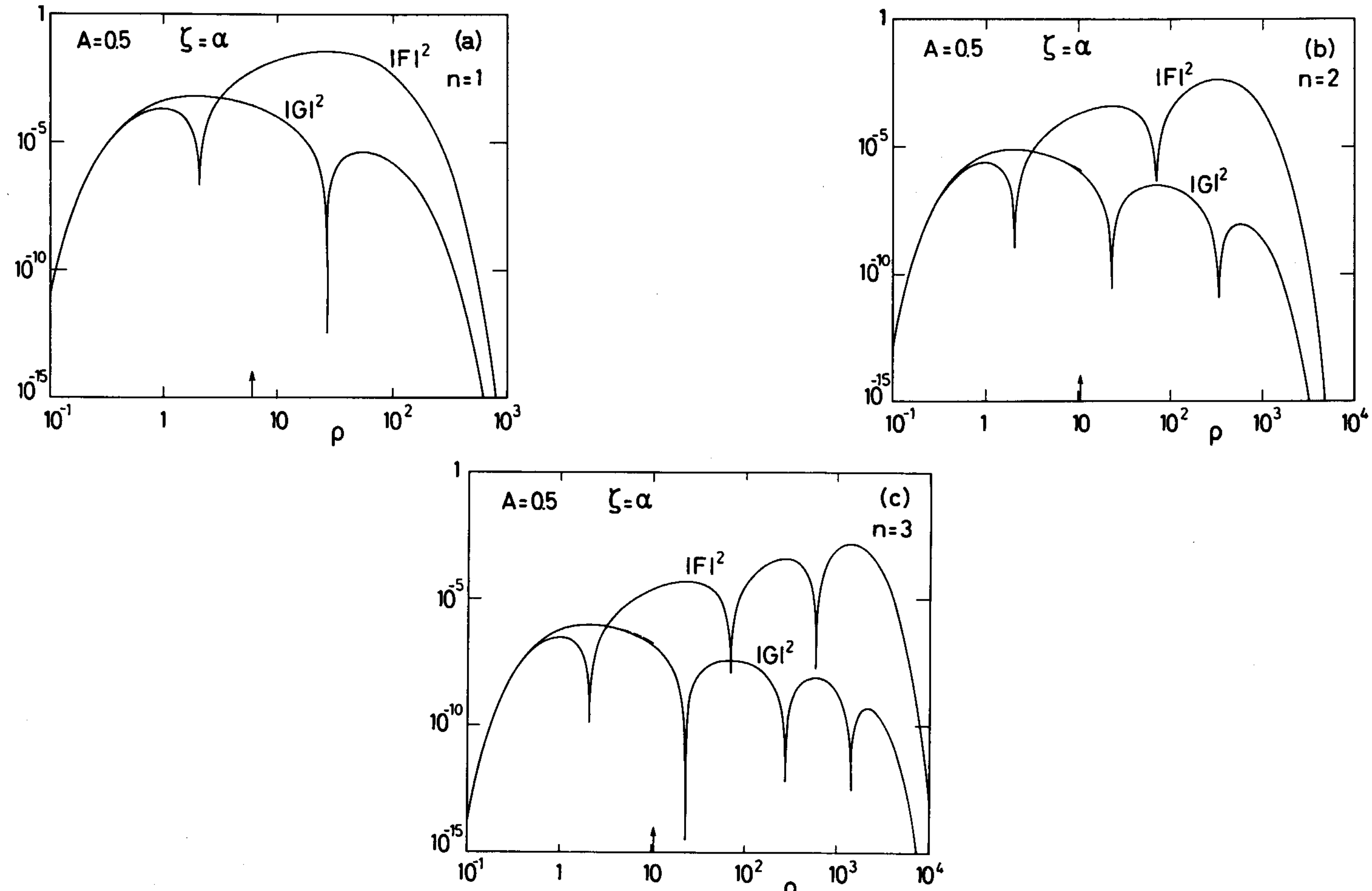

Fig. 1. Squares of the dyon-fermion radial wave functions F and G for $A = 0.5$ and $\zeta = \alpha \simeq \frac{1}{137}$. The three states $n = 1, 2$ and 3 are considered, with $j = |q| - \frac{1}{2}$. The wave functions are given by the explicit formulas of sects. 2 and 3, and matched at the points indicated by the arrows. Unless indicated by dashes, the present weak-binding results deviate from the more accurate ones determined by the method of paper I by less than the resolution of these plots. [The minima are actually zeros of the wave functions.]

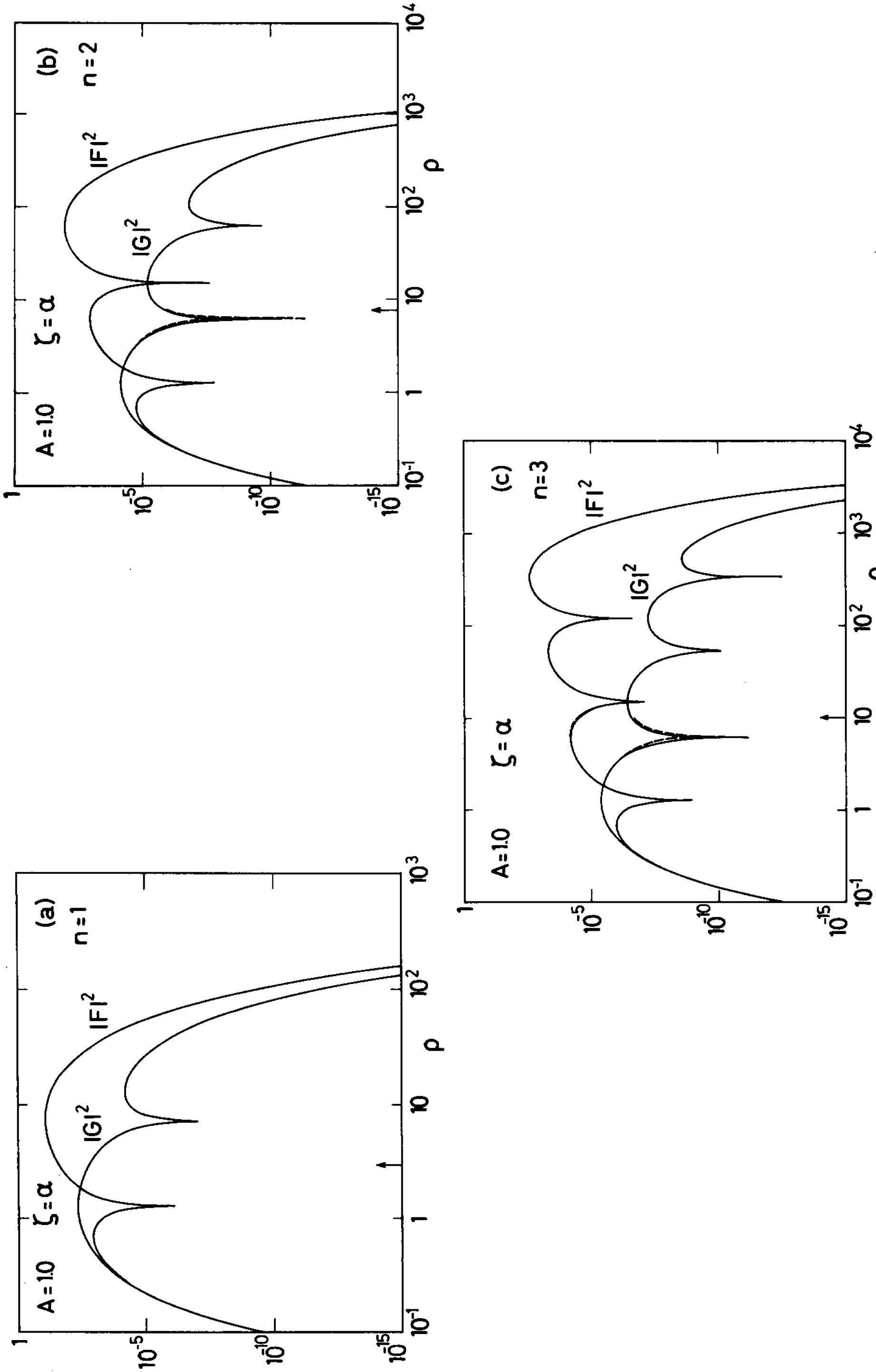

Fig. 2. Same as fig. 1 but for $A = 1.0$.

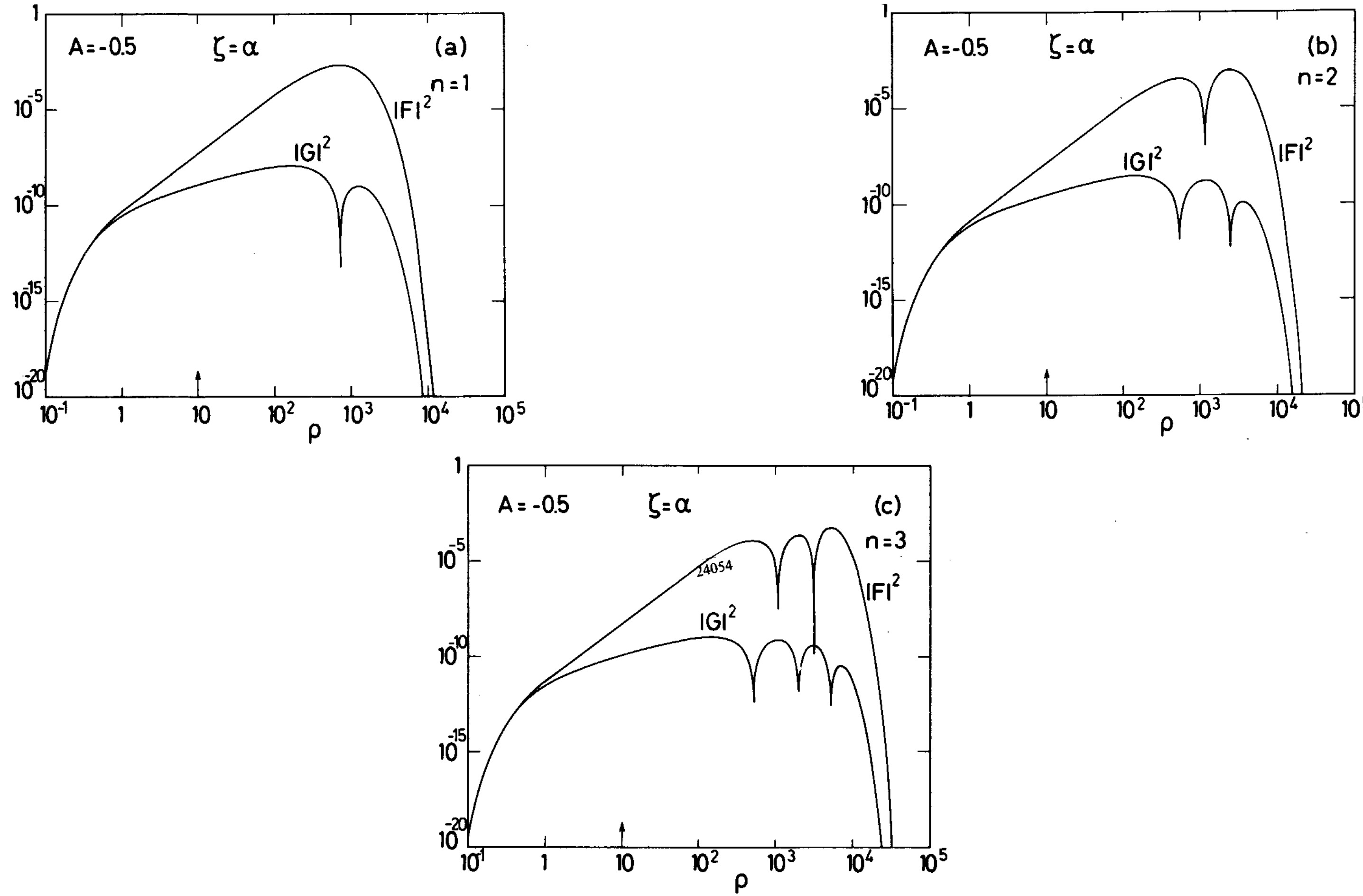

Fig. 3. Same as fig. 1 but for $A = -0.5$. In this case, as well as that considered in fig. 4, the Coulomb interaction provides the only attraction.

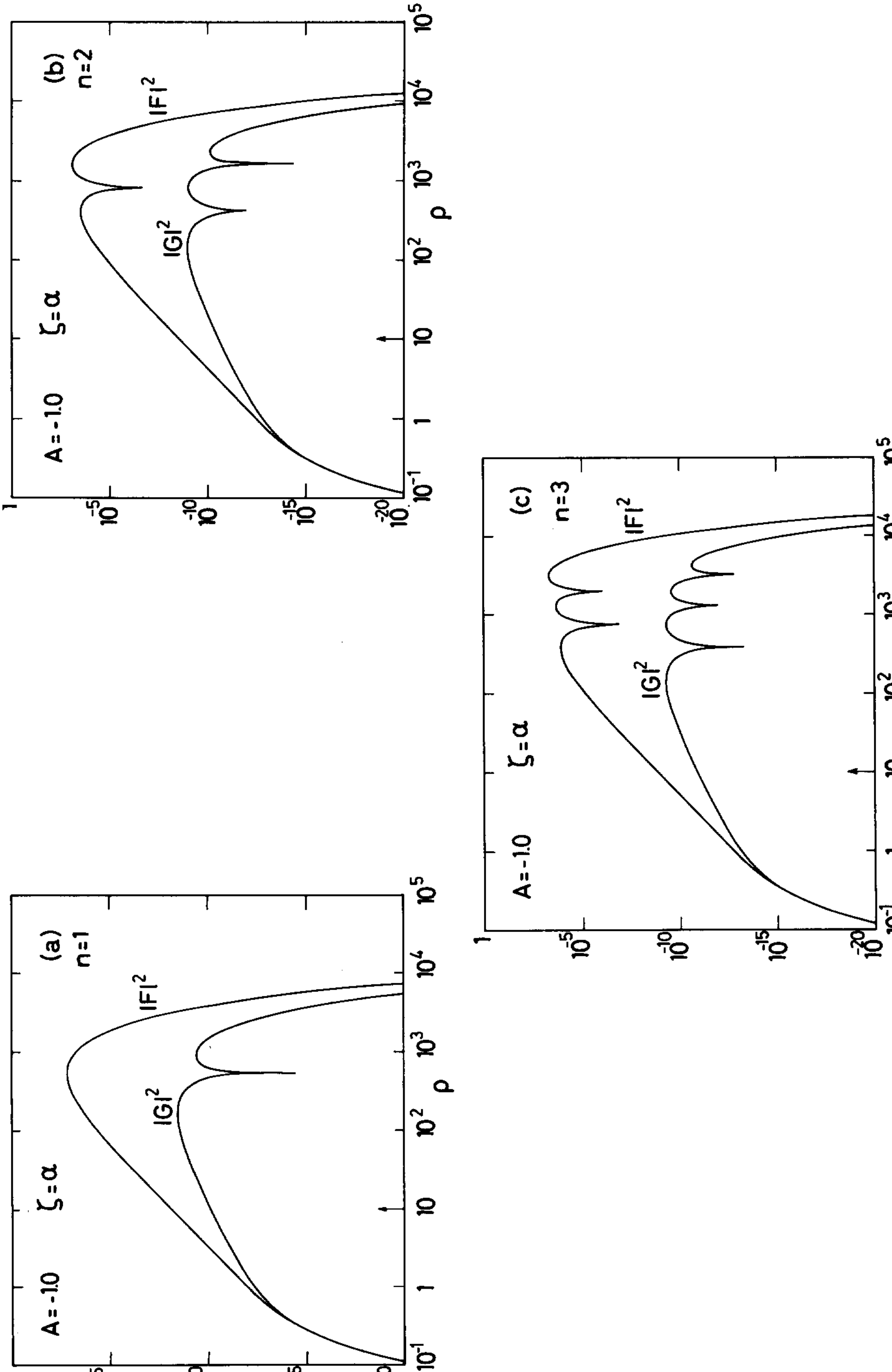

Fig. 4. Same as fig. 1 but for $A = -1.0$. (In this case, the wave functions are well approximated in terms of Laguerre polynomials.)

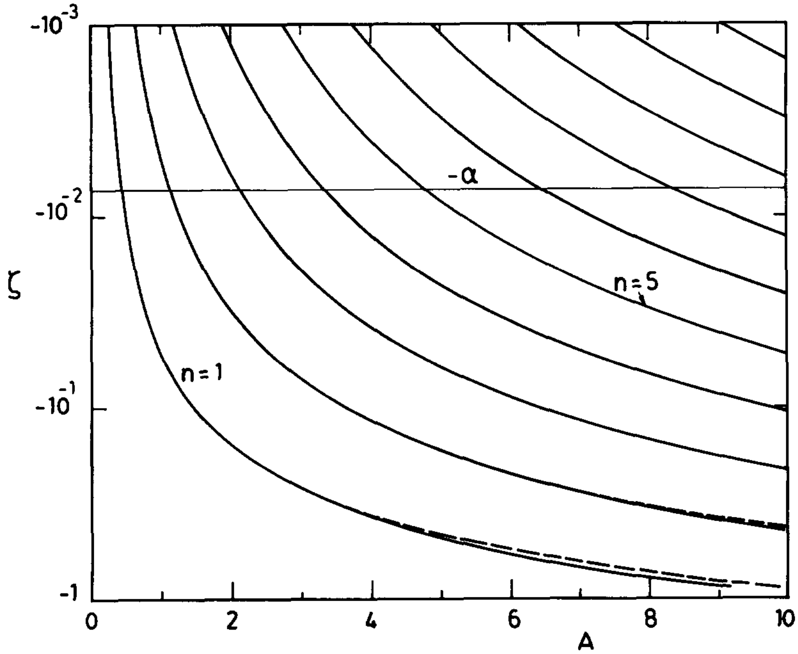

Fig. 5. Critical values of $\zeta<0$ which determine the onset of various levels n, as a function of A. The dashed curves are determined from eq. (5.15), which is valid only for $|\zeta| \ll \sqrt{A}$. The solid curves are exact, i.e., they are valid also for larger values of $|\zeta|$.

First, the validity of the approximation (2.3) depends on the magnitudes of ζ and $A-B$, but not on that of A. Let

$$\beta_0 = 1 + |A|^{1/2}, \tag{8.1}$$

then (2.10) holds when

$$x^2 \ll \beta_0 . \tag{8.2}$$

Therefore (2.13) should be generalized to

$$\beta_0^{-1/2} \ll \rho \ll \min\,[|\zeta|^{-1}, |A-B|^{-1/2}] . \tag{8.3}$$

This is non-empty if

$$|\zeta| \ll \beta_0^{1/2}, \tag{8.4}$$

$$|A-B| \ll \beta_0 . \tag{8.5}$$

With reference to sect. 3, (3.1) is to be replaced by

$$\rho \gg \max\,[|\zeta/A|, |A|^{-1/2}] \tag{8.6}$$

in order for the approximation (3.2) to hold in the exterior region. Since

$$M_{\lambda\nu}(z) = z^{1/2+\nu}\left[1 - \frac{\lambda}{1+2\nu} z + \frac{4\lambda^2+1+2\nu}{16(1+\nu)(1+2\nu)} z^2 + O(z^3)\right] \tag{8.7}$$

for small z, (3.8) holds when

$$z \ll \min\left[\beta_0/|\lambda|, \beta_0^{1/2}\right]. \tag{8.8}$$

Therefore the generalization of (3.11) is

$$\max\left[|\zeta/A|, |A|^{-1/2}\right] \ll \rho \ll \min\left[\frac{\beta_0}{|\zeta A|}, \left(\frac{\beta_0}{A(A-B)}\right)^{1/2}\right]. \tag{8.9}$$

This is non-empty when (8.4) and (8.5) hold, and furthermore

$$|\zeta| \ll \beta_0 |A|^{-1/2}, \tag{8.10}$$

$$\zeta^2 |A-B| \ll \beta_0 |A|. \tag{8.11}$$

In order to apply the procedure of this paper, the ranges (8.3) and (8.9) must overlap. This requires

$$\max\left[|\zeta/A|, |A|^{-1/2}\right] \ll \min\left[|\zeta|^{-1}, |A-B|^{-1/2}\right], \tag{8.12}$$

$$\beta_0^{-1/2} \ll \min\left[\frac{\beta_0}{|\zeta A|}, \left(\frac{\beta_0}{A(A-B)}\right)^{1/2}\right]. \tag{8.13}$$

The inequality (8.12) implies

$$|\zeta| \ll |A|^{1/2}, \tag{8.14}$$

$$|A-B| \ll |A|, \tag{8.15}$$

which is the original condition of small ε. Similarly, (8.13) implies that

$$|\zeta| \ll \beta_0^{3/2}/|A|, \tag{8.16}$$

$$|A-B| \ll \beta_0^2/|A|. \tag{8.17}$$

It only remains to simplify the conditions (8.4), (8.5), (8.10), (8.11) and (8.14)–(8.17). First, (8.5) and (8.14) imply (8.11). By (8.1) the conditions (8.4), (8.10), (8.14) and (8.16) give simply

$$|\zeta| \ll \min\left[|A|^{1/2}, (1+|A|^{1/2})^{3/2}/|A|\right], \tag{8.18}$$

while (8.5), (8.15) and (8.17) give

$$|A-B| \ll \min\left[|A|, (1+|A|^{1/2})^2/|A|\right]. \tag{8.19}$$

These are the required conditions.

It is somewhat more explicit to rewrite these conditions (8.18) and (8.19) as allowing the following two cases:

(i) When $|A| = O(1)$, then the conditions are

$$\frac{A-B}{A+B} \ll 1, \tag{8.20}$$

$$|\zeta| \ll |A|^{1/2}. \tag{8.21}$$

(ii) When $A \gg 1$, then the conditions are

$$|A-B| \ll 1\,, \tag{8.22}$$

$$|\zeta| \ll A^{-1/4}\,. \tag{8.23}$$

The conditions (8.22) and (8.23) are rather stringent. They will be relaxed in paper IV.

Appendix

We shall here present an alternative evaluation of the normalization integral

$$I_{\lambda\nu} = \int_0^\infty \mathrm{d}x\, W^2_{\lambda\nu}(x)\,. \tag{A.1}$$

An integral of this type is given in ref. [9] in terms of generalized hypergeometric functions:

$$\begin{aligned} J^\rho_{\lambda\nu} &= \int_0^\infty \mathrm{d}x\, x^{\rho-1} W^2_{\lambda\nu}(x) \\ &= \frac{\Gamma(1+\rho+2\nu)\Gamma(1+\rho)\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\lambda-\nu)\Gamma(\frac{3}{2}+\rho-\lambda+\nu)} \\ &\quad \times {}_3F_2(1+\rho+2\nu, 1+\rho, \tfrac{1}{2}-\lambda+\nu; 1+2\nu, \tfrac{3}{2}+\rho-\lambda+\nu; 1) + (\nu \to -\nu). \end{aligned} \tag{A.2}$$

For the left-hand side to be defined one must require [9] $2|\mathrm{Re}\,\nu| < \mathrm{Re}\,\rho + 1$.

We need this integral for $\rho = 1$. The above condition is satisfied for $A > -\frac{1}{8}$, to which we restrict the present discussion. However, the right-hand side is not defined for $\rho \geq 0$. This may be seen by writing it out in terms of Γ-functions:

$$\begin{aligned} J^\rho_{\lambda\nu} &= \frac{-\pi}{\sin(2\pi\nu)}[\Gamma(\tfrac{1}{2}-\lambda+\nu)\Gamma(\tfrac{1}{2}-\lambda-\nu)]^{-1} \\ &\times \sum_{n=0}^\infty \left[\frac{\Gamma(1+\rho+2\nu+n)\Gamma(1+\rho+n)\Gamma(\frac{1}{2}-\lambda+\nu+n)}{\Gamma(1+2\nu+n)\Gamma(\frac{3}{2}+\rho-\lambda+\nu+n)\Gamma(1+n)} - (\nu \to -\nu)\right]. \end{aligned} \tag{A.3}$$

Using the Stirling formula we find that the above sum is convergent only for $\rho < 0$, whereas we need it for $\rho = 1$.

Here comes the subtle point. We need only the part of the sum that is odd under $(\nu \to -\nu)$. That part is convergent, at large n and $\rho = 1$ it goes like $1/n^2$. *However, it converges to an incorrect result.* The reason is that if one sets $\rho = 1$ in eq. (A.3), one drops a term that for large n behaves like

$$(\rho-1)\sum_n n^{-2+\rho}\,, \tag{A.4}$$

and *has a finite limit as* $\rho \to 1^-$.

Let us now evaluate these two contributions to $J^{\rho}_{\lambda\nu}$. We define the two sums S_1 and S_2 through

$$J^{\rho}_{\lambda\nu}=\frac{-\pi}{\sin(2\pi\nu)}[\Gamma(\tfrac{1}{2}-\lambda+\nu)\Gamma(\tfrac{1}{2}-\lambda-\nu)]^{-1}(S_1+S_2)\,, \tag{A.5}$$

where the first sum, S_1, is obtained from (A.3) by putting $\rho=1$:

$$\begin{aligned}S_1&=\sum_{n=0}^{\infty}\left[\frac{\Gamma(2+2\nu+n)\Gamma(2+n)\Gamma(\frac{1}{2}-\lambda+\nu+n)}{\Gamma(1+2\nu+n)\Gamma(\frac{5}{2}-\lambda+\nu+n)\Gamma(1+n)}-(\nu\to-\nu)\right]\\&=\sum_{n=0}^{\infty}\left[\frac{(1+n)(1+2\nu+n)}{(\frac{3}{2}-\lambda+\nu+n)(\frac{1}{2}-\lambda+\nu+n)}-(\nu\to-\nu)\right].\end{aligned} \tag{A.6}$$

Resolving the ratio in partial fractions, we find convergent sums that are recognized in terms of ψ-functions:

$$S_1=-2\nu-2\lambda[\psi(\tfrac{1}{2}-\lambda+\nu)-\psi(\tfrac{1}{2}-\lambda-\nu)]\,. \tag{A.7}$$

The second sum will be of the form (A.4). In order to identify it we use the Stirling formula to expand

$$\frac{\Gamma(a+n)}{\Gamma(b+n)}\doteq n^{a-b}\left\{1+\frac{1}{2n}[a(a-1)-b(b-1)]\right\}, \tag{A.8}$$

valid for $n\gg|a|$, $|b|$. The "1" corresponds to the sum S_1, whereas the subdominant terms have a power $n^{-2+\rho}$ and coefficients that add up to $2\nu(\rho-1)$. Thus

$$S_2=\lim_{\rho\to1^-}\sum_n 2\nu(\rho-1)n^{-2+\rho}\,. \tag{A.9}$$

This may be evaluated in the following way:

$$S_2=2\nu\lim_{\rho\to1^-}(\rho-1)\left[\int_1^{\infty}\frac{dy}{y^{2-\rho}}+\text{finite}\right]=-2\nu\,. \tag{A.10}$$

Adding (A.7) and (A.10) we see that the expression (A.5) is the same as that in eq. (6.12).

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