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THE SIZE OF FINITE SIZE EFFECTS IN LATTICE GAUGE THEORIES

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If a quantum field is enclosed in a spatial box of finite volume, its mass spectrum depends on the box size L. For field theories in the continuum Lüscher has shown to all orders in perturbation theory that for large L this dependence is related to certain scattering amplitudes of the infinite volume theory. We derived the corresponding relations for lattice field theories. Assuming their validity for lattice gauge theory outside the perturbative region the magnitude of finite size effects on the spectrum is determined by a glueball coupling constant. This quantity is estimated by strong coupling methods.

1. Introduction and summary

Consider quantum field theory in a finite spatial volume, whereas time is assumed to be unrestricted. For definiteness we take the case of a periodic box of linear extent L. Then the spectrum of particle masses M_i will depend on L. We assume that

$$\lim_{L\to\infty} M_i = m_i \tag{1}$$

exists and yields the mass spectrum of the infinite volume theory. In particular we shall concentrate on the mass gap M_0 and the corresponding infinite volume mass gap m_0, which we assume to be nonzero.

For small L the behaviour of $M_i(L)$ is accessible to perturbation theory. For gauge field theory in the continuum this has been studied in [1].

On the other hand one is interested in the way the masses approach their infinite volume limits. Knowledge of the relative deviation

$$\delta_0 = (M_0 - m_0)/m_0 \tag{2}$$

would be important for all attempts to obtain the mass gap m_0 from calculations in a finite volume, as it is the case for the Monte Carlo method.

For scalar field theories in the continuum a relation between δ_0 and a forward scattering amplitude of the infinite volume theory has been established by Lüscher to all orders in perturbation theory [2]. It implies the asymptotic behaviour

$$\delta_0 \sim -C\zeta^{-p} \exp(-\alpha\zeta), \tag{3}$$

where

$$\zeta = m_0 L, \tag{4}$$

$$\tfrac{1}{2}\sqrt{3} \leqslant \alpha \leqslant 1, \qquad p \geqslant 0, \tag{5}$$

and C is some constant.

It is the purpose of this paper to study the large-L behaviour of δ_0 for euclidean field theories on a lattice. I derived a relation between δ_0 and the four-point function at certain on-shell momenta, analogous to the one in the continuum, in perturbation theory. Furthermore an expression for the asymptotic decay of δ_0 similar to (3) is obtained.

These results refer to the case of a scalar field in perturbation theory. They do, however, not depend on the detailed kind of interactions and are of a purely kinematical nature. Therefore it appears reasonable that they also apply in more general situations. In particular I believe that they are also true to all orders in the strong coupling expansion of lattice gauge theory. A rigorous proof of this would, however, be laborious and I did not carry it through.

Assuming the relations mentioned above also to hold for lattice gauge theory, the constant C in (3), which determines the magnitude of finite size effects on the mass gap, and the other two parameters α and p are in the scaling region given by

$$C = \frac{3}{16\pi}\frac{\lambda^2}{m_0^2}, \qquad \alpha = \tfrac{1}{2}\sqrt{3}, \qquad p = 1, \tag{6}$$

where λ is a 3-glueball coupling constant, i.e. the value of a three-point function at certain on-shell momenta. I studied this quantity in the framework of the strong coupling expansion. Its order of magnitude is estimated to be

$$\lambda^2/m_0^2 \approx 2 \cdot 10^3, \tag{7}$$

implying

$$C \approx 100. \tag{8}$$

2. The mass shift in scalar lattice field theories

We consider a $(d+1)$-dimensional hypercubical lattice of spatial extent L and impose periodic boundary conditions for the real scalar field $\phi(x)$. The lattice is infinite in the time direction x_0. The euclidean action is

$$S = \sum_x \left\{ \tfrac{1}{2}\sum_\mu (\nabla_\mu \phi(x))^2 + \tfrac{1}{2}m^2\phi^2(x) + \mathscr{L}_1(\phi(x), \nabla\phi(x)) \right\}, \tag{9}$$

where

$$\nabla_\mu \phi(x) = \phi(x+\hat{\mu}) - \phi(x), \qquad \mu = 0, \dots, d, \tag{10}$$

$$\hat{\mu}_\nu = \delta_{\mu\nu}, \tag{11}$$

and the interaction $\mathscr{L}_I$ should not depend on L. $\mathscr{L}_I$ does not include a kinetic term; otherwise it is an arbitrary polynomial in the field and its derivatives and will in general contain a mass term.

Let

$$G_L(x) = \langle\phi(x)\phi(0)\rangle_L - \langle\phi(0)\rangle_L^2 \tag{12}$$

be the full propagator. The mass gap $M_0(L)$ governs the exponential decay of G_L in the time direction. In the absence of an interaction the propagator equals the free lattice propagator

$$\Delta_L(x) = \int_{-\pi}^{\pi} \frac{dp_0}{2\pi} L^{-d} \sum_p e^{ip\cdot x} \left\{ 2\sum_\mu (1-\cos p_\mu) + m^2 \right\}^{-1} . \tag{13}$$

Here the momentum sum ranges over

$$p_k = \frac{2\pi}{L}\nu_k , \qquad \nu_k \in \mathbb{Z} , \qquad -\tfrac{1}{2}L \leq \nu_k < \tfrac{1}{2}L , \tag{14}$$

$$\text{for } k = 1, \ldots, d .$$

In this case the mass gap m_0 does not depend on L and we have

$$\cosh m_0 = 1 + \tfrac{1}{2}m^2 . \tag{15}$$

In the interacting case the mass counterterm contained in $\mathscr{L}_I$ is chosen in such a way that in the limit $L \to \infty$ the mass gap $M_0(L)$ coincides with the one of the free theory:

$$M_0(\infty) = m_0 . \tag{16}$$

The self-energy $\Sigma_L(p)$ and wave-function renormalization Z_3 are then given by

$$\tilde{G}_L^{-1}(p) = Z_3^{-1}\tilde{\Delta}^{-1}(p) - \Sigma_L(p) , \tag{17}$$

where $\tilde{G}_L$ is the Fourier transform of G_L and

$$\tilde{\Delta}^{-1}(p) = 2\sum_\mu (1-\cos p_\mu) + m^2 . \tag{18}$$

We have

$$\frac{\partial}{\partial p_0}\Sigma_\infty(p_0, \mathbf{0})\bigg|_{p_0 = im_0} = 0 , \tag{19}$$

and Z_3 does not depend on L. The relative mass shift can now be obtained from

$$\delta_0 = -Z_3(2m_0 \sinh m_0)^{-1}\Sigma_L(\hat{p}) + O(\delta_0^2) \tag{20}$$

with

$$\hat{p} = (im_0, \mathbf{0}) . \tag{21}$$

As in the continuum case [2] Σ_L is in perturbation theory given by the sum of all one-particle-irreducible amputated Feynman graphs with two external vertices. The analysis of [2] can be carried over to the lattice case and one can isolate a class of graphs which dominate for large L, leading to

$$\begin{aligned}\Sigma_L(\hat{p}) = d(2\pi)^{-d-1} \int_{-\pi}^{\pi} &\mathrm{d}^{d+1}p\, \mathrm{e}^{ip_1 L} \tilde{G}_\infty(p) \\ &\times \{\Gamma_3(-\hat{p}, \hat{p}, 0)\tilde{G}_\infty(0)\Gamma_3(p, -p, 0) \\ &+ \Gamma_3(-\hat{p}, p, \hat{p}-p)\tilde{G}_\infty(\hat{p}-p)\Gamma_3(\hat{p}, -p, p-\hat{p}) \\ &+ \Gamma_4(-\hat{p}, \hat{p}, p, -p)\} + \mathrm{O}(\exp(-\sqrt{\tfrac{3}{2}}\, m_0 L)) \,. \end{aligned} \tag{22}$$

Γ_3 and Γ_4 are vertex functions of the $L=\infty$ theory, i.e. full propagator amputated, one-particle-irreducible 3-point and 4-point functions. I do not give the details of the derivation of (22) here, because it is analogous to the continuum case. The main new ingredient on a lattice is the following estimate on the lattice propagator.

Lemma: Let $\Delta = \Delta_\infty$ be the free lattice propagator for bare mass m, as in (13), and m_0 be defined by (15). Then there exists a constant $K(m) > 0$ such that for all $x \in \mathbb{Z}^{d+1}$

$$|\nabla_{\mu_1} \cdots \nabla_{\mu_j} \Delta(x)| \leq 2^j K(m) \exp(-m_0|x|) \,. \tag{23}$$

This estimate is needed to get the bound on the remainder term in (22).

The next step is to shift the contour of the p_1 integration into the upper half-plane and to collect contributions from poles in the integrand. For convenience I interchange p_0 and p_1, such that now

$$\hat{p} = (0, im_0, 0, \ldots, 0) \,. \tag{24}$$

Again I do not bother the reader with the details of the calculation but merely state the result. For this we have to define some quantities. Let $E(\boldsymbol{p})$ be the energy-momentum dispersion relation. This means that for fixed real $\boldsymbol{p}$ the propagator $\tilde{G}_\infty(p)$ has in the upper half p_0 plane a first simple pole at $p_0 = iE(\boldsymbol{p})$. The minimum of $E(\boldsymbol{p})$ for real $\boldsymbol{p}$ is

$$E(\boldsymbol{o}) = m_0 \,. \tag{25}$$

Furthermore $E(\boldsymbol{p})$ is assumed to be analytic in p_1 in the region $|\mathrm{Im}\, p_1| \leq \frac{1}{2}m_0$, when $p_2, \ldots, p_d$ are real. Let the residuum of $\tilde{G}_\infty$ be given by

$$\left.\frac{\partial}{\partial p_0} \tilde{G}_\infty^{-1}(p)\right|_{p_0 = iE(\boldsymbol{p})} = iZ_3^{-1} A^{-1}(\boldsymbol{p}) \,, \tag{26}$$

where

$$A(\boldsymbol{o}) = (2 \sinh m_0)^{-1} \,. \tag{27}$$

Finally let

$$q_1 = -\tfrac{1}{2} i m_0 \,, \qquad \boldsymbol{q} = (q_1, \ldots, q_d) \,, \qquad q_0 = iE(\boldsymbol{q}) \,. \tag{28}$$

Then one obtains

$$\Sigma_{\rm L}(\hat{p}) = Z_3 d(2\pi)^{-d} \int_{-\pi}^{\pi} {\rm d}^d p\, A(p)\, {\rm e}^{-E(p)L} \tau(\hat{p}, -\hat{p}, p, -p)\Big|_{p_0 = iE(p)}$$
$$+ iZ_3 d(2\pi)^{-d+1} \int_{-\pi}^{\pi} {\rm d}^{d-1} q\, A(q)\, {\rm e}^{-E(q)L}\, {\rm Res}\, \tau(\hat{p}, -\hat{p}, p, -p)\Big|_{p=q}$$
$$+ {\rm O}({\rm e}^{-\bar{\alpha}L}), \tag{29}$$

where τ is the amputated four-point function. It is expressed [3] in terms of vertex functions as

$$\tau(p_1, p_2, p_3, p_4) = \Gamma_4(p_1, p_2, p_3, p_4)$$
$$+ \Gamma_3(p_1, p_2, -p_1 - p_2) \tilde{G}_\infty(p_1 + p_2) \Gamma_3(p_3, p_4, -p_3 - p_4)$$
$$+ (p_2 \leftrightarrow p_3) + (p_2 \leftrightarrow p_4) \quad \text{for } p_1 + p_2 + p_3 + p_4 = 0\,. \tag{30}$$

In the second term of (29) the residuum of τ at its pole $p = q$ appears, which is due to the last term in (30). For the remainder term we have

$$\bar{\alpha} = \min\,(\sqrt{\tfrac{3}{2}}\, m_0, m_{\rm B})\,, \tag{31}$$

where $m_{\rm B}$ is the lowest bound state mass above m_0 giving rise to a pole in the propagator. Above eq. (29) combined with (20) represents δ_0 in terms of the amputated four-point function at certain on-shell momenta, which actually is a forward scattering amplitude.

The asymptotic behaviour of δ_0 for large L is obtained from (29) by a saddle-point integration. Let

$$E(\boldsymbol{p}) = m_0 + (2m_1)^{-1} \boldsymbol{p}^2 + {\rm O}(p^4)\,, \qquad m_1 > 0\,, \tag{32}$$

$$E(\boldsymbol{q}) = \bar{E} + (2m_2)^{-1} \sum_{i=2}^{d} q_i^2 + {\rm O}(q^4)\,, \qquad m_2 > 0\,, \tag{33}$$

with

$$\bar{E} = E(-\tfrac{1}{2} i m_0, 0, \ldots, 0)\,, \tag{34}$$

and let

$$F = Z_3^2 \tau(\hat{p}, -\hat{p}, 0, 0)\,, \tag{35}$$

$$\lambda = Z_3^{3/2} \Gamma_3(\hat{p}, k, -\hat{p} - k)\,, \tag{36}$$

where

$$\boldsymbol{k} = (-\tfrac{1}{2} i m_0, 0, \ldots, 0)\,, \qquad k_0 = i\bar{E}\,. \tag{37}$$

F is a forward scattering amplitude and λ is a three-particle coupling constant.

Finally define

$$B^{-1} = 2iZ_3 \frac{\partial}{\partial p_1} \tilde{G}_\infty^{-1}(p)\bigg|_{p=k}$$
$$= 2iA(k)^{-1} \frac{\partial E}{\partial p_1}(k)\,. \tag{38}$$

The saddle-point integration leads to

$$\delta_0 = -d(2m_0 \sinh m_0)^{-1}$$
$$\times\{F\cdot A(o)(m_1/2\pi L)^{d/2}\,\mathrm{e}^{-m_0 L}$$
$$+\lambda^2 A(k)B(m_2/2\pi L)^{(d-1)/2}\,\mathrm{e}^{-\bar{E}L}\}\{1+\mathrm{O}(L^{-1})\}\,. \tag{39}$$

For comparison let us consider the continuum case. Euclidean invariance implies

$$E(p) = (m_0^2 + p^2)^{1/2}\,,$$
$$\bar{E} = m_2 = \tfrac{1}{2}\sqrt{3}\, m_0\,,$$
$$A^{-1}(p) = 2E(p)\,, \qquad B^{-1} = 2m_0\,, \qquad m_1 = m_0\,, \tag{40}$$

and in the prefactor we have to substitute

$$\sinh m_0 \equiv a^{-1} \sinh(am_0) \xrightarrow[am_0 \to 0]{} m_0\,, \tag{41}$$

where a is the lattice spacing. In this way the asymptotic formulae (16) of ref. [2] are recovered. For theories with a nonvanishing three-point function the second term in (39) dominates and yields the asymptotic behaviour (3) with

$$\alpha = \tfrac{1}{2}\sqrt{3}\,, \qquad p = \tfrac{1}{2}(d-1)\,,$$
$$C = \frac{d}{16\pi}\left(\frac{\sqrt{3}\, m_0^2}{4\pi}\right)^{(d-3)/2} \frac{\lambda^2}{m_0^2}\,. \tag{42}$$

The results discussed above are obtained in the framework of perturbation theory. Naturally one wonders whether they are valid also nonperturbatively. For the two-dimensional Ising model the asymptotic behaviour (3) can in fact be derived from the exact solution [2]. In general it is also true in the framework of the high-temperature expansion for the d-dimensional Ising model, and for other spin systems with vanishing three-point function an analogous derivation is certainly possible. In case the three-point function does not vanish a rigorous proof would be more complicated, and I did not attempt to carry it through. However, the results above are of a purely kinematical nature and it appears reasonable that they are valid generally.

3. The mass shift in lattice gauge theory

The most interesting case is of course quantum chromodynamics. For perturbative calculations of the spectrum in a finite volume [1] as well as for Monte Carlo calculations it would be desirable to supplement them by information about finite size effects. In the following I consider pure gauge theory without quarks in $d+1=4$ dimensions, the gauge group being SU(N). Since there are not yet any data available about glueball-glueball scattering, I study the relevant quantities in the framework of the strong coupling expansion.

The calculations are based on the assumption that the kinematical structure of finite size effects, as expressed in (39), also holds for lattice gauge theory. However, we have to take into account that the propagator as well as the vertex functions are no longer scalar quantities but are matrices, whose indices account for the different orientations of plaquettes in the lattice.

For lattice gauge theory, in particular concerning the strong coupling expansion, I use the notation and conventions of [4]. The three-plaquette correlation function does not vanish and the relevant piece from (39) is

$$\delta_0 = -3(2m_0 \sinh m_0)^{-1}\lambda^2 AB \times (m_2/2\pi L)\exp(-\bar{E}L)\{1+\mathrm{O}(L^{-1})\}, \tag{43}$$

$$A = A(\boldsymbol{k}). \tag{44}$$

Let us consider the kinematical quantities $\bar{E}$, m_2, A and B. $\bar{E}$ and m_2 follow from the energy-momentum dispersion relation for the lowest 0^+ glueball, see (33). For SU(2) the dispersion relation has been calculated in [5] in the strong coupling expansion in the form

$$E = m_0 + (2m_1)^{-1}\sum_{\mu=1}^{3} 2(1-\cos p_\mu) + \mathrm{O}((1-\cos p_\mu)^2). \tag{45}$$

In our case we have

$$1-\cos p_1 = 1-\cosh \tfrac{1}{2}m_0 = \tfrac{1}{2}u^{-2}(1+\mathrm{O}(u)), \tag{46}$$

where u is the usual strong coupling expansion parameter. Therefore (45) is not sufficient for our purpose and we have to know the higher powers of $(1-\cos p_1)$ in this expansion in order to get a systematic expansion for $\bar{E}$. I solved this problem by deriving an effective transfer matrix for the low-lying 0^+ and 2^+ glueball states in the strong coupling expansion [6]. This effective transfer matrix acts on wave functions in the space of all space-like plaquettes at fixed time x_0. When diagonalized with respect to eigenvalues of momentum it reduces to a 3×3 matrix. Further diagonalization of this matrix yields the dispersion relations for the glueball states under consideration. It also yields the proper wave functions of these states for arbitrary momentum in the strong coupling expansion. Taking the 0^+ glueball and momentum $\boldsymbol{q}$ as in (28) the result is

$$E = \bar{E} - u^4(\hat{q}_2^2-\hat{q}_3^2)^2(8-\hat{q}_2^2-\hat{q}_3^2)^{-1} + \mathrm{O}(u^6), \tag{47}$$

$$\bar{E} = -4\log u - \log(1+W) - u^2 - \tfrac{17}{2}u^4 + \mathrm{O}(u^6) \text{ for SU(2)},$$

$$\bar{E} = -4\log u - \log(1+W) - u^2 - \tfrac{3}{2}u^3 - \tfrac{49}{8}u^4 + \mathrm{O}(u^5) \text{ for SU(3)}, \tag{48}$$

where (see [4])

$$W = 3v - 4u^2 \text{ for SU(2)}, \tag{49}$$

$$W = 3u + 6v_1 + 8v_2 - 18u^2 \text{ for SU(3)},$$

$$\hat{p}_\mu^2 = 2(1 - \cos p_\mu). \tag{50}$$

(47) implies that m_2^{-1} is of order u^6, which cannot be calculated from the presently available strong coupling graphs of the ordinary eighth-order calculation.

Next we turn to the quantities A and B. Let p_1 and p_2 be two plaquettes and

$$G(p_1, p_2) = \langle \mathrm{Re\,tr}\, U(p_1)\, \mathrm{Re\,tr}\, U(p_2)\rangle_c \tag{51}$$

the connected plaquette-plaquette correlation function. Let x_i be the center of p_i and σ_i its orientation, where

$$\sigma = (\mu, \nu), \qquad 0 \leqslant \mu < \nu \leqslant 4, \tag{52}$$

can take six different values. Then the propagator in momentum space is defined by

$$\tilde{G}_{\sigma_1\sigma_2}(q) = \sum_{x_2} \mathrm{e}^{-iq(x_1 - x_2)} G(p_1, p_2). \tag{53}$$

When the momentum p is on-shell,

$$p_0 = iE(\boldsymbol{p}), \tag{54}$$

one of the eigenvalues of the propagator $\tilde{G}(p)$ has a simple pole. Its residuum yields $Z_3^{-1}A^{-1}(\boldsymbol{p})$ according to (26). If we denote the corresponding eigenvector $v(\boldsymbol{p})$, this means

$$iZ_3^{-1}A^{-1}(\boldsymbol{p}) = \langle v(\boldsymbol{p})| \frac{\partial}{\partial p_0} \tilde{G}_\infty^{-1}(p)\Big|_{p_0 = iE(\boldsymbol{p})} |v(\boldsymbol{p})\rangle. \tag{55}$$

In the strong coupling expansion I obtained for the particular momentum $\boldsymbol{k}$, as in (37), for the gauge group SU(N),

$$Z_3 A \doteq (2 - \delta_{N2}) \left(N^2 \frac{\mathrm{d}u}{\mathrm{d}\beta} \right)^2 (1 + W)^{-1} \{1 + \mathrm{O}(u^4)\}. \tag{56}$$

Using the dispersion relation one derives $Z_3 B$ from (38):

$$Z_3 B = \tfrac{1}{2}(2 - \delta_{N2}) \left(N^2 \frac{\mathrm{d}u}{\mathrm{d}\beta} \right)^2 (1 + W)^{-1} \cdot K,$$

$$K = \begin{cases} 1 - u^2 + \mathrm{O}(u^4), \ \mathrm{SU}(2) \\ 1 - \frac{3}{2}u + \frac{29}{8}u^2 + \mathrm{O}(u^3), \ \mathrm{SU}(3). \end{cases} \tag{57}$$

The wave-function renormalization constant Z_3 follows from

$$D \equiv 2Z_3 A(\boldsymbol{o}) = 2(2 - \delta_{N2}) \left(N^2 \frac{\mathrm{d}u}{\mathrm{d}\beta} \right)^2 (1 + W)^{-1} \{1 + 10u^4 + \mathrm{O}(u^5)\}, \tag{58}$$

and (27) which implies

$$D = Z_3(\sinh m_0)^{-1}$$
$$= Z_3 2u^4(1+W)\{1+34u^4+O(u^5)\}\,. \tag{59}$$

In particular Z_3 cancels out in

$$A \sinh m_0 = Z_3 A/D = \tfrac{1}{2}\{1+O(u^4)\}\,, \tag{60}$$

$$B \sinh m_0 = Z_3 B/D = \tfrac{1}{4}K\,. \tag{61}$$

Fortunately we know a priori the continuum limits of these quantities. According to (40)

$$\bar{E}/m_0,\quad m_2/m_0 \to \tfrac{1}{2}\sqrt{3} = 0.866\,,$$
$$A \sinh m_0 \to \tfrac{1}{3}\sqrt{3} = 0.577\,,$$
$$B \sinh m_0 \to \tfrac{1}{2}\,. \tag{62}$$

For $A \sinh m_0$ the leading term (60) is already near its limit. $B \sinh m_0$ is plotted in fig. 1 for the gauge group SU(3) according to (61). It is a smooth function of β and in the strong coupling region it is a factor of 2 smaller than its continuum limit. Also plotted is $\bar{E}/m_0$ with $\bar{E}$ from (48) and

$$m_0 = -4 \log u - \log(1+W) - 34u^4 + O(u^5) \tag{63}$$

up to the same order. This function is near 1 in the strong coupling region and then starts rising. This is of course the wrong direction, but the situation might change in higher orders of the expansion. The case of SU(2) is quite similar.

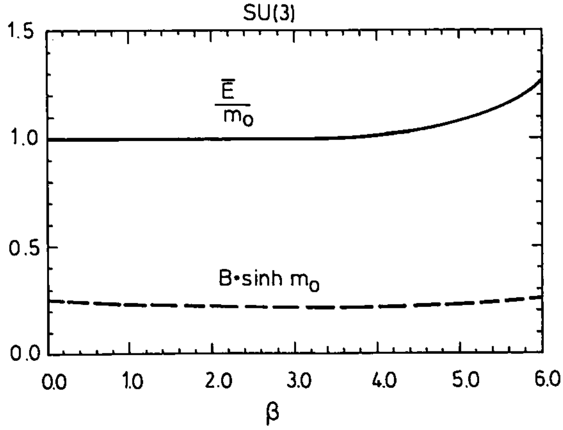

Fig. 1. $\bar{E}/m_0$ and $B \sinh m_0$ for the gauge group SU(3) in the strong coupling expansion, as explained in the text following eq. (62).

Finally we consider the most important quantity, namely the three-particle coupling constant λ. In the scaling region the mass shift is asymptotically

$$\delta_0 = -C(m_0 L)^{-1} \exp(-\tfrac{1}{2}\sqrt{3}\, m_0 L)\{1 + O(L^{-1})\}, \tag{64}$$

$$C = \frac{3}{16\pi}\frac{\lambda^2}{m_0^2}, \tag{65}$$

and λ/m_0 sets the scale for finite size effects on the mass gap. In order to get λ one has to calculate the vertex Γ_3. Analogous to the case of the propagator (53) the vertex function as well as the connected three-plaquette correlation function $\tilde{G}^{(3)}$ carry indices for the orientations of plaquettes. They are related through

$$\Gamma_{\sigma_1\sigma_2\sigma_3}(q_1, q_2, q_3) = \tilde{G}^{-1}_{\sigma_1\tau_1}(q_1)\tilde{G}^{-1}_{\sigma_2\tau_2}(q_2)\tilde{G}^{-1}_{\sigma_3\tau_3}(q_3)\tilde{G}^{(3)}_{\tau_1\tau_2\tau_3}(q_1, q_2, q_3). \tag{66}$$

The propagator and $\tilde{G}^{(3)}$ can be expanded in powers of u by the usual graphical methods. Inserting the results into (66) I calculated the vertex function up to order u^4. The result is most compactly expressed in real space. Let p_1, p_2 and p_3 be plaquettes on the lattice. Then

$$\begin{aligned}\Gamma(p_1, p_2, p_3) = {} & \left(N\frac{\mathrm{d}u}{\mathrm{d}\beta}\right)^{-3}\frac{\mathrm{d}^2u}{\mathrm{d}\beta^2}\delta(p_1, p_2, p_3) + N^{-1}(2-\delta_{N2})u^3\delta_{\mathrm{W}}(p_1, p_2, p_3) \\ & - N(2-\delta_{N2})\left(N\frac{\mathrm{d}u}{\mathrm{d}\beta}\right)^{-2}\frac{\mathrm{d}^2u}{\mathrm{d}\beta^2}u^4 \\ & \times\left[\sum_{p_1'}\delta(p_1', p_2, p_3)\delta_{\mathrm{W}}(p_1', p_1) + 2\text{permutations}\right] + O(u^5),\end{aligned} \tag{67}$$

where

$$\delta(p_1, p_2, p_3) = \begin{cases} 1 & \text{if } p_1 = p_2 = p_3 \\ 0 & \text{else}.\end{cases} \tag{68}$$

$$\delta_{\mathrm{W}}(p_1, p_2, p_2) = \begin{cases} 1 & \text{if } p_i \neq p_j \text{ pairwise} \\ & \text{and there exists a cube W such that} \\ & p_1, p_2, p_3 \in \partial\mathrm{W}, \\ 0 & \text{else},\end{cases} \tag{69}$$

and $\delta_{\mathrm{W}}(p_1, p_2)$ correspondingly. The Fourier transform of this expression, which is $\Gamma_{\sigma_1\sigma_2\sigma_3}(q_1, q_2, q_3)$, yields λ according to (compare (36))

$$Z_3^{-3/2}\lambda = \sum_{\sigma,\rho,\tau} v_\sigma(\boldsymbol{p})v_\rho(\boldsymbol{k})v_\tau(-\boldsymbol{p}-\boldsymbol{k})\Gamma_{\sigma\rho\tau}(\hat{p}, k, -\hat{p}-k). \tag{70}$$

Here $v(\boldsymbol{p})$ is the eigenvector which already appeared in (55). It is only known up to order u for the momenta under consideration. The final result is

$$Z_3^{-3}\lambda^2 = N^{-2}(2-\delta_{N2})^2\tfrac{1}{3}u^{-6}(1+W)^{-3}\{1+O(u^2)\}. \tag{71}$$

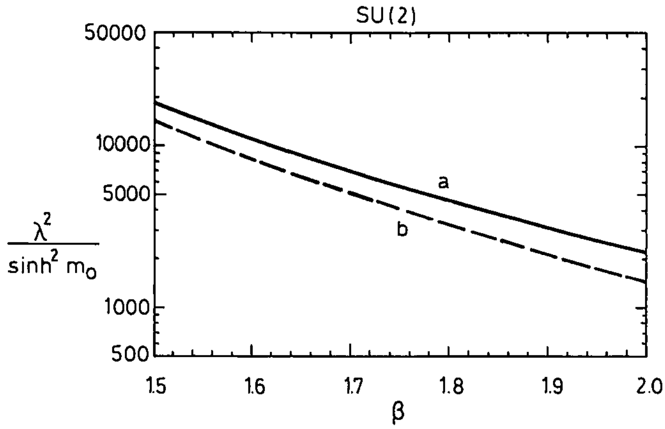

Fig. 2. The coupling constant $\lambda^2/\sinh^2 m_0$ for the gauge group SU(2) in the strong coupling expansion. Curve a represents eq. (72), whereas curve b shows the leading term not containing the factor $(1+W)^{-1}$.

To this expression only the second term in (67) contributes. In order to get rid of the renormalization constant Z_3 and to have a dimensionless quantity, we consider

$$Z_3^{-3}\lambda^2 D^3 \sinh m_0 = \lambda^2/\sinh^2 m_0$$
$$= \tfrac{4}{3}N^{-2}(2-\delta_{N2})^{-1}u^{-10}(1+W)^{-1}\{1+O(u^2)\}\,. \qquad (72)$$

This is plotted as a function of β for gauge group SU(2) and SU(3) in figs. 2 and 3, respectively. In the scaling region the true function $\lambda^2/\sinh^2 m_0$ approaches its continuum limit λ^2/m_0^2. The strong coupling expansion of other quantities and the results of Monte Carlo calculations suggest that in a crossover region around $\beta \approx 2$ for SU(2) and $\beta \approx 5$ for SU(3) we might obtain an estimate for the continuum limit

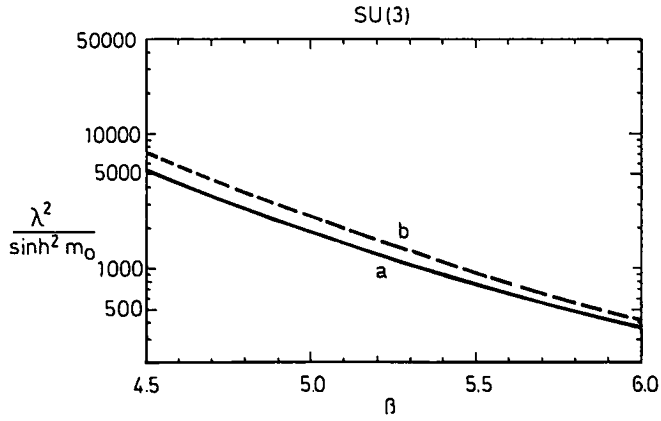

Fig. 3. The same as fig. 2, but for the gauge group SU(3).

from the strong coupling expansion. Here for both gauge groups we read off the values

$$\lambda^2/m_0^2 \approx 2 \cdot 10^3 , \tag{73}$$

corresponding to

$$C \approx 10^2 . \tag{74}$$

I consider these as rough estimates of the order of magnitude of the true values. The error of these estimates is completely unknown. Compared to other strong coupling expansions the uncertainty is relatively large, due to the fact that we only dispose of the leading term and the first correction.

The values in (73) and (74) appear rather large at first sight, which may shed some doubt on them. However, the magnitude of finite size effects, which are observed in Monte Carlo calculations [7], is compatible with this order of magnitude.

If we assume that the estimates are roughly right, what would this imply? First of all it would mean that in the gauge field case the obstacles for an extrapolation of the small-L expansions [1] of single masses to their large-L limits are much bigger than in the case of the two-dimensional nonlinear sigma model. Whether this is also true for mass ratios is unknown. Secondly it means that glueballs do interact very strongly. This may be illustrated by the strength of the Yukawa potential

$$V(r) = -\frac{g^2}{4\pi} \frac{e^{-m_0 r}}{r} , \tag{75}$$

which effectively describes the interaction between nonrelativistic glueballs. It is related to the coupling constant λ through

$$\frac{g^2}{4\pi} = \frac{\lambda^2}{16\pi m_0^2} = \frac{C}{3} \approx 40 . \tag{76}$$

This number might be compared to the corresponding one for nucleon-nucleon interactions, which is [8]

$$2f^2 = \frac{m_\pi^2}{2m_N^2} \frac{g_{\pi N}^2}{4\pi} = 0.16 . \tag{77}$$

The pion-nucleon coupling constant

$$\frac{g_{\pi N}^2}{4\pi} \approx 14 \tag{78}$$

is the direct analogue of $g^2/4\pi$, but in the Yukawa potential between nucleons an additional factor involving the pion-nucleon mass ratio enters due to the pseudoscalar nature of the interaction.

The 2^{++} glueball of course also contributes to δ_0. Its contribution in the scaling region is asymptotically

$$\Delta\delta_0 = -C_1 (m_1 L)^{-1} \exp\left(-\tfrac{1}{2}\sqrt{3}\, m_1 L\right)\{1 + O(L^{-1})\} , \tag{79}$$

where m_l is its mass. In the strong coupling expansion I obtained

$$C_l = C. \tag{80}$$

How much this contribution (79) is suppressed relative to the leading term depends on the mass difference $m_l - m_0$.

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